PROCEEDINGS OF THE
AMERICAN MATHEMATICAL SOCIETY
Volume 126, Number 3, March 1998, Pages 715–719
S 0002-9939(98)04545-6

THE SCHATTEN SPACE S_4 IS A Q-ALGEBRA

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(Communicated by Palle E. T. Jorgensen)

ABSTRACT. For any $1 \leq p \leq \infty$, let S_p denote the classical p-Schatten space of operators on the Hilbert space ℓ_2. It was shown by Varopoulos (for $p \geq 2$) and by Blecher and the author (full result) that for any $1 \leq p \leq \infty, S_p$ equipped with the Schur product is an operator algebra. Here we prove that S_4 (and thus S_p for any $2 \leq p \leq 4$) is actually a Q-algebra, which means that it is isomorphic to some quotient of a uniform algebra in the Banach algebra sense.

This note deals with the classical problem of determining when a commutative complex Banach algebra is (isomorphic to) a Q-algebra. Let us first recall some classical terminology and notation.

Given a compact set T and a Banach space X, we denote by $C(T)$ (resp. $C(T;X)$) the commutative C^*-algebra (resp. the Banach space) of all continuous functions from T into the complex field $\mathcal{C}$ (resp. X). By a uniform algebra, we mean a closed subalgebra of some commutative C^*-algebra $C(T)$.

Definition. Let A be a commutative complex Banach algebra. We say that A is a Q-algebra provided that there exist a uniform algebra C, a closed ideal I of C and a Banach algebra isomorphism from A onto the quotient algebra C/I.

Given $1 \leq p \leq +\infty$, let ℓ_p denote the Banach space of all p-summable complex sequences. Let us equip ℓ_p with the pointwise product. Then ℓ_p is clearly a Banach algebra. In 1972, Davie and Varopoulos [6], [10] proved that, actually, ℓ_p is a Q-algebra for any $1 \leq p \leq +\infty$. One of the main features of Q-algebras, discovered by Cole [13], is that they are operator algebras. Namely for any Q-algebra A, there exist a Hilbert space H and a Banach algebra isomorphism from A onto some (closed) subalgebra of $B(H)$. Thus Davie and Varopoulos obtained that all the ℓ_p's are operator algebras.

In view of these results it is tempting to consider Schatten spaces S_p which are the non-commutative analogues of ℓ_p spaces. We recall that, by definition, S_∞ denotes the Banach space of all compact operators on ℓ_2 (with operator norm) and for any $1 \leq p < +\infty$, S_p denotes the Banach space of all T in S_∞ such that $\operatorname{tr}|T|^p < \infty$, equipped with the (complete) norm $||T||_p = (\operatorname{tr}|T|^p)^{\frac{1}{p}}$.

Any bounded operator on ℓ_2 is represented by a bi-infinite matrix $(x_{ij})_{i,j\geq 1}$ with respect to the canonical basis of ℓ_2. By definition, the Schur product $*$ of two such matrices $(x_{ij})_{i,j\geq 1}$ and $(y_{ij})_{i,j\geq 1}$ is given by the formula:

$$(x_{ij}) * (y_{ij}) = (x_{ij}y_{ij})\,.$$

Received by the editors June 26, 1996.
1991 *Mathematics Subject Classification.* Primary 47D25; Secondary 47A80, 46B70.

It turns out that the Schur product is a Banach algebra multiplication on each S_p, whence the question of whether $(S_p,*)$ is an operator algebra or even a Q-algebra. In [12], Varopoulos showed that for any $2\leq p\leq\infty$, $(S_p,*)$ is an operator algebra and it was proved recently by Blecher and the author [4] that, in fact, $(S_p,*)$ is an operator algebra for all $1\leq p\leq\infty$. The question of whether S_p is a Q-algebra remained open except for $p=2$. Indeed $(S_2,*)$ is a Q-algebra because as Banach algebras, $(S_2,*)$ and ℓ_2 (equipped with the pointwise product) are the same object. The aim of this note is to provide the following partial answer.

