

ON THE JACOBIAN MODULE ASSOCIATED TO A GRAPH

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ABSTRACT. We consider the *jacobian module* of a set $\mathbf{f} := \{f_1, \dots, f_m\} \in R := k[X_1, \dots, X_n]$ of squarefree monomials of degree 2 corresponding to the edges of a connected bipartite graph G . We show that for such a graph G the number of its primitive cycles (i.e., cycles whose chords are not edges of G) is the second Betti number in a minimal resolution of the corresponding jacobian module. A byproduct is a graph theoretic criterion for the subalgebra $k[G] := k[\mathbf{f}]$ to be a complete intersection.

INTRODUCTION

Let $\mathbf{f} = \{f_1, \dots, f_m\} \subset R = k[\mathbf{X}] = k[X_1, \dots, X_n]$ be a set of polynomials, where k is a field. The *jacobian module* $\mathfrak{J}(\mathbf{f})$ of \mathbf{f} is the cokernel of the map $R^m \rightarrow R^n$ defined by the transposed jacobian matrix ${}^t\Theta(\mathbf{f}) = {}^t\Theta(f_1, \dots, f_m)$ of the polynomials \mathbf{f} .

If \mathbf{f} is a set of linearly independent homogeneous polynomials of the same degree, then $\mathfrak{J}(\mathbf{f})$ does not depend on the choice of a minimal set of homogeneous generators. The underlying theme of this work is the understanding of how combinatorial properties of an ideal (or k -subalgebra) of R minimally generated by quadrics $\mathbf{f} = \{f_1, \dots, f_m\}$ translate to the numerical invariants of the minimal homogeneous free resolution of $\mathfrak{J}(\mathbf{f})$.

One finds that these considerations have a degree of success in the case where $\mathbf{f} = \{f_1, \dots, f_m\}$ correspond to the edges of a simple connected graph G , generating the so-called edge-ideal $I(G)$ (cf. [3], [4]). In this setup, the simpler notation $\mathfrak{J}(G) := \mathfrak{J}(\mathbf{f})$, ${}^t\Theta(G) := {}^t\Theta(\mathbf{f})$ will be used. It is shown that if G is a connected bipartite graph then the second module of syzygies of $\mathfrak{J}(G)$ is generated by the so-called polar syzygies (to be defined below). As a consequence, one also obtains that the second Betti number in the minimal free resolution of $\mathfrak{J}(G)$ over the polynomial ring R coincides with the number of primitive (i.e., chordless) cycles of G , and that $\mathfrak{J}(G)$ has homological dimension at most 2 if and only if the number of primitive cycles of G coincides with the usual rank of G in the sense of graph theory. In particular, one obtains a way of translating the computation of the number of primitive cycles of a connected bipartite graph into the computation of the number of minimal generating relations of a matrix whose entries are indeterminates. Moreover,

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from each such generating relation one can read off the order of the corresponding primitive cycle.

A byproduct is a criterion for the subalgebra $k[G]$ to be a complete intersection. This relates to [2], where the authors consider the case of bipartite planar graphs by entirely different methods.

An earlier version of this work underwent important criticism by someone who wishes to remain anonymous; the present version reflects partially this criticism, for which I am indebted.

1. PRELIMINARIES

We collect a few facts concerning the jacobian matrix of a set of monomials for which a suitable reference in the literature has not been found. Let $\mathbf{X} = X_1, \dots, X_n$ be indeterminants over a field k . By a *monomial* in $k[\mathbf{X}]$ we mean a power product $\mathbf{X}^{\mathbf{a}} = X_1^{a_1} \dots X_n^{a_n}$, with $a_i \in \mathbb{N}$.

The following result is quite particular about the jacobian matrix of a set of monomials.

