

FINITISTIC DIMENSION AND ZIEGLER SPECTRUM

HENNING KRAUSE

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ABSTRACT. Given a two-sided artinian ring Λ , it is shown that the Ziegler spectrum of Λ forms a test class for certain homological properties of Λ . We discuss the finitistic dimension of Λ , Nunke's condition, and also the relation between the big and the little finitistic dimension.

Let Λ be a two-sided artinian ring. Denote by $\text{Mod } \Lambda$ the category of (right) Λ -modules and by $\text{mod } \Lambda$ the full subcategory of all finitely presented Λ -modules. Given a Λ -module M , we denote by $\text{pd } M$ its projective dimension, and the *finitistic dimension* $\text{Fin. dim } \Lambda$ of Λ is the supremum of the projective dimensions of the Λ -modules with finite projective dimension. The *Ziegler spectrum* $\text{Zsp } \Lambda$ of Λ is by definition the set of isomorphism classes of indecomposable pure-injective Λ -modules [14].

The aim of this note is to show that the Ziegler spectrum forms a test class for certain homological properties of Λ . Furthermore, the Ziegler spectrum carries a topology, and we shall use its compactness to obtain an equivalent formulation of Nunke's condition. In the final part of this note we study the relation between $\text{Fin. dim } \Lambda$ and the little finitistic dimension $\text{fin. dim } \Lambda$ of Λ .

Our first result is easily stated as follows.

Theorem 1. $\text{Fin. dim } \Lambda = \sup\{\text{pd } M \mid M \in \text{Zsp } \Lambda \text{ and } \text{pd } M < \infty\}$.

We postpone the proof and first discuss some related homological properties of Λ . Suppose that Λ has a self-duality D between $\text{mod } \Lambda$ and $\text{mod } \Lambda^{\text{op}}$. We are interested in *Nunke's condition* [9]:

$$\text{Nd } M < \infty \text{ for every } \Lambda\text{-module } M \neq 0,$$

where $\text{Nd } M$ denotes the smallest integer n (or ∞ if such an integer does not exist) such that $\text{Ext}_{\Lambda}^n(D\Lambda, M) \neq 0$. It is convenient to define the *Nunke dimension* of Λ as follows:

$$\text{Nun. dim } \Lambda = \sup\{\text{Nd } M \mid M \in \text{Mod } \Lambda \text{ and } M \neq 0\}.$$

The following relation between Nunke dimension and finitistic dimension is essentially due to Jans [10] (see also [5]).

Proposition 2. $\text{Nun. dim } \Lambda \leq \text{Fin. dim } \Lambda$.

Proof. We use the fact that the functor

$$\text{Inj } \Lambda \longrightarrow \text{Proj } \Lambda, \quad X \mapsto \text{Hom}_{\Lambda}(D\Lambda, X)$$

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is an equivalence between the full subcategories of injective and projective Λ -modules. Suppose now that $\text{Nun. dim } \Lambda \geq n$ and choose a non-zero Λ -module M with $\prod_{i=0}^{n-1} \text{Ext}_{\Lambda}^i(D\Lambda, M) = 0$. Take an injective coresolution

$$0 \longrightarrow M \longrightarrow I_0 \xrightarrow{\psi_0} I_1 \xrightarrow{\psi_1} I_2 \xrightarrow{\psi_2} \dots$$

and put $N = \text{Coker Hom}_{\Lambda}(D\Lambda, \psi_{n-1})$. It is easily checked that $\text{pd } N = n$. Thus $\text{Fin. dim } \Lambda \geq n$. □

Remark. Given any $n \in \mathbb{N}_0$, there is a finite dimensional algebra Λ over a field (with $\text{rad}^2 \Lambda = 0$ and $\text{Nun. dim } \Lambda = 2$) such that $\text{Fin. dim } \Lambda - \text{Nun. dim } \Lambda = n$ [7].

Theorem 3. $\text{Nun. dim } \Lambda = \sup\{\text{Nd } M \mid M \in \text{Zsp } \Lambda\}$, and Nunke’s condition holds if and only if $\text{Nun. dim } \Lambda < \infty$.

Some further definitions are needed in order to give the proof of both theorems. An additive functor $F: \text{Mod } \Lambda \rightarrow \text{Ab}$ into the category of abelian groups is *finitely presented* if there is a presentation

$$\text{Hom}_{\Lambda}(Y, \) \longrightarrow \text{Hom}_{\Lambda}(X, \) \longrightarrow F \longrightarrow 0$$

with X and Y in $\text{mod } \Lambda$. We denote by $\text{Ker } F$ the full subcategory of Λ -modules M with $F(M) = 0$. A full subcategory \mathcal{X} of $\text{Mod } \Lambda$ is *definable* if there is a family $F_i, i \in I$, of finitely presented functors such that $\mathcal{X} = \bigcap_{i \in I} \text{Ker } F_i$. A definable subcategory is automatically closed under direct limits, products and pure submodules, and it can be shown that this property characterizes a definable subcategory [12]. The notion of a definable subcategory allows us to describe the topology on $\text{Zsp } \Lambda$ which was first introduced by Ziegler in model theoretic terms [14].

