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PROBABILITY MEASURES IN W^*J-ALGEBRAS IN HILBERT SPACES WITH CONJUGATION

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Abstract. Let $\mathcal{M}$ be a real W^*-algebra of J-real bounded operators containing no central summand of type I_2 in a complex Hilbert space H with conjugation J. Denote by P the quantum logic of all J-orthogonal projections in the von Neumann algebra $\mathcal{N} = \mathcal{M} + i\mathcal{M}$. Let $\mu : P \to [0, 1]$ be a probability measure. It is shown that $\mathcal{N}$ contains a finite central summand and there exists a normal finite trace τ on $\mathcal{N}$ such that $\mu(p) = \tau(p)$, $\forall p \in P$.

1. Introduction

One of the basic problems related to the propositional calculus approach to the foundations of quantum mechanics is the description of probability measures (called states, in the physical terminology) on the set of experimentally verifiable propositions regarding a physical system. The set of propositions form an orthomodular partially ordered set, where the order is induced by a relation of implication, and is called a *quantum logic*.

An important interpretation of a quantum logic is the set Π of all orthogonal projections on a Hilbert space H. A remarkable Gleason theorem [1] says: *Let H be a Hilbert space*, $\dim H \geq 3$, *and let $\mu : \Pi \to R$ be a probability measure. There exists a positive trace class operator T such that* $\mu(p) = tr(Tp)$, $\forall p \in \Pi$.

The problem of the construction of a quantum field theory leads to indefinite metric spaces [2]. In this case, the set $\mathcal{P}$ of all $\overline{J}$-orthogonal projections serves to be an analog to the logic Π. There is an indefinite analog to the Gleason theorem [3].

Theorem 1. *Let H be a $\overline{J}$-space,* $\dim H \geq 3$, *and let $\mu : \mathcal{P} \to R$ be an indefinite measure. Then there exist a $\overline{J}$-selfadjoint trace class operator T and a semitrace μ_0 such that $\mu(p) = tr(Tp) + \mu_0(p)$, $\forall p \in \mathcal{P}$. Moreover, if the indefinite rank of H is equal to $+\infty$, then $\mu_0(.) \equiv 0$.*

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2. Some notation

Let H be a complex Hilbert space with the inner product $(.,.)$. Denote by S the unit sphere in H. Let J be an operator of *conjugation* in H (i.e., $J^2 = I$; $(Jx, Jy) = (y, x)$, $\forall x, y \in H$; $J(\lambda x + \beta y) = \overline{\lambda} Jx + \overline{\beta} Jy$, $\forall \lambda, \beta \in C$). A vector $x \in H$ is said to be *J-real* if $Jx = x$. The vectors $x_{\Re} \equiv \frac{1}{2}(x + Jx)$ and $x_{\Im} \equiv \frac{1}{2i}(x - Jx)$ $(= -\frac{1}{2}(ix + Jix))$ are J-real, $\forall x \in H$ and $x = x_{\Re} + ix_{\Im}$. The set $H_{\Re}$ of all J-real vectors is a real Hilbert space with respect to the inner product $(.,.)$. For every conjugation J there exists a J-real orthonormal base $\{e_\iota\}$ in H. Hence $\dim H_{\Re} = \dim H$. Put $\langle x, y\rangle \equiv (Jx, y)$. It is clear that an operator $A \in B(H)$ is *J-selfadjoint*, i.e., $\langle Ax, y\rangle = \langle x, Ay\rangle$, $\forall x, y \in H$ iff $A = JA^*J$.

An operator $A \in B(H)$ is said to be *J-real* if $JAJ = A$. Note that A is J-real iff $AH_{\Re} \subseteq H_{\Re}$. Clearly $(JAJ)^* = JA^*J$. Hence if A is J-real, then A^* is J-real also. For every $A \in B(H)$ denote by $A_{\Re}$ and by $A_{\Im}$ the operators $\frac{1}{2}(A + A^*)$ and $\frac{1}{2i}(A - A^*)$, respectively. If A is a J- selfadjoint bounded operator, then $A_{\Re}$ and $A_{\Im}$ are J-real. Let v be a partial J-real isometry with the initial projection e_i and the final one e_f. Then e_i and e_f are J-real projections. Now, let A be a selfadjoint (positive) operator on the Hilbert space $H_{\Re}$. It is easy to see that the linear extension of A over H is a selfadjoint (positive) operator, too.

A von Neumann algebra $\mathcal{A}$ acting in H is called a *W^*J-algebra* if $\mathcal{A}$ is closed with respect to the J-adjunction (i.e., $A \in \mathcal{A}$ implies $A^0 \equiv JA^*J \in \mathcal{A}$).

Let $\mathcal{A}$ be a W^*J-algebra, and $A \in \mathcal{A}$. The operator $A' \equiv A + A^* + JAJ + JA^*J \in \mathcal{A}$ is J-real, selfadjoint, and J-selfadjoint. Hence the orthogonal projections in the spectral decomposition for A' are J-real, too.

It is clear that the set $\mathcal{M}$ of all J-real operators in $\mathcal{A}$ is a real weakly closed *-subalgebra of $\mathcal{A}$ with the unit I.

