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ON CARDINAL INVARIANTS FOR CCC σ-IDEALS

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ABSTRACT. We show several results about cardinal invariants for σ-ideals of the reals. In particular we show that for every CCC σ-ideal on the real line $p \leq \mathrm{cof}(J)$.

In this paper we will consider σ-ideals on the real line with Borel basis; i.e., every element from the σ-ideal is contained in a Borel set from the σ-ideal. We assume also that every singleton is in the σ-ideal.

We say that σ-ideal $\mathcal{J}$ is CCC (countable chain condition) if the quotient Boolean algebra $B(\mathbf{R})/\mathcal{J}$ is CCC, where $B(\mathbf{R})$ is the σ-algebra of all Borel sets of the reals.

We say that a σ-ideal $\mathcal{J}$ is invariant if for each $A \in \mathcal{J}$ and $t \in \mathbf{R}$, $A + t \in \mathcal{J}$ and $-A \in \mathcal{J}$.

We define the following cardinal invariants of $\mathcal{J}$:
$\mathrm{cov}(\mathcal{J}) = \min\{|F| : F \subset \mathcal{J} \text{ and } \bigcup F = \mathbf{R}\}$,
$\mathrm{add}(\mathcal{J}) = \min\{|F| : F \subset \mathcal{J} \text{ and } \bigcup F \notin \mathcal{J}\}$,
$\mathrm{non}(\mathcal{J}) = \min\{|X| : X \subset \mathbf{R} \text{ and } X \notin \mathcal{J}\}$,
$\mathrm{cof}(\mathcal{J}) = \min\{|F| : F \subset \mathcal{J} \text{ and } \forall_{X \in \mathcal{J}} \exists_{Y \in F} X \subset Y\}$,
$\mathrm{cov}_l(\mathcal{J}) = \min\{|F| : F \subset \mathcal{J} \text{ and } \exists_{B \notin \mathcal{J} \text{Borel}} B \subset \bigcup F\}$.

Observe that $\mathrm{cov}_l(\mathcal{J})$ can be smaller than $\mathrm{cov}(\mathcal{J})$, for example, in some models of set theory for the σ-ideal generated by meagre sets on $(-\infty, 0]$ and null sets on $[0, \infty)$.

$\mathbf{p} = \min\{|F| : F \subset [\omega]^\omega$ such that for every finite subset F_0 of F, $|\bigcap F_0| = \omega$ and there is no $A \in [\omega]^\omega$ such that for every $B \in F, |B \setminus A| < \omega\}$ (see [D]).

Definition. $X \subset \mathbf{R}$ is a Q-set if every subset of X is relatively G_δ in X.

Theorem 1. *Let $\mathcal{J} \subset P(\mathbf{R})$ be a σ-ideal with Borel basis and CCC in $B(\mathbf{R})$. Then $\mathbf{p} \leq \mathit{cof}(\mathcal{J})$.*

Proof. In fact, in the proof we use only that every set of size less than $\mathbf{p}$ is a Q-set. Assume that $\mathrm{cof}(\mathcal{J}) < \mathbf{p}$. We know that every set of size $\mathrm{cof}(\mathcal{J})$ is a Q-set (see [MS]). Let $X \subset \mathbf{R}$ with $X \notin \mathcal{J}$ of size $\mathrm{cof}(\mathcal{J})$. Let B be a Borel set such that $X \subset B$ and for each Borel set $D \subset (B \setminus X)$, $D \in \mathcal{J}$. Such a set exists by CCC. Observe that $B \setminus X \notin \mathcal{J}$. Otherwise, there is a Borel set $B_1 \supset (B \setminus X)$ with $B_1 \in \mathcal{J}$; thus $B \setminus B_1 \subset X$ and $B \setminus B_1$ is Borel and $B \setminus B_1 \notin J$ which is

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impossible. Let $\{A_\alpha : \alpha < \mathrm{cof}(\mathcal{J})\}$ be a basis for $\mathcal{J}$. Let $y_\alpha \in B \setminus (A_\alpha \cup X)$. Then let $Y = \{y_\alpha : \alpha < \mathrm{cof}(\mathcal{J})\}$. Then $X \cup Y$ is a Q-set so Y is a G_δ-set in $X \cup Y$. There is a G_δ-set G such that $Y \subset (G \cap B)$ and $(G \cap B) \cap X = \emptyset$. Of course $G \cap B \notin \mathcal{J}$. Contradiction.

P. Zakrzewski observed that from the proof of Theorem 1 follows the following result. Let $\mathrm{non}^*(\mathcal{J}) = \min\{\kappa : \forall_{X \notin \mathcal{J}} \exists_{Y \subset X} |Y| = \kappa \wedge Y \notin \mathcal{J}\}$. Let $\kappa_0 = \min\{|X| : \exists_{Y \subset X} Y \notin B(X)\}$. Then $\kappa_0 \leq \mathrm{non}^*(\mathcal{J})$. It is easy to see that $\mathbf{p} \leq \kappa_0$ and $\mathrm{non}^*(\mathcal{J}) \leq \mathrm{cof}(\mathcal{J})$.

Definition. Let $\mathcal{J} \subset P(\mathbf{R})$ be a σ-ideal with Borel basis and CCC in $B(\mathbf{R})$. We say that $\mathcal{J}$ is regular if for each Borel set $B \subset \mathbf{R}^2$ $\{x : B_x \in \mathcal{J}\} \equiv C(\mathrm{mod}\mathcal{J})$ for some Borel set C.

