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A WEAK-TYPE INEQUALITY OF SUBHARMONIC FUNCTIONS

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ABSTRACT. We prove the weak-type inequality $\lambda\mu(u+|v|\geq\lambda)\leq(\alpha+2)\int_{\partial D}u\,d\mu$, $\lambda>0$, between a non-negative subharmonic function u and an $\mathbb{H}$-valued smooth function v, defined on an open set containing the closure of a bounded domain D in a Euclidean space $\mathbb{R}^n$, satisfying $|v(0)|\leq u(0)$, $|\nabla v|\leq|\nabla u|$ and $|\Delta v|\leq\alpha\Delta u$, where $\alpha\geq 0$ is a constant. Here μ is the harmonic measure on ∂D with respect to 0. This inequality extends Burkholder's inequality in which $\alpha=1$ and $\mathbb{H}=\mathbb{R}^\nu$, a Euclidean space.

1. A weak-type inequality

Let Ω be an open subset of $\mathbb{R}^n$ where n is a positive integer. Let D be a bounded subdomain of Ω with $0\in D$ and $\partial D\subset\Omega$. Let μ be the harmonic measure on ∂D with respect to 0. Let $\mathbb{H}$ be a Hilbert space over $\mathbb{R}$. For $x,y\in\mathbb{H}$ we denote by $x\cdot y$ the inner product of x and y and put $|x|^2=x\cdot x$. We consider two smooth functions u and v on Ω; that is, u and v have continuous partial derivatives up to the second order. Here, u is real-valued and v is $\mathbb{H}$-valued. By ∇u we denote the gradient of u and by Δu, the Laplacian of u. Write u_i for the partial derivative of u with respect to the ith variable. Thus, $\nabla v=(v_1,\dots,v_n)\in\mathbb{H}^n$, the standard product Hilbert space. Let $\alpha\geq 0$ be a constant.

Theorem. *If u is a non-negative subharmonic function on Ω and*
(i) $|v(0)|\leq u(0)$,
(ii) $|\nabla v|\leq|\nabla u|$ *on* Ω,
(iii) $|\Delta v|\leq\alpha\Delta u$ *on* Ω,
then, for all $\lambda>0$, we have

$$\lambda\mu(u+|v|\geq\lambda)\leq(\alpha+2)\int_{\partial D}u\,d\mu.$$

Corollary. *If u and v are as in the Theorem, then for all $\lambda>0$*

$$\lambda\mu(|v|\geq\lambda)\leq(\alpha+2)\int_{\partial D}u\,d\mu.$$

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Remark 1.1. In [1] Burkholder proved the inequality in the Theorem when $\alpha = 1$ and $\mathbb{H} = \mathbb{R}^\nu$, a Euclidean space. For basic facts of harmonic measures and subharmonic functions one may see [2].

2. Technical lemmas

Put $S = \{(x,y)\colon x > 0 \text{ and } y \in \mathbb{H} \text{ with } |y| > 0\}$. Define two functions U and V on S by

$$U(x,y) = \begin{cases} (|y| - (\alpha+1)x)(x+|y|)^{1/(\alpha+1)} & \text{if } x + |y| < 1, \\ 1 - (\alpha+2)x & \text{if } x + |y| \geq 1 \end{cases}$$

and

$$V(x,y) = \begin{cases} -(\alpha+2)x & \text{if } x + |y| < 1, \\ 1 - (\alpha+2)x & \text{if } x + |y| \geq 1. \end{cases}$$

Observe that U is continuous on S.

Lemma 1. (a) *$V \leq U$ on S.*
(b) *$U(x,y) \leq 0$ if $x \geq |y|$.*

Proof. For (a) we may assume $x + |y| < 1$. Write $x + |y| = r^{\alpha+1}$. Since $0 < r < 1$, we have

$$V(x,y) - U(x,y) = -r^{\alpha+2} - (1-r)(\alpha+2)x < 0.$$

In order to prove (b) assume $|y| \leq x$. If $x + |y| < 1$, then $|y| - (\alpha+1)x \leq |y| - x \leq 0$, hence $U(x,y) \leq 0$. If $x + |y| \geq 1$, then $U(x,y) = 1 - (\alpha+2)x \leq x + |y| - (\alpha+2)x = |y| - (\alpha+1)x \leq |y| - x \leq 0$. □

Lemma 2. *If $x + |y| < 1$, then $U_x(x,y) + \alpha|U_y(x,y)| \leq 0$.*

Proof. If $x + |y| < 1$, then differentiation gives

$$\begin{cases} U_x(x,y) = -\dfrac{(\alpha+1)(\alpha+2)x + \alpha(\alpha+2)|y|}{(\alpha+1)(x+|y|)^{\alpha/(\alpha+1)}}, \\ U_y(x,y) = \dfrac{(\alpha+2)y}{(\alpha+1)(x+|y|)^{\alpha/(\alpha+1)}}. \end{cases}$$

Now the lemma is clear. □

For differentiating vector functions one may see [3].