Denote by P $(=P(\mathcal{A}))$ the set of all J-selfadjoint $(= J$-orthogonal) projections $\{p \in \mathcal{A} : p = p^2, \langle px, y\rangle = \langle x, py\rangle, \forall x, y \in H\}$. Note that $p \in P$ iff $p^* \in P$. With respect to the standard relations, the ordering $p \leq g$ iff $p = gp$ $(= pg)$, and the orthocomplementation $p \to p^\perp \equiv I - p$, the set P is a *quantum logic*. The set of all orthogonal projections in P is denoted by Π. It is clear that $p \in P$ is J-real iff $p \in \Pi$. Hence Π is isomorphic to the lattice of all orthogonal projections on $H_{\Re}$.

Note that for every von Neumann algebra K the set of all J-selfadjoint projections in K is a quantum logic also.

In the terminology of [6], a weakly closed real *-algebra $M \subset B(H)$ with $M \cap iM = \{0\}$ is said to be a *real W^*-algebra*. By [6, Corollary, page 22], if M is a real W^*-algebra, then $N \equiv M + iM$ is a W^*-algebra (= von Neumann algebra).

It is clear that the set $\mathcal{M}$ of all J-real operators in a W^*J-algebra $\mathcal{A}$ is a real W^*-algebra. Let $\mathcal{M}_s$ be the set of all selfadjoint operators in $\mathcal{M}$. Then $\mathcal{M}_s$ is a Jordan algebra with respect to the product $A \circ B \equiv \frac{1}{2}(AB + BA)$. The $\mathcal{M}_s$ has type I (II, III) iff the von Neumann algebra $\mathcal{N} \equiv \mathcal{M} + i\mathcal{M}$ has type I (II, III).

3. The structure of the logic P

Proposition 2. *The logic P is not a σ-logic.*

Proof. Let a W^*J-algebra $\mathcal{A}$ be equal to $B(H)$, where H is an infinite-dimensional separable Hilbert space. We will construct a sequence of mutually orthogonal projections $\{e_n\}_1^\infty \subset P$ such that the supremum $\sum e_n$ does not exist in P.

Let H^+ and H^- be infinite-dimensional subspaces of $H_{\Re}$, $H^+ \perp H^-$, and $H^+ \oplus H^- = H_{\Re}$. Let $\{\phi_n^+\}_1^\infty$ and $\{\phi_n^-\}_1^\infty$ be orthonormal bases in H^+ and H^-. Put $f_n \equiv (n+1)^{\frac{1}{2}}\phi_n^+ + n^{\frac{1}{2}} i\phi_n^-$. Then $\langle f_n, f_n\rangle = (Jf_n, f_n) = 1$, $p_{f_n} \equiv (., Jf_n)f_n \in P$, and $\{p_{f_n}\}_1^\infty$ is an orthogonal sequence in P.

Now, assume that there exists the supremum $q \equiv \sum_1^\infty p_{f_n} \in P$. Put

$$p_m \equiv \sum_1^m p_{f_n} + \sum_{m+1}^\infty [(., \phi_n^+)\phi_n^+ + (., \phi_n^-)\phi_n^-].$$

It is clear that $p_m \in P$, $p_m \geq p_{m+1}$, $\forall m$, and $p_m \geq p_{f_n}$, $\forall m, n$. Hence $p_m \geq q = \sum_1^\infty p_{f_n} \geq \sum_1^m p_{f_n}$. Thus $p_m - \sum_1^m p_{f_n} \geq q - \sum_1^m p_{f_n}$ and $p_m - \sum_1^m p_{f_n} \geq (q - \sum_1^m p_{f_n})^*$. Finally,

$$qq^* = (q - \sum_1^m p_{f_n})(q - \sum_1^m p_{f_n})^* + \sum_1^m p_{f_n}(p_{f_n})^*$$

$$= (q - \sum_1^m p_{f_n})(q - \sum_1^m p_{f_n})^* + \sum_1^m (2n+1)(., f_n)f_n, \forall m.$$

Thus the operator q is unbounded. This is a contradiction. Q.E.D.

Definition 3. We say that $p \in P$ has an *orthogonal component* if there is an orthogonal projection e, $e \neq 0$ such that $e \leq p$.

Proposition 4. *Let p be an idempotent bounded operator. Denote by e the orthogonal projection onto $pH \cap p^*H$. Then e is the greatest orthogonal projection with $e \leq p$. If $p \in P$, then $e \in \Pi$.*

Proof. It is clear that $ep = (p^*e)^* = e^* = e = pe$. Thus $e \leq p$. Assume that r is an orthogonal projection such that $r \leq p$. Then $r \leq p^*$. Therefore, $rH \subseteq pH \cap p^*H$. This means that $r \leq e$.

Now, assume that $p \in P$. Let $y \in pH \cap p^*H$, and let $x, x_0 \in H$ be such that $y = px$ and $y = p^*x_0$. Clearly $pJx_0 = Jp^*x_0 = Jy = Jpx = p^*Jx$. Thus $Jy \in pH \cap p^*H$. This means that $y_{\Re}, y_{Im} \in eH$ and $JeH = eH$. The operator JeJ is an orthogonal projection. For any $y \in eH$ we have $JeJy = JJy = y$. Hence $e \leq JeJ$. This implies $JeJ \leq e$. Thus $e = JeJ$ and $e \in \Pi$. Q.E.D.