Lemma 1.1. *Let $\mathbf{f} = \{f_1, \dots, f_m\}$ be a set of monomials in $k[\mathbf{X}]$. Then every minor of the jacobian matrix of \mathbf{f} is of the form $\alpha \mathbf{X}^{\mathbf{a}}$, where α belongs to the prime ring of k .*

Proof. One endows the ring $k[\mathbf{X}]$ with the \mathbb{N}^n -gradation in which $k[\mathbf{X}]_{\mathbf{a}} = k \mathbf{X}^{\mathbf{a}}$. A homogeneous element in this gradation will be called *multihomogeneous*. Now, we claim that the jacobian matrix is also multihomogeneous, i.e., that any 2×2 minor is a multihomogeneous element of $k[\mathbf{X}]$. This will imply that any minor is multihomogeneous, hence must be of the form $\alpha \mathbf{X}^{\mathbf{a}}$, for some $\mathbf{a} \in \mathbb{N}^n$ and some $\alpha \in k$. To conclude, the definition of derivatives easily implies that α actually belongs to the prime ring of k . Thus, let, say, the 2×2 minor be that of columns 1, 2 and rows 1, 2. Letting $f_1 = \mathbf{X}^{\mathbf{a}}$ and $f_2 = \mathbf{X}^{\mathbf{b}}$, the minor is a binomial with terms of multidegrees $(a_1 - 1 + b_1, a_2 + b_2 - 1, \dots)$ and $(a_1 + b_1 - 1, a_2 - 1 + b_2, \dots)$, respectively. Since the latter are equal, the minor is multihomogeneous. \square

Consider the map \mathcal{L} that associates to a monomial $\mathbf{X}^{\mathbf{a}}$ its exponent vector $\mathbf{a} \in \mathbb{N}^n$. Given a finite set $\mathbf{f} = \mathbf{X}^{\mathbf{a}_1}, \dots, \mathbf{X}^{\mathbf{a}_m}$ of monomials, one can look at the integer matrix $\mathcal{L}(\mathbf{f}) = (\mathbf{a}_1, \dots, \mathbf{a}_m)$. One may call $\mathcal{L}(\mathbf{f})$ the *log-matrix* of \mathbf{f} . Note that if G is a simple graph then the log-matrix of the generators of $k[G]$ is precisely the incidence matrix of G .

A consequence is as follows.

Proposition 1.2 (char $k = 0$). *Let \mathbf{f} be a set of monomials. Then the jacobian matrix of \mathbf{f} and the log-matrix of \mathbf{f} have the same rank.*

Proof. Identify \mathbb{Z} with a subring of k . Then ${}^t\Theta(\mathbf{f})$ has entries in \mathbb{Z} and the canonical homomorphism $\mathbb{Z}[\mathbf{X}] \rightarrow \mathbb{Z}$ implies that $\text{rank}(\mathcal{L}(\mathbf{f})) \leq \text{rank}({}^t\Theta(\mathbf{f}))$. Conversely, let $r = \text{rank}({}^t\Theta(\mathbf{f}))$ and let Δ be a nonzero $r \times r$ minor of ${}^t\Theta(\mathbf{f})$. Then Δ is of the form $\alpha \mathbf{X}^{\mathbf{a}}$, with $\alpha \in \mathbb{Z} \setminus \{0\}$ by Lemma 1.1. Therefore, $\epsilon(\Delta) = \alpha \neq 0$. But $\epsilon(\Delta)$ is an $r \times r$ minor of $\mathcal{L}(\mathbf{f})$. \square

Note that the proposition fails in positive characteristic on taking

$$\mathbf{f} = \{X_1X_2, X_1X_3, X_2X_3\}.$$

The previous results allow for the following alternative characterization of a connected bipartite graph.

Proposition 1.3 (char $k = 0$). *Let G be a connected graph. The following conditions are equivalent:*

- (i) G is bipartite.
- (ii) $\mathfrak{J}(G)$ has rank one.
- (iii) $\mathfrak{J}(G)$ has positive rank.

Proof. It is well-known that, for a connected graph G on n vertices, its incidence matrix has rank at least $n - 1$, and that it has exactly rank $n - 1$ if and only if G is bipartite. Therefore, the result is a straightforward consequence of Proposition 1.2. □

2. THE SYZYGIES OF THE JACOBIAN MODULE

The following result gives a graph-theoretic characterization of a connected graph whose jacobian module has the smallest possible homological dimension.