Lemma 3. *If $h \in \mathbb{R}$, $k \in \mathbb{H}$, $(x,y) \in S$ and $x + |y| < 1$, then*

$$\begin{aligned} &U_{xx}(x,y)h^2 + 2U_{xy}(x,y) \cdot hk + U_{yy}(x,y)k \cdot k \\ &\leq (|k|^2 - h^2)\frac{(\alpha+2)}{(\alpha+1)}(x+|y|)^{-\alpha/(\alpha+1)}. \end{aligned}$$

Proof. Put $I = \{t \in \mathbb{R}\colon x + th > 0,\ |y + tk| > 0 \text{ and } x + th + |y + tk| < 1\}$ and observe that $0 \in I$ and I is an open set. Define a function G on I by

$$G(t) = U(x + th, y + tk).$$

From the chain rule we have

$$G''(0) = U_{xx}(x,y)h^2 + 2U_{xy}(x,y) \cdot hk + U_{yy}(x,y)k \cdot k.$$

Thus it suffices to show

$$G''(0) \le (|k|^2 - h^2)\frac{(\alpha+2)}{(\alpha+1)}(x+|y|)^{-\alpha/(\alpha+1)}.$$

For this we define more functions K, Q and R on I by $K = K(t) = x + th$, $Q = |y + tk|$ and $R = K + Q$. We omit the argument $t \in I$ in the following computations. On I we have

$$G = R^{(\alpha+2)/(\alpha+1)} - (\alpha+2)KR^{1/(\alpha+1)}.$$

Differentiating G, we get

$$G' = \frac{\alpha+2}{\alpha+1}R'R^{1/(\alpha+1)} - (\alpha+2)hR^{1/(\alpha+1)} - \frac{\alpha+2}{\alpha+1}KR'R^{-\alpha/(\alpha+1)}$$

and

$$\eta G'' = R''R^2 + \frac{1}{\alpha+1}(R')^2R - 2hR'R - KR''R + \frac{\alpha}{\alpha+1}K(R')^2$$

where

$$\eta = \frac{\alpha+1}{\alpha+2}R^{(2\alpha+1)/(\alpha+1)}.$$

Rearranging terms and inserting $(R')^2R - R(R')^2$, we have

$$\begin{aligned}\eta G'' &= (R''R - KR'' - 2hR' + (R')^2)R + \left(-R + \frac{1}{\alpha+1}R + \frac{\alpha}{\alpha+1}K\right)(R')^2 \\ &= (|k|^2 - h^2)R - \frac{\alpha}{\alpha+1}Q(R')^2 \le (|k|^2 - h^2)R.\end{aligned}$$

Here we used the observation that $K' = h$, $Q' = R' - h$, $QQ' = k \cdot (y + tk)$ and $QR'' = QQ'' = |k|^2 - (Q')^2$. Putting $t = 0$ we get the desired inequality and this proves Lemma 3. $\square$

Lemma 4. *If $(x_0, y_0) \in S$ and $x_0 + |y_0| = 1$, then $U(x, y) \le 1 - (\alpha+2)x$ for all (x, y) in a neighborhood of (x_0, y_0).*

Proof. Let $(x_0, y_0) \in S$ and $x_0 + |y_0| = 1$. Define $r = r(x, y)$ on S by $x + |y| = r^{\alpha+1}$. Observe that $U(x, y) = 1 - (\alpha+2)x$ if $r(x, y) \ge 1$. Now assume $0 < r(x, y) < 1$. Then, one can write

$$\begin{aligned}U(x, y) - (1 - (\alpha+2)x) &= r^{\alpha+2} - (\alpha+2)xr - (1 - (\alpha+2)x) \\ &= (r-1)C(x, y)\end{aligned}$$

where $C(x, y) = (r^{\alpha+2} - 1)/(r - 1) - (\alpha+2)x$. As (x, y) tends to (x_0, y_0), we see that $r(x, y)$ tends to 1 and $C(x, y)$ tends to $(\alpha+2)(1 - x_0) = (\alpha+2)|y_0| > 0$. Thus the lemma follows.