Theorem. *Equipped with the Schur product, S_4 is a Q-algebra.*

In the proof of this result we will use the fact, proved and exploited by Varopoulos in [10], that the class of Q-algebras is stable under complex interpolation. More precisely we have:

Lemma. *Let (A_0,A_1) be a compatible couple of complex Banach algebras. For any $0\leq\theta\leq 1$, we denote by $A_\theta=[A_0,A_1]_\theta$ the Banach algebra obtained by the complex interpolation method (see [1]). If A_0 and A_1 are Q-algebras, then A_θ is a Q-algebra.*

Note that this Lemma allows us to strengthen our Theorem as follows.

Corollary. *For any $2\leq p\leq 4$, $(S_p,*)$ is a Q-algebra.*

Proof of the Theorem. We use the classical notation $\overset{\vee}{\otimes}$ to denote the injective tensor product in the Banach space category. Clearly the two Banach spaces $\ell_1\overset{\vee}{\otimes}\ell_2$ and $\ell_\infty\overset{\vee}{\otimes}\ell_2$ can be regarded as compatible for complex interpolation. The key fact in our proof is the following interpolation result which will be proved later.

$$[\ell_1\overset{\vee}{\otimes}\ell_2,\ell_\infty\overset{\vee}{\otimes}\ell_2]_{\frac{1}{2}}\simeq S_4\qquad\text{isomorphically.}\tag{1}$$

Let A and B be two Banach algebras. Their algebraic tensor product $A\otimes B$ has a canonical multiplication obtained as the tensor product of the multiplications on A and B respectively. It is given for any finite families $(a_i)_{i\geq 1},\ (c_j)_{j\geq 1}$ in A and $(b_i)_{i\geq 1},\ (d_j)_{j\geq 1}$ in B by:

$$\Big(\sum_i a_i\otimes b_i\Big)\Big(\sum_j c_j\otimes d_j\Big)=\sum_{i,j}a_ic_j\otimes b_id_j\,.\tag{2}$$

In general, this multiplication is not bounded when $A\otimes B$ is equipped with the injective tensor product. We analyze the situation in some particular cases in order to establish (3) and (4) below.

Let us assume that B is a Q-algebra. By definition, we can write $B\simeq C/I$ where $C\subset C(T)$ is a uniform algebra and I is an ideal of C. If $A\subset C(K)$ is a uniform algebra (K being a compact set), we have:

$$A\overset{\vee}{\otimes}B\subset C(K)\overset{\vee}{\otimes}C/I=C\big(K;C/I\big)\,.$$

Thus $A\overset{\vee}{\otimes}B$ is a Banach algebra in this case. Moreover the well-known identification $C\big(K;C/I\big)=\dfrac{C(K;C)}{C(K;I)}$ shows that $C\big(K;C/I\big)$ is a Q-algebra, hence $A\overset{\vee}{\otimes}B$ itself is a Q-algebra (any subalgebra of a Q-algebra is a Q-algebra).

We now assume that A is a Q-algebra, i.e. $A\simeq D/J$ for some uniform algebra D and $J\subset D$ a closed ideal. From above, we have that $D\overset{\vee}{\otimes}B$ is a Q-algebra.

Moreover $J \overset{\vee}{\otimes} B$ is an ideal of $D \overset{\vee}{\otimes} B$, hence $\dfrac{D \overset{\vee}{\otimes} B}{J \overset{\vee}{\otimes} B}$ is a Q-algebra (any quotient of a Q-algebra is a Q-algebra).