Proposition 2.1 (char $k = 0$). *Let G be a connected graph. The following conditions are equivalent:*

- (i) $\mathfrak{J}(G)$ has homological dimension one.
- (ii) G has at most one cycle, and this cycle is odd.

Proof. (i) \Rightarrow (ii). Let n be the number of vertices of G . Since G is connected, the number m of edges is at least $n - 1$. If this number is $n - 1$ then G is a tree. So, assume that $m \geq n$. The homological dimension being 1, we must have $m = n$. But in this case it is well-known or clear that G admits at most one cycle. By Proposition 1.3, this cycle must be odd.

(ii) \Rightarrow (i). The case of a tree G is easy, directly. Indeed, induction on the number of vertices, passing to a subtree $G \setminus \{X_i\}$ with X_i a vertex of degree one, shows that in fact any individual $(n - 1) \times (n - 1)$ minor of ${}^t\Theta(G)$ is nonzero. Thus, assume that G contains a unique cycle and that cycle is odd. In this case the jacobian matrix of $I(G)$ is a square matrix. By Proposition 1.3, the determinant of ${}^t\Theta(G)$ is nonzero; hence $\mathfrak{J}(G)$ has homological dimension one. □

Remark 2.2. By [4], condition (ii) above characterizes a connected graph G for which the edge-ideal $I(G)$ is of linear type, i.e., for which the symmetric algebra of $I(G)$ is a torsion-free R -algebra. Thus, Proposition 2.1 shows that this condition is also equivalent to saying that the jacobian module $\mathfrak{J}(G)$ has homological dimension one.

2.1. Polar syzygies. Now, quite generally for the sake of definition, let $\mathbf{f} = \{f_1, \dots, f_m\}$ be a set of polynomials in $R = k[\mathbf{X}]$ such that $f_j(\mathbf{0}) = 0, \forall j$. Let $\mathcal{D}(I) := \text{Im}({}^t\Theta(\mathbf{f})) \subset \sum_{i=1}^n R dX_i$ stand for the submodule generated by the differentials of \mathbf{f} . One is interested in finding a structured set of generators of its first syzygy module $\mathcal{Z} := Z_1(\mathcal{D}(I)) \subset R^m = \sum_j RT_j$.

For that purpose, one considers the k -linear map $\lambda : k[\mathbf{T}] = S(R^m) \rightarrow R^m$ given by $\lambda(F) := \sum_j \frac{\partial F}{\partial T_j}(f_1, \dots, f_m) T_j \in R^m$. Moreover, let $J \subset k[\mathbf{T}]$ denote the ideal of polynomial relations of \mathbf{f} . By the usual rules of composite derivatives, if $F \in J$ then $\sum_{j=1}^m \frac{\partial F}{\partial T_j}(f_1, \dots, f_m) df_j = 0$. This means that restriction induces a map $J \rightarrow \mathcal{Z}$. Any element in the image of J will be called a *polar syzygy*.

Finally, we claim that the latter map still induces a quotient map

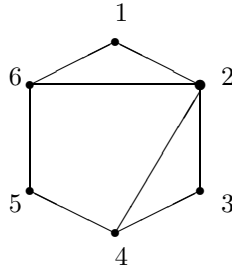
$$(1) \quad \bar{\lambda} : J/(\mathbf{T})J \rightarrow \mathcal{Z}/(\mathbf{X})\mathcal{Z}.$$

To see this, let $F \in (\mathbf{T})J$, say, $F = \sum_k G_k F_k$, with $G_k \in (\mathbf{T})$ and $F_k \in J$ for every k . Then, by the derivative rules and the vanishing $F_k(\mathbf{f}) = 0$,

$$\begin{aligned} \lambda(F) &= \sum_j \frac{\partial F}{\partial T_j}(\mathbf{f}) T_j = \sum_k \left(\sum_j \frac{\partial F_k}{\partial T_j}(\mathbf{f}) T_j \right) G_k(\mathbf{f}) \\ &= \sum_k G_k(\mathbf{f}) \lambda(F_k) \in (\mathbf{X})\lambda(J) \subset (\mathbf{X})\mathcal{Z}. \end{aligned}$$

The map (1) will be called the *polar map* (associated to \mathbf{f}). If \mathbf{f} are homogeneous polynomials of the same degree, one says that $\mathfrak{J}(\mathbf{f})$ is *polarizable* if the polar map is an isomorphism. When \mathbf{f} are homogeneous of the same degree, J (resp. \mathcal{Z}) is a homogeneous ideal of $k[\mathbf{T}]$ (resp. a graded submodule of $\sum_j RT_j$). When needed, one denotes by J_q and \mathcal{Z}_q the respective q th graded pieces.