Proposition 4. (1) *The assignment $\mathcal{X} \mapsto \mathcal{X} \cap \text{Zsp } \Lambda$ defines a bijection between the definable subcategories of $\text{Mod } \Lambda$ and the closed subsets of $\text{Zsp } \Lambda$.*

(2) *$\text{Zsp } \Lambda$ is a quasi-compact space.*

Proof. (1) This result is due to Crawley-Boevey [3], and he relies on work of Herzog [6] and Ziegler [14]. We refer to [12] for a proof which uses the localization theory for locally coherent categories developed in [11]. However, the inverse of the assignment $\mathcal{X} \mapsto \mathcal{X} \cap \text{Zsp } \Lambda$ is easily constructed; it sends a closed subset \mathcal{U} of $\text{Zsp } \Lambda$ to $\bigcap_{F(\mathcal{U})=0} \text{Ker } F$.

(2) See [14] or [11]. □

The next lemma shows that some relevant subcategories of $\text{Mod } \Lambda$ are definable.

Lemma 5. *Let $n \in \mathbb{N}_0$.*

- (1) *The full subcategory \mathcal{P}_n of Λ -modules M with $\text{pd } M \leq n$ is definable.*
- (2) *The full subcategory \mathcal{Q}_n of Λ -modules M with $\text{Nd } M \geq n$ is definable.*

Proof. Projective dimension and flat dimension coincide for any module over an artinian ring, and therefore $\mathcal{P}_n = \bigcap_{X \in \text{mod } \Lambda^{\text{op}}} \text{Ker Tor}_{n+1}^{\Lambda}(\ , X)$ since $\text{Tor}_{n+1}^{\Lambda}(M, \)$ commutes with direct limits and every module can be written as a direct limit of finitely presented modules. Therefore \mathcal{P}_n is definable, and \mathcal{Q}_n is definable since $\mathcal{Q}_n = \text{Ker } \prod_{i=0}^{n-1} \text{Ext}_{\Lambda}^i(D\Lambda, \)$. □

An easy consequence of the preceding lemma is the following.

Proposition 6. *Let \mathcal{U} be a subset of $\text{Zsp } \Lambda$. Then $\text{pd } M \leq \sup\{\text{pd } N \mid N \in \mathcal{U}\}$ for every module M in the closure of \mathcal{U} .*

Proof of Theorem 1. Suppose that \mathcal{P}_n is properly contained in \mathcal{P}_{n+1} . The first part of Proposition 4 then implies there is $M \in \text{Zsp } \Lambda$ with $M \in \mathcal{P}_{n+1} \setminus \mathcal{P}_n$, since \mathcal{P}_n and \mathcal{P}_{n+1} are definable. Thus the assertion follows. \square

Proof of Theorem 3. The proof of the first part of the assertion is analogous to that of Theorem 1. Suppose now that Nunke’s condition holds. Thus $\bigcap_{n \geq 0} \mathcal{Q}_n = 0$, and the compactness of $\text{Zsp } \Lambda$ then implies the existence of $d \in \mathbb{N}_0$ with $\bigcap_{n=0}^d \mathcal{Q}_n = 0$. We conclude that $\text{Nun. dim } \Lambda < d$. \square

Remark. The finitistic dimension of Λ is finite if and only if the full subcategory of all Λ -modules with finite projective dimension is definable. Therefore one might conjecture that $\text{Fin. dim } \Lambda < \infty$ if and only if the modules in the Ziegler spectrum having finite projective dimension form a closed subset of $\text{Zsp } \Lambda$.

The rest of this paper is devoted to studying the relation between the finitistic dimension $\text{Fin. dim } \Lambda$ and the *little finitistic dimension* $\text{fin. dim } \Lambda$, which is the supremum of the projective dimensions of the finitely presented Λ -modules with finite projective dimension. Note that there are examples of finite dimensional algebras where $\text{fin. dim } \Lambda$ is different from $\text{Fin. dim } \Lambda$ [15]. Our results are motivated by recent work of Huisgen-Zimmermann and Smalø, who have shown that both dimensions coincide for an interesting class of Artin algebras [8].

In order to state the next result we recall from [2] that for any subcategory \mathcal{C} of $\text{Mod } \Lambda$ a morphism $M \rightarrow M'$ is a *left \mathcal{C} -approximation* of M if M' lies in \mathcal{C} and the induced map $\text{Hom}_\Lambda(M', X) \rightarrow \text{Hom}_\Lambda(M, X)$ is surjective for every X in \mathcal{C} . We denote by \mathcal{P}_∞ the full subcategory of all Λ -modules with finite projective dimension.