From Varopoulos's work [11], we know that ℓ_1 (with pointwise product) can be represented as a Q-algebra $\ell_1 \simeq D/J$ in such a way that J is a complemented subspace of D. When J is complemented in D, the quotient space $\dfrac{D \overset{\vee}{\otimes} B}{J \overset{\vee}{\otimes} B}$ is canonically isomorphic to $(D/J) \overset{\vee}{\otimes} B$, hence $\ell_1 \overset{\vee}{\otimes} B$ is a Q-algebra. With $B = \ell_2$ we thus obtain that:

$$\ell_1 \overset{\vee}{\otimes} \ell_2 \quad \text{is a } Q\text{-algebra.} \tag{3}$$

The arguments above also show (but it is obvious in this case) that:

$$\ell_\infty \overset{\vee}{\otimes} \ell_2 \quad \text{is a } Q\text{-algebra.} \tag{4}$$

Let $(e_n)_{n\geq 1}$ be the canonical basis of ℓ_2. Let $(x_{ij})_{i,j\geq 1}$ and $(y_{ij})_{i,j\geq 1}$ be two finite matrices and let $x = \sum_{i,j} x_{ij}\ e_i \otimes e_j$, $\quad y = \sum_{i,j} y_{ij}\ e_i \otimes e_j$ be considered as elements of $\ell_\infty \otimes \ell_\infty$, say. The product of x and y as defined by (2) is:

$$xy = \sum_{i,j} x_{ij}\ y_{ij}\ e_i \otimes e_j\,.$$

Thus the products on $\ell_1 \overset{\vee}{\otimes} \ell_2$ and $\ell_\infty \overset{\vee}{\otimes} \ell_2$ for which we proved (3) and (4) coincide with the Schur product. Consequently, our Theorem readily follows from Varopoulos's Lemma and the assertion (1).

We now turn to the proof of (1). By a standard approximation argument, it suffices to show that

$$[\ell_1^N \overset{\vee}{\otimes} \ell_2^N, \ell_\infty^N \overset{\vee}{\otimes} \ell_2^N]_{\frac{1}{2}} \simeq S_4^N, \tag{1$'$}$$

with isomorphic constant not depending on $N \geq 1$. This result is a consequence of some recent work of Pisier on complex interpolation theory for operator spaces [9]. The proof requires some background on this theory and we will assume the reader is familiar with the notions of operator spaces, completely bounded maps and the duality theory of operator spaces (see [2], [3], [5], [7], [9]). We merely recall that given a Banach space X, there is, among all the operator space structures on X, a largest one called $\mathrm{Max}(X)$, and a smallest one called $\mathrm{Min}(X)$. They can be defined as follows. For any $n, m \geq 1$ and any $n \times m$ matrix $x = [x_{ij}]$ with entries in X,

$$\begin{aligned} ||[x_{ij}]||_{M_{n,m}(\mathrm{Max}(X))} = \mathrm{Sup}\, \big\{ ||[T(x_{ij})]||_{M_{n,m}(B(H))} \mid H \quad &\text{is a Hilbert space,} \\ T\colon X \to B(H) \quad &\text{is a linear contraction} \big\}, \end{aligned}$$

$$||[x_{ij}]||_{M_{n,m}(\mathrm{Min}(X))} = \mathrm{Sup}\, \big\{ ||[\xi(x_{ij})]||_{M_{n,m}} \mid \xi \in X^*,\ ||\xi|| \leq 1 \big\}\,.$$

Moreover for any Banach space X, $\mathrm{Max}(X^*)$ is the operator space dual of $\mathrm{Min}(X)$ in the sense of [3], [5], [7].

Pisier [9] introduced complex interpolation for operator spaces as follows. Let E_0 and E_1 be two operator spaces, and assume that (E_0, E_1) is a compatible couple in the sense of Banach space interpolation. For any integers $n, m \geq 1$,

$(M_{n,m}(E_0), M_{n,m}(E_1))$ is then a compatible couple. For any $0 \leq \theta \leq 1$ we can equip the interpolation space $E_\theta = [E_0, E_1]_\theta$ with matrix norms by letting:

$$M_{n,m}(E_\theta) = [M_{n,m}(E_0), M_{n,m}(E_1)]_\theta. \tag{5}$$

It turns out that (5) defines an operator space structure on E_θ.