The central question in this regard asks when the polar map is an isomorphism. That it may fail to be so in general is shown by the edge-ideal of the non-bipartite graph below:



An easy calculation shows here that $\mu(J) = 3$ and $\mu(\mathcal{Z}) = 2$; hence the polar map fails to be an isomorphism. Here, the “extra” minimal generator of J coming from the even walk (not a cycle!) with vertices $X_1, X_2, X_3, X_4, X_2, X_6$ maps to a deep combination of the 2 minimal generators of the module \mathcal{Z} . Of these, one is of degree 3, hence cannot be a polar syzygy. Thus, the polar map is neither injective nor surjective.

In view of this example, the following result seems to be of interest.

Theorem 2.3. *If G is a connected bipartite graph, then $\mathfrak{J}(G)$ is polarizable.*

In order to prove this result, one first needs a good grip on a set of generators of the ideal $J \subset k[\mathbf{T}]$ of relations. For bipartite graphs, the peculiar form of the generators of J as obtained in [5] allows for a substantial reduction of the number of generators in terms of the so-called primitive cycles (a cycle C of a graph G is a *primitive cycle* of G if no chord of C is an edge of G). Recall that the order of a cycle C of G is the number of edges of G belonging to C .

Here is the precise statement.

Lemma 2.4. *Let G be a connected bipartite graph. Then the residue classes of the (binomial) relations corresponding to the primitive cycles of G of order $2q$ form a vector basis of $(J/(\mathbf{T})J)_q = J_q / \sum_j T_j J_{q-j}$. In particular, J is minimally generated by the relations corresponding to the primitive cycles.*

Proof. By [5, Proposition 3.1] and since G is bipartite, one knows that J is generated by the binomial relations coming from the cycles of G . Now, if C_{2q} is not a primitive cycle, then it is subdivided into two other cycles $C'_{2q'}, C''_{2q''}$ (necessarily even) with $2q = 2q' + 2q'' - 2$, the two sharing a common edge which is a chord of C_{2q} . In this situation, one can see that the relation coming from C_{2q} is in the ideal generated by those coming from the two smaller cycles. This shows that the relations coming from the primitive cycles generate the ideal J ; hence the (images of) these relations corresponding to the-primitive cycles of order $2q$ span the vector space $J_q / \sum_j T_j J_{q-j}$. It remains to see that they are linearly independent. Now, an even primitive cycle is determined by a complete set of its alternate edges, i.e., two primitive cycles in a bipartite graph sharing a common 1-factor are equal. Therefore, for any two primitive cycles of order $2q$, the corresponding binomial relations are such that their constituent monomials are all pairwise distinct (i.e., have different supports). Thus, a k -linear combination of such distinct binomials is also one of distinct monomials. Now, by induction on q , one may assume that the primitive cycles of any order $2p < 2q$ already form a k -basis of $J_p / \sum_j T_j J_{p-j}$. Let it be given that

$$(2) \quad \sum_{l=1}^s \nu_l F_l = \sum_{k_1} G_{k_1} F_{k_1} + \dots + \sum_{k_{q-2}} G_{k_{q-2}} F_{k_{q-2}},$$