Note that $pH \cap p^*H \neq \{0\}$ iff $H_{\Re} \cap pH \neq \{0\}$.

Definition 5. The projection e in Proposition 4 is said to be *the orthogonal component* of p and is denoted by p_{or}. A projection $p \in P$ is said to be a *skew projection* if $p_{or} = 0$.

It is clear that for every $p \in P\backslash\Pi$ the projection $p - p_{or}$ is a skew projection.

For every $p \in P$ denote by e_p the orthogonal projection onto pH.

Proposition 6. *Let p be a bounded projection. Denote by $(p+p^*)_+$ and by $(p+p^*)_-$ the positive and the negative parts of $p + p^*$. Let e_+ be the orthogonal projection onto $\overline{(p+p^*)_+H}$. Then $e_+pe_+ = \frac{1}{2}(p+p^*)_+$ and $e_+pe_+ \geq e_+$.*

Proof. It is obvious that $p = e_p p = e_p + e_p p e_p^{\perp}$. Hence $pp^* = e_p + e_p p e_p^{\perp} p^* e_p \geq e_p$. Thus the bounded operator $(e_p^{\perp} + pp^*)^{-\frac{1}{2}}$ exists. Let $|p^*| \equiv (pp^*)^{\frac{1}{2}}$, and let $e_p^{\perp} p^* e_p = v_p |e_p^{\perp} p^* e_p|$ be the polar decomposition for $e_p^{\perp} p^* e_p$. Clearly $|e_p^{\perp} p^* e_p| = (pp^* - e_p)^{\frac{1}{2}}$. For nonnegative operators a and b ($ab = ba$) and a partial isometry v, we put $f(a, b, v) \equiv a + (ab)^{\frac{1}{2}} v^* + v(ab)^{\frac{1}{2}} + vbv^*$. Now, let

$$a_p \equiv \frac{1}{2}(e_p + |p^*|)(e_p^{\perp} + pp^*)^{-\frac{1}{2}}.$$

It is straightforward that $f(a_p, e_p - a_p, v_p)$ is an orthogonal projection. Next,

$$f((e_p + |p^*|)a_p, (e_p + |p^*|)(e_p - a_p), v_p))$$

$$+ f((e_p - |p^*|)(e_p - a_p), (e_p - |p^*|)a_p, -v_p)$$

$$= 2e_p + (pp^* - e_p)^{\frac{1}{2}} v_p^* + v_p (pp^* - e_p)^{\frac{1}{2}} = p + p^*.$$

The operators

$$f((e_p + |p^*|)a_p, (e_p + |p^*|)(e_p - a_p), v_p)$$

and

$$f((e_p - |p^*|)(e_p - a_p), (e_p - |p^*|)a_p, -v_p)$$

are positive and negative, respectively. It can be easily verified that their product is equal to zero. Hence

$$f((e_p + |p^*|)a_p, (e_p + |p^*|)(e_p - a_p), v_p) = (p + p^*)_+. \tag{1}$$

The cover projection of the left hand side of (1) is equal to $f(a_p, e_p - a_p, v_p)$. Hence $e_+ = f(a_p, e_p - a_p, v_p)$. Then

$$e_+ p e_+ = e_+ (e_p + (pp^* - e_p)^{\frac{1}{2}} v_p^*) e_+$$

$$= \frac{1}{2} f((e_p + |p^*|)a_p, (e_p + |p^*|)(e_p - a_p), v_p) = by(1) = \frac{1}{2}(p + p^*)_+,$$

$$e_+ p^* e_+ = \frac{1}{2}(p + p^*)_+, \text{ and } e_+ p e_+ \geq e_+. \quad \text{Q.E.D.}$$

Remark 7. Let $p \in P$. Then the operator $e_+ p e_+$ is J-real.

Proof. Let $p + p^* = \int \lambda de_\lambda$ be the spectral decomposition for $p + p^*$. The operator $p + p^* (= p + JpJ)$ is J- real. By the uniqueness of the spectral decomposition, e_λ is J-real for all λ. Hence e_+ is J-real, too. Finally, since $e_+ p e_+ = e_+ p^* e_+$, it follows that $J e_+ p e_+ J = e_+ J p J e_+ = e_+ p^* e_+ = e_+ p e_+$. Q.E.D.

Put $e_- \equiv I - e_+$. Now, denote by F_y ($= e_p$, $p \in P$) the orthogonal projection onto $\overline{yH}$, $\forall y \in B(H)$.

Proposition 8. *Let $p \in P$, and let $e_- p e_+ = w|e_- p e_+|$ be the polar decomposition for $e_- p e_+$. Then $x \equiv e_+ p e_+$ ($\geq e_+$) and $v \equiv \frac{1}{i} w$ are J-real operators in $\mathcal{A}$, and the following formula holds:*

$$p = x + iv(x^2 - x)^{\frac{1}{2}} + i(x^2 - x)^{\frac{1}{2}} v^* - v(x - F_x)v^*. \tag{2}$$

Conversely, let $x \in \mathcal{A}$ be an arbitrary J-real operator such that $x \geq F_x$, and let $v \in \mathcal{A}$ be a J-real isometry with the initial projection F_x and the final J-real projection e such that $e \perp F_x$. Then (2) defines a projection in P.