Let E be an operator space of finite dimension N. We denote by $\overline{E}$ the complex conjugate of E. Fixing a basis on $\mathcal{C}^N$, we can then regard $(\overline{E}^*, E)$ as a compatible couple. Then Pisier proved [9, Corollary 2.4] that with the operator space structure given by (5), $[\overline{E}^*, E]_{\frac{1}{2}}$ is completely isometric to the operator space OH_N, which is a special Hilbertian operator space studied in [9]. In particular, if F is another N-dimensional operator space,

$$[\overline{E}^*, E]_{\frac{1}{2}} = [\overline{F}^*, F]_{\frac{1}{2}} \qquad \text{completely isometrically.}$$

Restricting to column matrices we obtain that for any $n \geq 1$:

$$[C_n(\overline{E}^*), C_n(E)]_{\frac{1}{2}} = [C_n(\overline{F}^*), C_n(F)]_{\frac{1}{2}} \tag{6}$$

isometrically for any two operator spaces E, F of the same finite dimension where, as usual, we use the notation C_n for $M_{n,1}$.

We now claim that $(1)'$ will follow from (6) applied with $E = C_N$ and $F = \mathrm{Min}(\ell_\infty^N)$.

The operator space $E = C_N \subset M_N$ is ℓ_2^N as a Banach space. It is well-known that $\overline{E}^* = R_N \ (:= M_{1,N})$ in the operator space sense and moreover:

$$C_N\ (C_N) = S_2^N,$$
$$C_N\ (R_N) = S_\infty^N = M_N.$$

This follows from the general identity $M_{n,m}(M_{p,q}) = M_{np,mq}$. Thus we have:

$$[C_N(\overline{E}^*), C_N(E)]_{\frac{1}{2}} = S_4^N \qquad \text{isometrically.}$$

Let us now turn to the right-hand side of (6) for $F = \mathrm{Min}(\ell_\infty^N)$. The very definition of the minimal operator space structure means $M_{n,m}(\mathrm{Min}(X)) = M_{n,m} \overset{\vee}{\otimes} X$ for any Banach space X, hence:

$$C_N(\mathrm{Min}(\ell_\infty^N)) = \ell_\infty^N \overset{\vee}{\otimes} \ell_2^N\,.$$

Furthermore the operator space dual of $\mathrm{Min}(\ell_\infty^N)$ is $\mathrm{Max}(\ell_1^N)$, hence it remains to check that

$$C_N(\mathrm{Max}(\ell_1^N)) \simeq \ell_1^N \overset{\vee}{\otimes} \ell_2^N \tag{7}$$

isomorphically with an isomorphic constant not depending on N.

Let $\varphi_1, \dots, \varphi_N$ be elements of ℓ_1^N and let $\varphi = (\varphi_1, \dots, \varphi_N)^t \in C_N(\mathrm{Max}(\ell_1^N))$ be the corresponding column matrix. Let us consider the linear map $u \colon \ell_\infty^N \to \ell_2^N$ defined by $u(x) = (\langle \varphi_j, x\rangle)_{1 \leq j \leq N}$ for any $x \in \ell_\infty^N$. It is not hard to derive from the definition of the maximal operator space structure that:

$$||\varphi|| = \mathrm{Sup}\big\{\big(\sum_{i=1}^N ||w(\varphi_i)||^2\big)^{\frac{1}{2}}\big\}$$

where the supremum runs over all contractions $w \colon \ell_1^N \to H$. Now observe that for such a map w, the quantity $\big(\sum_{i=1}^N ||w\varphi_i||^2\big)^{\frac{1}{2}}$ is the Hilbert-Schmidt norm of uw^*.

Consequently, $||\varphi|| = \pi_2(u)$, where $\pi_2(u)$ denotes the 2-summing norm of u (see e.g. [8, Chapter 1]). By the so-called little Grothendieck Theorem (see e.g. [8, Theorem 5.4]), we have $\pi_2(u) \leq K||u||$ for some universal constant K. Hence we have $||u|| \leq ||\varphi|| \leq K||u||$. From the identification $B(\ell_\infty^N, \ell_2^N) = \ell_1^N \overset{\vee}{\otimes} \ell_2^N$, we then deduce (7). □

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