Observe that for each Borel set $B \subset \mathbf{R}^2$ $\{x : B_x \in \mathcal{J}\}$ is Borel (see [K]) where $\mathcal{J}$ is a σ-ideal of null sets or a σ-ideal of meagre sets so these σ-ideals are regular.

Theorem 2. *Let $\mathcal{J}$ be regular. Then* $\min\{\mathbf{p},\ cov_l(\mathcal{J})\} \leq non(\mathcal{J})$.

Proof. Assume that $\mathrm{non}(\mathcal{J}) < \mathbf{p}$. Then there is Q-set $X \subset \mathbf{R}$ with $X \notin \mathcal{J}$. We can assume that every subset of size less than $|X|$ of X is in $\mathcal{J}$. Let us consider $A = \{(x, y) \in X \times X : y < x\}$ where $<$ is a well order on X of type $|X|$. For each $x \in X$ A_x is a G_δ-set in X. By [BBM] we have that A is relatively G_δ in $X \times X$. Let G be a G_δ in $\mathbf{R}^2$ such that $G \cap (X \times X) = A$. Let K be a Borel set such that $X \subset K$ and for each Borel set $C \subset (K \setminus X)$, $C \in \mathcal{J}$. Let $H = G \cap (K \times K)$. Then each section $H_x \in \mathcal{J}$ for $x \in X$. Assume that $|X| < \mathrm{cov}_l(\mathcal{J})$. So $\{y \in K : y \notin \bigcup_{x \in X} H_x)\} \notin \mathcal{J}$ so also $\{y \in K : H^y \in \mathcal{J}\} \notin \mathcal{J}$. Thus by regularity of $\mathcal{J}$ it contains a Borel set $W \notin \mathcal{J}$. Then $W \cap X \neq \emptyset$ so we have that for some $y \in X$ $H^y \in \mathcal{J}$. Contradiction.

Remark. In fact we show that if a Q-set $X \subset \mathbf{R}$ is of size less than $\mathrm{cov}(\mathcal{J})$ for a regular $\mathcal{J}$. Then $X \in \mathcal{J}$.

Corollary. *Assume Martin's Axiom. Then $p = \mathbf{c}$ and $cov_l(\mathcal{J}) = \mathbf{c}$ So for regular ideals we have $non(\mathcal{J}) = \mathbf{c}$.*

Observe that it is not true for arbitrary CCC σ-ideal. If it is consistent that there is a measurable cardinal then the following is consistent:

Martin's Axiom holds and there exists $\kappa < \mathbf{c}$ such that $P(\kappa)$ contains a proper uniform ω_1-saturated, κ-additive ideal $\mathcal{K}$. We define $\mathcal{J} = \{B \in B(\mathbf{R}) : B \cap \kappa \in \mathcal{K}\}$. Then $\mathcal{J}$ is CCC so by Martin's Axiom $\mathrm{cov}(\mathcal{J}) = c$ and $\kappa \notin \mathcal{J}$ so $\mathrm{non}(\mathcal{J}) \leq \kappa$.

Question. *Is it true that for every regular σ-ideal $\mathcal{J}$ $\mathbf{p} \leq non(\mathcal{J})$?*

We say that a set $X \subset \mathbf{R}$ is of strong measure zero if for every sequence of positive numbers $\{\epsilon_n\}_{n\in\omega}$ there is a sequence of intervals $\{L_n\}_{n\in\omega}$ such that $X \subset \bigcup_n L_n$ and $|L_n| < \epsilon_n$ for each n.

By [GMS] X is strong measure zero set iff for each meagre set F, $X + F \neq \mathbf{R}$.

We say that $X \subset \mathbf{R}$ is strongly meagre if for each Lebesgue null set G, $X + G \neq \mathbf{R}$ (see [M])

Theorem 3. *Assume that X is a meagre, strong measure zero set or X is null, strongly meagre set. Then X is in every invariant CCC σ-ideal.*

Proof. Assume that X is meagre with strong measure zero. Assume that $X \notin \mathcal{J}$ where $\mathcal{J}$ is an invariant CCC σ-ideal. Let $t_0 = 0$. Then there is an F_σ meagre set F_0

such that $X \subset F_0$. We construct a sequence of F_σ, meagre sets F_α and a sequence of real numbers t_α for $\alpha < \omega_1$. There is $t_\alpha \in \mathbf{R}$ such that $(X + t_\alpha) \cap \bigcup_{\beta<\alpha} F_\beta = \emptyset$. Let F_α be an F_σ meager set containing $\bigcup_{\beta<\alpha} F_\beta \cup (X + t_\alpha)$.

Let $G_\alpha = F_\alpha \setminus \bigcup_{\beta<\alpha} F_\beta$. This is a family of pairwise disjoint Borel sets such that none of them belongs to ideal because each contains some $X + t_\alpha$. Contradiction.

The proof for Lebesgue measure is similar.

Theorem 4. *Let $\mathcal{J}$, $\mathcal{I}$ be invariant CCC σ-ideals. Then* $\min\{cov(\mathcal{I}), non(\mathcal{I})\} \leq non(\mathcal{J})$.

Proof. For $\mathcal{I} = \mathcal{M}$ or $\mathcal{I} = \mathcal{N}$ this is a corollary of Theorem 3; for the general case it is a corollary of the proof of Theorem 3.

Observe that for an invariant CCC σ-ideal $\mathcal{J}$ the question has positive answer because we apply Theorem 4 for $\mathcal{I} = \mathcal{M}$ and by known facts that $\mathbf{p} \leq \min\{\mathrm{cov}(\mathcal{M}), \mathrm{non}(\mathcal{M})\}$ we get that $\mathbf{p} \leq \mathrm{non}(\mathcal{J})$.

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