Proof. Let $p \in P$. We have

$$e_+pe_- = e_+p_{\Re}e_- + \frac{1}{2}(e_+pe_- - e_+p^*e_-) = 0 + \frac{1}{2}(e_+pe_- - e_+p^*e_-).$$

Hence

$$i(e_+p_{\Im}e_-) = e_+pe_- = -e_+p^*e_- = -(e_-pe_+)^*. \tag{3}$$

Similarly,

$$i(e_-p_{\Im}e_+) = e_-pe_+ = -e_-p^*e_+ = -(e_+pe_-)^*. \tag{4}$$

This implies that $\frac{1}{i}e_-pe_+$, $\frac{1}{i}e_+pe_-$, and $|e_-pe_+|$

$$(= ((\frac{1}{i}e_-pe_+)^*(\frac{1}{i}e_-pe_+))^{\frac{1}{2}})$$

are J-real operators. It follows from the uniqueness of the polar decomposition that $\frac{1}{i}w$ is a J-real isometry.

By (3), we have

$$|e_-pe_+| = ((e_-pe_+)^*(e_-pe_+))^{\frac{1}{2}} = (-(e_+pe_-)e_-pe_+)^{\frac{1}{2}}$$

$$= (e_+p(e_+ - I)pe_+)^{\frac{1}{2}} = ((e_+pe_+)(e_+pe_+) - e_+pe_+)^{\frac{1}{2}} = (x^2 - x)^{\frac{1}{2}}.$$

Thus $e_-pe_+ = iv(x^2 - x)^{\frac{1}{2}}$ and $e_+pe_- = -(e_-pe_+)^* = i(x^2 - x)^{\frac{1}{2}}v^*$. It is clear that $x|e_-pe_+| = |e_-pe_+|x$.

Now, we show that $e_-pe_- = -v(x - F_x)v^*$. We have

$$e_+pe_- = (e_+pe_+)(e_+pe_-) + (e_+pe_-)(e_-pe_-),$$

i.e., $(x^2 - x)^{\frac{1}{2}}v^* = x(x^2 - x)^{\frac{1}{2}}v^* + (x^2 - x)^{\frac{1}{2}}v^*(e_-pe_-)$. Hence

$$(x^2 - x)^{\frac{1}{2}}(F_x - x)v^* = (x^2 - x)^{\frac{1}{2}}v^*(e_-pe_-). \tag{5}$$

If $z \in e_-H \ominus \overline{e_-pH}$, then $v^*z = 0$ and $e_-pe_-z = 0$. This means that $-v(x - -F_x)v^*z = e_-pe_-z$. If $z \in \overline{e_-pH}$, then $v^*z \in \overline{(x^2 - x)^{\frac{1}{2}}H}$. By (5), $(F_x - x)v^*z = v^*(e_-pe_-)z$, i.e., $-v(x - F_x)v^*z = e_-p_-z$, $\forall z \in H$. The proof of (2) is completed.

Let $x \in \mathcal{A}$ be a J-real operator such that $x \geq F_x$, and let $v \in \mathcal{A}$ be an arbitrary J-real isometry with the initial projection F_x and the final J-real projection e, where $e \perp F_x$. Using the righthand side of (2), define p. It can be easily verified that $p^2 = p$ and $Jp^*J = p$. Hence $p \in P$. Q.E.D.

To emphasize that p in (2) depends on x and v, we shall use the notation $p(x, v)$ as well. Note that $p(x, v)$ is a skew projection iff $x > F_x$ on F_xH. We say that a projection $p \in P$ is *simple* if $e_+pe_+ = \alpha e_+$ $\alpha \geq 1$. It simply follows from Proposition 8 that

Corollary 9. $P \subset \mathcal{M} + i\mathcal{M}$.

Corollary 10. *Suppose that $p \in P$ is represented by (2). Then*

(i) $\|p\| = \|2x - I\| = 2\|x\| - 1$;

(ii) $F_p = x(2x - I)^{-1} + v(x^2 - x)^{\frac{1}{2}}(2x - I)^{-1} + (2x - I)^{-1}(x^2 - x)^{\frac{1}{2}}v^* + v(x - e_x)(2x - I)^{-1}v^x$;

(iii) *the projection p is simple iff $e_+F_pe_+ = te_+$ and $F_pe_+F_p = tF_p$, $0 < t < 1$.*

Let $p(x,v) \in P$ and let $x = \int\limits_1^a \lambda de_\lambda$ be the spectral decomposition for x. Denote $x_n \equiv \sum \frac{\iota}{n}(e_{\frac{(\iota+1)}{n}} - e_{\frac{\iota}{n}})$.