Theorem 7. *The following conditions are equivalent.*

- (1) *A Λ -module has finite projective dimension if and only if it is a direct limit of finitely presented modules having finite projective dimension.*
- (2) *The little finitistic dimension $\text{fin. dim } \Lambda$ is finite, and every finitely presented Λ -module has a left \mathcal{P}_∞ -approximation which is finitely presented.*

Moreover, if (1)–(2) hold, then $\text{fin. dim } \Lambda = \text{Fin. dim } \Lambda < \infty$.

Examples of rings which satisfy the conditions in this theorem are Artin algebras Λ where every finitely presented Λ -modules has a right $\mathcal{P}_\infty \cap \text{mod } \Lambda$ -approximation [1, 8]. There are also examples of Artin algebras Λ which show that in condition (2) the term “ \mathcal{P}_∞ -approximation” cannot be replaced by “ $\mathcal{P}_\infty \cap \text{mod } \Lambda$ -approximation” [13]. The proof of Theorem 7 is based on the following lemma.

Lemma 8. *Let \mathcal{C} be any additive subcategory of $\text{mod } \Lambda$. Then a Λ -module M is a direct limit of objects in \mathcal{C} iff every morphism $X \rightarrow M$ with $X \in \text{mod } \Lambda$ factors through an object in \mathcal{C} .*

Proof. See [4, Lemma 4.1]. \square

Proof of Theorem 7. (1) \Rightarrow (2) In order to show that $\text{fin. dim } \Lambda < \infty$ assume there is a family $M_i, i \in \mathbb{N}$, in $\text{mod } \Lambda$ with $\text{pd } M_i < \text{pd } M_{i+1} < \infty$ for all i . Then $\coprod_{i \in \mathbb{N}} M_i$ is a direct limit of finitely presented objects in \mathcal{P}_∞ , but $\text{pd } \coprod_{i \in \mathbb{N}} M_i = \infty$. Thus (1)

implies $\text{fin. dim } \Lambda = n < \infty$ and therefore $\text{Fin. dim } \Lambda = n$, since \mathcal{P}_n is closed under direct limits by Lemma 5. To show the second part of the assertion we use an idea from [4]. Let M be in $\text{mod } \Lambda$ and choose a representative set $\varphi_i: M \rightarrow N_i, i \in I$, of morphisms in $\text{mod } \Lambda$ such that the domain has finite projective dimension. Then $N = \prod_{i \in I} N_i$ lies in \mathcal{P}_∞ , since $\mathcal{P}_\infty = \mathcal{P}_n$ is closed under products by Lemma 5. Thus the induced morphism $\varphi: M \rightarrow N$ factors through a finitely presented object M' in \mathcal{P}_∞ by the preceding lemma. It is easily checked that the corresponding morphism $M \rightarrow M'$ is a left \mathcal{P}_∞ -approximation of M .

(2) \Rightarrow (1) It follows immediately from the preceding lemma that every module in \mathcal{P}_∞ is a direct limit of finitely presented modules in \mathcal{P}_∞ . Conversely, the projective dimension of a direct limit of finitely presented Λ -modules in \mathcal{P}_∞ is bounded by $\text{fin. dim } \Lambda = n$, since \mathcal{P}_n is closed under direct limits by Lemma 5. \square

If the ring Λ has the property that every finitely presented Λ -module is pure-injective, for instance if Λ is an Artin algebra, then condition (1) in the preceding theorem implies that the finitely presented modules form a dense subset of $\{M \in \text{Zsp } \Lambda \mid \text{pd } M < \infty\}$. We point out that this condition is already sufficient for the equality $\text{fin. dim } \Lambda = \text{Fin. dim } \Lambda$.

Corollary 9. *Let $n \in \mathbb{N}_0$ and suppose that the finitely presented modules $M \in \text{Zsp } \Lambda$ with $\text{pd } M \leq n$ form a dense subset of $\{M \in \text{Zsp } \Lambda \mid \text{pd } M < \infty\}$. Then $\text{fin. dim } \Lambda = \text{Fin. dim } \Lambda \leq n$.*

Proof. Let $\mathcal{U}_n = \{M \in \text{Zsp } \Lambda \mid \text{pd } M \leq n \text{ and } M \in \text{mod } \Lambda\}$, and we may assume that $n \in \mathbb{N}_0$ is the minimal choice with $\{M \in \text{Zsp } \Lambda \mid \text{pd } M < \infty\} \subseteq \overline{\mathcal{U}_n}$. It follows from Proposition 6 and Theorem 1 that $\text{Fin. dim } \Lambda \leq n$, and we therefore obtain $n \leq \text{fin. dim } \Lambda \leq \text{Fin. dim } \Lambda \leq n$. Thus $\text{fin. dim } \Lambda = \text{Fin. dim } \Lambda = n$. \square

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FAKULTÄT FÜR MATHEMATIK, UNIVERSITÄT BIELEFELD, 33501 BIELEFELD, GERMANY
E-mail address: `henning@mathematik.uni-bielefeld.de`