where F_l (resp. F_{k_u}) is the relation yielded by a primitive cycle of order $2q$ (resp. $2(u + 1)$) and $\nu_l \in k$. Moreover, one may assume that the left-hand side has the smallest number s of nonzero coefficients ν_l for which a relation (2) holds. In particular, $\nu_l \neq 0$ for every l . Let $F_1 = M_1 - N_1$, with M_1, N_1 monomials such that $\gcd(M_1, N_1) = 1$. Then there exist $F_{k_u} = M_{k_u} - N_{k_u}, F_{k_v} = M_{k_v} - N_{k_v}$ such that $M_1 = L_{k_u} M_{k_u}, N_1 = -L_{k_v} N_{k_v}$, for suitable monomials L_{k_u}, L_{k_v} . Clearly, $L_{k_u} F_{k_u} + L_{k_v} F_{k_v} \in J_q$, and since

$$(3) \quad L_{k_u} F_{k_u} + L_{k_v} F_{k_v} = F_1 - (L_{k_u} N_{k_u} - L_{k_v} M_{k_v}),$$

one has $L_{k_u} N_{k_u} - L_{k_v} M_{k_v} \in J_q$. If this binomial actually belongs to $\sum_j T_j J_{q-1}$, then by cancelling $\nu_1 F_1$ on both sides of (2), one would get a relation of the same form as (2) only with the left-hand side having fewer terms. Therefore, it must be the case that $F := L_{k_u} N_{k_u} - L_{k_v} M_{k_v} \in J_q \setminus \sum_j T_j J_{q-1}$. But then (3) gives a relation such as (2) whose left-hand side is $\nu_1 F_1 + \nu F$, for suitable $\nu \in k$. So, the number s of terms in the left-hand side of (2) is at most 2, and one may assume the relation is of the form

$$\nu_1 F_1 + \nu_2 F_2 = L_{k_u} F_{k_u} + L_{k_v} F_{k_v},$$

with F_{k_u} and F_{k_v} corresponding as before to smaller cycles. Since no further cancelling is possible in the right-hand side alone as the monomials in the left-hand side are all distinct, it follows that the smaller cycles could only be produced by chords of the cycles corresponding to the binomials F_1, F_2 . This is a contradiction, as they are primitive cycles by hypothesis. \square

Proof of the theorem. Since we are dealing with a bipartite graph, the two sets of variables giving the appropriate bipartition will be object of emphasis. Thus, \mathbf{X} will be replaced by \mathbf{X}, \mathbf{Y} . The transposed jacobian matrix of $I(G)$ can then be

$X_{i_1}Y_{v_{j_1}} \in I(G)$, then $v_{j_1} \neq u_j$ as well, by our present assumption. Since we already had $X_{i_1}Y_{u_j} | g_{1,u_1}$, we now have in fact that $X_{i_1}Y_{u_j}Y_{v_{j_1}} | g_{1,u_1}$.

In this situation, the claim is that there exists an entry X_{i_2} appearing in the Y_{u_j} -row, but not in the Y_{u_1} -row, such that $X_{i_2} | g_{1,u_1}$. For that, one resorts to the Y_{u_j} -row and coordinate g_{1,u_j} as pivot. Since there are no 4-cycles as just stated, then no X -entry in this row, other than X_1 , can appear simultaneously in the Y_{u_1} -row. Therefore, there exists indeed X_{i_2} such that $i_2 \neq 1, i_2 \neq i_1$ and such that $X_{i_2} | g_{1,u_j}$. On the other hand, in *Step 1*, as remarked before, the monomials $g_{1,u_1}/Y_{u_j}$ and $g_{1,u_j}/Y_{u_1}$ share the same \mathbf{X} -part. Hence, $X_{i_2} | g_{1,u_1}$, as claimed.

Thus, summing up, one has so far obtained that $X_{i_1}Y_{v_{j_1}}X_{i_2}Y_{u_j}$ divides g_{1,u_1} . Now it is time to loop back to *Alternative 1.1*, namely:

Alternative 2.1. $X_{i_2}Y_{v_{j_1}} \in I(G)$.

In this case, there is a 6-cycle in G , namely, the one with edges

$$X_1Y_{u_1}, X_{i_1}Y_{u_1}, X_{i_1}Y_{v_{j_1}}, X_{i_2}Y_{v_{j_1}}, X_{i_2}Y_{u_j}, X_1Y_{u_j}.$$

One then considers the syzygy $\mathbf{z} = f_{1,u_1}\lambda(F)$, where $F \in k[\mathbf{T}]$ denotes the binomial (of degree 3) corresponding to the above 6-cycle and

$$g_{1,u_1} = f_{1,u_1}X_{i_1}Y_{v_{j_1}}X_{i_2}Y_{u_j}g_{1,u_1}.$$

As before, we are done by the inductive hypothesis.