Corollary 11. *The operator $p(x_n,v)$ is the sum of the mutually orthogonal simple projections $p(\frac{\iota}{n}(e_{\frac{(\iota+1)}{n}} - e_{\frac{\iota}{n}}),v)$ in P, $\iota \geq n$; $p(x_n,v) \in P$, and*

$$\lim_{n\to\infty} \|p(x,v) - p(x_n,v)\| = 0.$$

4. Hyperbolic sublogics of the logic P

Let $e \in \Pi$ $(0 \neq e \neq I)$. Denote the real subspace $lin_R\{eH_\Re \oplus ie^\perp H_\Re\}$ by $\mathcal{H}_e$. The set $\mathcal{H}_e$ is a real Hilbert space with respect to the product $(.,.)$. It is evident that H is equal to the direct sum $\mathcal{H}_e + i\mathcal{H}_e$.

Denote by $\overline{J}$ the restriction of J to $\mathcal{H}_e$. Clearly $\overline{J} = (e - e^\perp)/\mathcal{H}_e$ and $\overline{J}$ is a symmetry (i.e., $\overline{J} = \overline{J}^*$ in $\mathcal{H}_e$).

Consequently, we have: Every $b \in B(\mathcal{H}_e)$ can be uniquely extended to a bounded linear operator b_H on H, $(b_H)^* = (b^*)_H$, and if p is a bounded projection on $\mathcal{H}_e$, then p_H is a bounded projection, too. In addition, if a projection p is $\overline{J}$-selfadjoint (i.e., $p = \overline{J}p^*\overline{J}$), then p_H is J-selfadjoint.

Conversely, if $q \in P$ and $q\mathcal{H}_e \subseteq \mathcal{H}_e$, then $q/\mathcal{H}_e$ is a $\overline{J}$-selfadjoint projection.

With respect to the product $[x,y] \equiv (Jx,y)$, $\forall x,y \in \mathcal{H}_e$, the set $\mathcal{H}_e$ is a real indefinite metric space (= Krein space = $\overline{J}$-space), and $\overline{J}$ is the canonical symmetry with respect to the canonical decomposition $\mathcal{H}_e = \mathcal{H}_e^+[\dot{+}]\mathcal{H}_e^-$, where $\mathcal{H}_e^+ \equiv eH_\Re$ and $\mathcal{H}_e^- \equiv ie^\perp H_\Re$ (see [7]). The indefinite rank of $\mathcal{H}_e$ is equal to $min\{dimeH, dime^\perp H\}$.

Let $p = p(x_p,v) \in P$ (see (2)). Put $e = F_{x_p}$ and $J_1 = p - p^\perp$ $(=2p-I)$. According to the theory developed in [7], the restriction of J_1 to $\mathcal{H}_e$ (with $[.,.]$) is the canonical symmetry with respect to the canonical decomposition $\mathcal{H}_e = p\mathcal{H}_e[\dot{+}]p^\perp\mathcal{H}_e$. We provide a direct proof to this.

1) Obviously $[z,y] = [pz,p^\perp y] = [p^\perp pz,y] = [0z,y] = 0$, $\forall z \in p\mathcal{H}_e$, $y \in p^\perp\mathcal{H}_e$. Thus $\mathcal{H}_e = p\mathcal{H}_e[\dot{+}]p^\perp\mathcal{H}_e$.

2) It remains to show that, with respect to the product $(z,y)_1 \equiv [J_1z,y]$, $\mathcal{H}_e$ is a Hilbert space, $p\mathcal{H}_e$ and $p^\perp\mathcal{H}_e$ its subspaces, $[z,z] > 0$, $\forall z \in p\mathcal{H}_e$, $z \neq 0$, and $[y,y] < 0$, $\forall y \in p^\perp\mathcal{H}_e$, $y \neq 0$.

Define $f(t) \equiv 2t - 1 - 2(t^2-t)^{\frac{1}{2}}$ $(= (t^{\frac{1}{2}} - (1-t)^{\frac{1}{2}})^2)$, $\forall t \geq 1$. Put

$$B \equiv 2x_p - e - 2(x_p^2 - x_p)^{\frac{1}{2}} + v(2x_p - e - 2(x_p^2 - x_p)^{\frac{1}{2}})v^* + (e^\perp - vev^*).$$

Observe that $f(t)$ decreases. Therefore, $B \geq \lambda I > 0$ $(\in B(H))$, where $\lambda \equiv 2\|x_p\| - 1 - 2(\|x_p\|^2 - \|x_p\|)^{\frac{1}{2}}$ (> 0). Next,

$$A \equiv (x_p^2 - x_p)^{\frac{1}{2}} - (x_p^2 - x_p)^{\frac{1}{2}}v^* - v(x_p^2 - x_p)^{\frac{1}{2}} + v(x_p^2 - x_p)^{\frac{1}{2}}v^* \qquad (\geq 0)$$

Let $y = y_e + iy_{e^\perp} \in \mathcal{H}_e$, where $y_e \in eH_\Re$ and $y_{e^\perp} \in e^\perp H_\Re$. A straightforward verification shows that

$$(\|2p - I\|\|y\|^2 \geq) \qquad [J_1y,y] = (JJ_1y,y) = (Jy,(2p-I)y)$$

$$= ((B+2A)(y_e + y_{e^\perp}), y_e + y_{e^\perp}) \geq \lambda\|y_e + y_{e^\perp}\|^2 = \lambda\|y\|^2.$$