3. Proof of the inequality

We may assume $\lambda = 1$ and $\int_{\partial D} u \, d\mu < \infty$; thus we are to prove the inequality

$$\mu(u + |v| \ge 1) \le (\alpha+2)\int_{\partial D} u \, d\mu. \tag{3.1}$$

We may further assume that

(iv) $$u > 0 \text{ and } |v| > 0.$$

Indeed, for each $\varepsilon > 0$, the functions $u + \varepsilon$ and (v, ε), where (v, ε) has value in the standard product Hilbert space $\mathbb{H} \times \mathbb{R}$, satisfy this extra assumption as well as the assumptions of the theorem. Now, the inequality

$$\mu(u + \varepsilon + |(v, \varepsilon)| \geq 1) \leq (\alpha + 2) \int_{\partial D} (u + \varepsilon)\, d\mu$$

yields, as $\varepsilon \to 0$, the inequality (3.1) because $\mu(u+|v| \geq 1) \leq \mu(u+\varepsilon+|(v,\varepsilon)| \geq 1)$.

Let the functions U and V be as in the previous section. Observe, from the assumption (iv), that $(u, v) \in S$ on Ω. The inequality (3.1) is equivalent to

$$\int_{\partial D} V(u, v)\, d\mu \leq 0.$$

According to (a) of Lemma 1 it suffices to prove

$$\int_{\partial D} U(u, v)\, d\mu \leq 0.$$

Also, (b) of Lemma 1 and the assumption (i) imply $U(u(0), v(0)) \leq 0$. Hence the proof is complete if we can show

$$\int_{\partial D} U(u, v)\, d\mu \leq U(u(0), v(0))$$

which follows from the superharmonicity of $U(u, v)$.

In order to show that $f = U(u, v)$ is superharmonic on Ω we define subsets Ω^+ and Ω^- of Ω by $\Omega^+ = \{\omega \colon u(\omega) + |v(\omega)| > 1\}$ and $\Omega^- = \{\omega \colon u(\omega) + |v(\omega)| < 1\}$. Observe that the continuity of u and v implies that the sets Ω^+ and Ω^- are open.

On Ω^+ we have $f = 1 - (\alpha + 2)u$, which is superharmonic because u is subharmonic by the assumption.

On Ω^- the smooth function f is superharmonic because $\Delta f \leq 0$, as is checked in the following. For each $1 \leq i \leq n$, the chain rule gives

$$f_i = U_x(u, v)u_i + U_y(u, v) \cdot v_i \text{ and } f_{ii} = U_x(u, v)u_{ii} + U_y(u, v) \cdot v_{ii} + A_i$$

where $A_i = U_{xx}(u, v)u_i^2 + 2U_{xy}(u, v) \cdot u_i v_i + U_{yy}(u, v)v_i \cdot v_i$. Thus

$$\Delta f = U_x(u, v)\Delta u + U_y(u, v) \cdot \Delta v + \sum_{i=1}^{n} A_i.$$

Lemma 2, the assumption (iii), the Cauchy-Schwarz inequality and the assumption that u is subharmonic, imply

$$\begin{aligned} U_x(u, v)\Delta u + U_y(u, v) \cdot \Delta v &\leq U_x(u, v)\Delta u + |U_y(u, v)|\, |\Delta v| \\ &\leq (U_x(u, v) + \alpha |U_y(u, v)|)\Delta u \leq 0. \end{aligned}$$

On the other hand, for $1 \leq i \leq n$ we put $x = u$, $h = u_i$, $y = v$ and $k = v_i$, and apply the assumption (iv) and Lemma 3 to get

$$\begin{aligned} &U_{xx}(u, v)u_i^2 + 2U_{xy}(u, v) \cdot u_i v_i + U_{yy}(u, v)v_i \cdot v_i \\ &\leq (|v_i|^2 - u_i^2)\frac{(\alpha + 2)}{(\alpha + 1)}(u + |v|)^{-\alpha/(\alpha+1)}. \end{aligned}$$

Hence

$$\sum_{i=1}^{n} A_i \leq (|\nabla v|^2 - |\nabla u|^2)\frac{(\alpha + 2)}{(\alpha + 1)}(u + |v|)^{-\alpha/(\alpha+1)} \leq 0$$

by the assumption (ii). This proves that f is superharmonic on Ω^-.

Finally assume $u(\omega) + |v(\omega)| = 1$. Then, by Lemma 4 and the continuity of u and v we have $f \leq 1 - (\alpha + 2)u$ in a neighborhood of ω. Thus, for all small $\rho > 0$ the subharmonicity of u implies

$$\mathrm{Avg}(f; \omega, \rho) \leq \mathrm{Avg}(1 - (\alpha + 2)u; \omega, \rho) \leq 1 - (\alpha + 2)u(\omega) = f(\omega).$$

Here, $\mathrm{Avg}(f; \omega, \rho)$ is the average of f over the ball of radius ρ centered at ω. Hence f is superharmonic at ω.

This proves superharmonicity of $f = U(u, v)$ on Ω and completes the proof of the inequality (3.1), hence that of the Theorem. □

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