Thus $\mathcal{H}_e$, with respect to $(z,y)_1 \equiv [J_1z,y]$, $\forall z$, $y \in \mathcal{H}_e$, is a Hilbert space. The topologies on $\mathcal{H}_e$ defined by $(.,.)$ and $(.,.)_1$ are equivalent. Therefore, $p\mathcal{H}_e$ and $p^\perp\mathcal{H}_e$ are subspaces of $\mathcal{H}_e$ endowed with $(.,.)_1$. Eventually, $[z,z] = [J_1z,z] = (z,z)_1 > 0$,

$\forall z \in p\mathcal{H}_e$ $(z \neq 0)$, and $[y,y] = -[J_1 y, y] = -(y,y)_1 < 0$, $\forall y \in p^{\perp}\mathcal{H}_e$ $(y \neq 0)$. We have accomplished the verification.

Let $\mathcal{M}$ be a real W^*-algebra of J-real operators. Denote by $\mathcal{N}_e$ the set $\{B \in (\mathcal{M} + i\mathcal{M}) : B\mathcal{H}_e \subseteq \mathcal{H}_e\}$. Obviously $\mathcal{N}_e$ is a real closed in the strong operator topology *-algebra. In addition, $B \in \mathcal{N}_e$ implies $B^0 \in \mathcal{N}_e$. In the terminology of [5], $\mathcal{N}_e$ is a $W^*\overline{J}$-algebra in the real indefinite metric space $\mathcal{H}_e$. Put $P_e \equiv P \cap \mathcal{N}_e$. Clearly P_e is a quantum sublogic of P. In [3], [4] the logic P_e is called a *hyperbolic* logic. Denote by P_e^+ (P_e^-) the set of all $p \in P_e$ for which the subspace $p\mathcal{H}_e$ of $\mathcal{H}_e$ is *positive* (i.e., $\forall z \in p\mathcal{H}_e$, $z \neq 0$, $[z,z] > 0$) (respectively, *negative*, i.e., $\forall z \in \mathcal{H}_e$, $z \neq 0$, $[z,z] < 0$). Note that $p \in P_e^+$ iff $\overline{J}p \geq 0$ on $\mathcal{H}_e$, and $p \in P_e^-$ iff $\overline{J}p \leq 0$. For instance, let $\mathcal{M}$ be the set $(\equiv B(H_{\Re}))$ of all J-real bounded operators in H. Then $(.,\overline{J}f)f \in P_e^+$, where $f = \alpha z + i\beta y$, $\alpha, \beta \in R$, $\alpha^2 - \beta^2 = 1$, and $z \in e\mathcal{H}_e \cap S$, $y \in e^{\perp}\mathcal{H}_e \cap S$. The projection $-(.,\overline{J}g)g$ belongs to P_e^-, where $g = \beta z + i\alpha y$. Note that $p(x,v)$ is positive on $\mathcal{H}_{F_x}$. Every projection $p \in P_e$ is representable (not uniquely!) in the form $p = p_- + p_+$, where $p_- \in P_e^-$, $p_+ \in P_e^+$ [7].

5. Measures on the logic P

Denote by $\mathcal{M}^u$ the set of all J-real symmetries (= unitary operators) in the Jordan algebra $\mathcal{M}_s$. Let $e, f \in \Pi$. We write $e \sim f$ if there exists $v \in \mathcal{M}^u$ with $e = vfv$ and $f = vev$. Let $(p_\iota)_{\iota \in I} \subset P$ be a set of mutually orthogonal projections. Assume that for every subset $X \subseteq I$ there exists $q = \sum_{\iota \in X} p_\iota$ (the sum being understood in the strong sense). Then a representation $p = \sum_{\iota \in I} p_\iota$ is said to be a *decomposition* of p.

A mapping $\mu : P \to R$ is said to be a *measure* (=*quantum measure*) if $\mu(p) = \sum \mu(p_\iota)$ for any decomposition $p = \sum p_\iota$.

Here, the convergence of an uncountable family of summands means that there exists a countable set of nonzero terms in the family and the usual series with these summands converges absolutely.

A nonnegative measure μ is said to be a *probability* measure if $\mu(I) = 1$.

On a hyperbolic logic, there exists a new class of measures. A measure $\mu : P_e \to R$ is said to be *indefinite* if $\mu/P_e^+ \geq 0$ and $\mu/P_e^- \leq 0$, a *semitrace* if there exists a normal semifinite trace τ on $\mathcal{N}_e$ and a number c such that $\mu(p) = c\tau(p_+)$, $\forall p \in P_e$ or $\mu(p) = c\tau(p_-)$, $\forall p \in P_e$. By definition, $0. + \infty \equiv 0$. If the indefinite rank of $\mathcal{H}_e$ is equal to $+\infty$, then every semitrace on P_e equals 0.

Theorem 12. *Let $\mathcal{M}_s$ be a JW-algebra containing no central summand of type I_2 and let $\mu : P(\mathcal{N}) \to [0,1]$ be a probability measure. Then $\mathcal{M}_s$ and $\mathcal{N} = \mathcal{M} + i\mathcal{M}$ contain finite central summand and there exists a normal finite trace τ on $\mathcal{N}$ such that $\mu(p) = \tau(p)$, $\forall p \in P(\mathcal{N})$.*

Proof. Let $\mu : P \to [0,1]$ be a probability measure. The proof will consist of several steps.

1) First suppose that $\mathcal{M}$ is the set $B(H_{\Re})$ of all J-reality bounded operators on H.

i) Let $3 \leq n \equiv \dim H_{\Re} < +\infty$. Let $e \in \Pi$ $(0 \neq e \neq I)$, and let μ_e be the restriction of μ to P_e, i.e., $\mu_e(p) = \mu(p_H)$, $\forall p \in P_e$. Let $\nu(p) \equiv tr(\mu(I)\overline{J}p) - \mu_e(p)$, $\forall p \in P_e$. If $p \in \mathcal{P}_e^+$ $(p \neq 0)$, then $p\overline{J} \geq e_p \geq 0$ and

$$\nu(p) = \mu(I)tr(p\overline{J}) - \mu_e(p) \geq \mu(I)tr(e_p) - \mu_e(p) \geq \mu(I) - \mu_e(p) \geq 0.$$

If $p \in \mathcal{P}^-$ $(p \neq 0)$, then $p\overline{J} \leq -e_p \leq 0$ and $\nu(p) \leq -\mu(I) - \mu_e(p) \leq 0$. Hence ν is an indefinite measure on P_e. By [3, Theorem 1], there exist a $\overline{J}$-selfadjoint operator $T \in B(\mathcal{H}_e)$ (i.e., $T = \overline{J}T^*\overline{J}$) and a number c such that $\nu(p) = tr(Tp) - c\dim(p_+\mathcal{H}_e)$. Hence $\mu_e(p) = tr((\mu(I)\overline{J} - T)p) + c\dim(p_+H)$, $\forall p \in P_e$. Obviously

$$\mu_e(p) = tr((\mu(I)J - T + cI)p) - c\dim(p_-\mathcal{H}_e), \forall p \in P_e.$$

Put $p_z \equiv (.,z)z$, $\forall z \in S$. Let $z \in eH_{\Re} \cap S$, $y \in ie^{\perp}H_{\Re} \cap S$, and $v = (.,z)y$. Put $B \equiv \mu(I)\overline{J} - T$. Since $B = \overline{J}B^*\overline{J}$, we get

$$(Bz,y) = (B\overline{J}z,y) = (\overline{J}B^*z,y) = (B^*z,\overline{J}y)$$

$$= -(B^*z,y) = -(z,By) = -(By,z).$$

It is clear that $p(\beta p_z, v)^* = p(\beta p_z, -v)$, $(\beta > 1)$, (see (2)). Hence

$$0 \leq \mu(p(\beta p_z, \pm v)) = tr(Bp(\beta p_z, \pm v)) + c$$

$$= \beta tr(Bp_z) + (\beta^2 - \beta)^{1/2}(tr(\pm Bv) - tr(\pm Bv^*)) - (\beta - 1)tr(Bp_y) + c$$

$$= \beta(Bz,z) + (\beta^2 - \beta)^{1/2}(\pm(By,z) - \pm(Bz,y)) - (\beta - 1)(By,y) + c$$

$$= \beta((Bz,z) - (By,y)) + 2(\beta^2 - \beta)^{1/2}(\pm(By,z)) + (Bz,z) + c \leq 1.$$

Since

$$0 \leq 4(\beta^2 - \beta)^{\frac{1}{2}}|(By,z)| = |\mu(p(\beta p_z, v)) - \mu(p(\beta p_z, v)^*)| \leq 1, \forall \beta \geq 1,$$

we have $(By,z) = 0$. This means that $(Bz,z) = (By,y)$, $\forall z \in eH_{\Re} \cap S$, $\forall y \in e^{\perp}H_{\Re} \cap S$. Hence $B = aI$, where $a \equiv (Bz,z)$. Therefore,

$$\mu(p_H) = \mu_e(p) = tr_{\mathcal{H}_e}(Bp) + c\dim(p_+\mathcal{H}_e) = atr_{\mathcal{H}_e}(p) + c\dim(p_+\mathcal{H}_e)$$

$$= (a + c)\dim(p_+\mathcal{H}_e) + a\dim(p_-\mathcal{H}_e), \forall p \in P_e.$$

Thus μ_e, on P_e, is a sum of semitraces. This also means that μ_e is a constant $\equiv a + c$ $(\equiv a)$ on the set of all one-dimensional positive (negative) projections in P_e. Since $\dim H \geq 3$, it follows that for every $z, y \in H_{\Re} \cap S$ there exists a projection $e \in \Pi$, $0 \neq e \neq I$ such that $z, y \in eH_{\Re}$. Hence

$$\mu(p_z) = \mu_e(p_z) = \mu_e(p_y) = \mu(p_y) \qquad (= \frac{1}{n}).$$

Thus $\mu(p) = \frac{1}{n}tr(p)$, $\forall p \in P$.

ii) Let $\dim H_{\Re} = +\infty$. Assume that there exists a probability measure $\mu : P(B(H)) \to [0,1]$. It follows from the step i) that $\mu(p_y) \equiv const$ on the set of all one-dimensional orthogonal projections. We have $1 = \mu(I) = \sum \mu(p_{\varphi_i})$, where (φ_i) is an orthonormal base in the Hilbert space $H_{\Re}$. This is a contradiction.

2) Now consider the general case. Let $\mathcal{M}_s$ be a Jordan algebra. Choose e, $f \in \Pi$ such that $e \perp f$ and $e \sim f$. Let $v \in \mathcal{M}^u$ be a J-real symmetry with $e = vfv$ and $f = vev$. Assume that there is a J-real symmetry $w \in \mathcal{M}^u$ with $wew \leq I - e - f$. The minimal $*$-algebra $\mathcal{A}(e,we,ve)$ generated by the operators e, we, and ve may be identified with a W^*J-factor of type I_3 acting in the Hilbert space $(e+f+wew)H$. It follows from Section 1i) that $\mu(e) = \mu(f)$ $(= \frac{1}{3}(\mu(e+f+wew))$. By the assumption

on $\mathcal{M}_s$, we conclude that $e \sim f$ and $e \perp f$ imply $\mu(e) = \mu(f)$. Thus

a) $e \sim f$ imply $\mu(e) = \mu(f)$, $\forall e, f \in \Pi$. Again, by the step 1i), we have

b) $\mu(p(x,v)) = \mu(F_x)$, $\forall p(x,v) \in P \cap \mathcal{A}(e, ve, we)$.

If there exists a properly infinite, with respect to $\mathcal{M}_s$, projection $e \in \Pi$, then $e = \sum_{n=1}^{\infty} e_n$, where $e_n \in \Pi$ and $e_n \sim e_m$, $\forall n$, m. Since $\mu(e_n) = \mu(e_m)$, we have $\mu(e) = 0$. Therefore, $0 = \mu(p)$ $(\leq \mu(e) = 0)$, $\forall p \in P$, $p \leq e$.

Note, that the following property (see the proof of Theorem 2, the steps 1) and 2)) was proved for the indefinite case in [5].

Proposition 13. *Let H ($\dim H \geq 3$) be an indefinite metric space with a canonical symmetry $\overline{J}$ and let $\mathcal{P}$ be the logic of all $\overline{J}$-selfadjoint projections. Then ν is constant on the set of all positive (negative) one-dimensional projections for every positive measure $\nu : \mathcal{P} \to R^+$.*

Let us continue the proof of Theorem 12. Since $\mu(I) = 1$, there exists a finite, with respect to $\mathcal{M}_s$, central projection $e \in \Pi$. Hence there exists a finite central summand. a) means that the restriction of μ to Π is unitarily invariant (i.e., $p \sim f \Rightarrow \mu(p) = \mu(f)$). By [8], the measure μ/Π extends to a normal finite trace τ on $\mathcal{M}_s$. By a definition of [6], a positive linear functional $\tau : \mathcal{M}_s \to R$ is said to be a *finite trace* if $\tau(SAS) = \tau(A)$, $\forall S \in \mathcal{M}^u$ and $\forall A \in \mathcal{M}_s$.

By [6, Theorem 4.3, p.32], one can extend the trace τ to a normal finite trace on $\mathcal{N} = \mathcal{M}_s + i\mathcal{M}_s$. (We denote the extension by τ again.) Let $p(x,v) \in P$. Without loss of generality, we may assume that $p(x,v)$ is a skew projection and there exists a partial isometry $w \in \mathcal{M}$ with the initial projection F_x and such that the projections F_x, ww^*, and vv^* are mutually orthogonal. Put $e \equiv F_x + ww^*$ $(\in \Pi)$, and $J_1 = 2p(x,v) - I$. In the Hilbert space $\mathcal{H}_e$ with the product $(.,.)_1 = [J_1.,.]$, the projections ww^* and $p(x,v)$ are orthogonal, positive, and mutually orthogonal. In addition, there exist partial isometries u_1, $u_2 \in \mathcal{H}_e$ with the initial projection $p(x,v)$ and the final projections ww^* and $F_x + vv^* - p(x,v)$. Then the *-algebra $\mathcal{A}(u_1, u_2)$ generated by u_1, u_2 may be identified with a $W^*\overline{J}_1$-factor of type I_3 acting in an indefinite metric space $\mathcal{E}$, $\dim \mathcal{E} = 3$. By this and Proposition 13, we have $\mu(ww^*) = \mu(p(x,v))$. By a), $\mu(ww^*) = \mu(F_x)$. Hence $\mu(F_x) = \mu(p(x,v))$. As τ is a trace on the algebra von Neumann, we have

$$\mu(p(x,v)) = \mu(F_x) = \tau(F_x) = \tau(e_p) = \tau(e_p + e_p p(x,v) e_p^{\perp}) = \tau(p(x,v)),$$

where e_p is the orthogonal projection onto pH. Theorem 12 is proved. Q.E.D.

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