Alternative 2.2. There are no 6-cycles involving the path $X_1Y_{u_j}, X_1Y_{u_1}, X_{i_1}Y_{u_1}, X_{i_1}Y_{v_{j_1}}$ for which an edge of the 1-factor of the cycle containing $X_1Y_{u_1}$ (other than $X_1Y_{u_1}$) divides the coordinate g_{1,u_1} .

We may, therefore, assume that we have a path $X_1Y_{u_1}, X_{i_1}Y_{u_1}, X_{i_1}Y_{v_{j_1}}, X_{i_2}Y_{u_j}, X_1Y_{u_j}$, but $X_{i_2}Y_{v_{j_1}} \notin I(G)$. We then loop back to the first step, namely:

Step 3. Apply *Step 1* to the X_{i_2} -block with $g_{i_1,v_{j_1}}$ as pivotal coordinate. By a similar token, one finds new variables $X_{i_3}, Y_{w_{j_2}}$ such that $X_{i_3}Y_{w_{j_2}}X_{i_1}Y_{v_{j_1}}X_{i_2}Y_{u_j}$ divides g_{1,u_1} (In particular, one has that if $\deg(g_{1,u_1}) \leq 5$ in *Step 2*, then *Alternative 2.1* would necessarily take place for some $X_{i_2}Y_{v_{j_1}}$). Next, one searches for an 8-cycle for which the edges

$$X_1Y_{u_1}, X_{i_1}Y_{v_{j_1}}, X_{i_2}Y_{w_{j_2}}, X_{i_3}Y_{u_j}$$

form a 1-factor and their product divides g_{1,u_1} .

The procedure comes to a halt since, if the total degree (in $k[\mathbf{X}]$) of the coordinate g_{1,u_1} is s , one can scan all cycles containing any (even, non-closed) path, whose order is at most $s + 2$. The parenthetical observation in *Step 3* just above illustrates why eventually one loops back to *Step 1*.

This proves the surjectivity of the polar map $\bar{\lambda}$, showing that \mathcal{Z} is generated by the polar syzygies.

One now proceeds to show that this map is injective.

For that, first note that, as a consequence of the surjectivity just proved and Lemma 2.4, \mathcal{Z} is generated by the polar syzygies coming from primitive cycles. Therefore we wish to show that, for each $q \geq 2$, the polar syzygies of degree $2(q-1)$ are k -linearly independent modulo $(\mathbf{X}, \mathbf{Y})_2 \mathcal{Z}_{2(q-2)}$. Thus, let $\mathbf{z} = \sum_s \eta_s \mathbf{z}_s$, where $\mathbf{z}_s \in \mathcal{Z}_{2(q-1)}$ are polar syzygies coming from distinct primitive cycles (of order $2q$), with $\eta_s \in k$, be such that $\mathbf{z} \in (\mathbf{X}, \mathbf{Y})_2 \mathcal{Z}_{2(q-2)}$. We induct on q . For $q = 2$, we are given that $\sum_s \eta_s \mathbf{z}_s = 0$. Since we are dealing with squares, corresponding

coordinates of distinct $z_s, z_{s'}$ are relatively prime. Therefore, no cancellation is possible; hence $\eta_s = 0$ for every s .

Thus, assume that the assertion is true for polar syzygies coming from cycles of order less than $2q$. Suppose one can write

$$(4) \quad \sum_s \eta_s \mathbf{z}_s = \sum_{k_1} G_{k_1} \mathbf{z}_{k_1} + \dots + \sum_{k_{q-2}} G_{k_{q-2}} \mathbf{z}_{k_{q-2}}$$

where $G_{k_t} \in (\mathbf{X}, \mathbf{Y})_{2t}$ and $\mathbf{z}_{k_t} \in \mathcal{Z}_{2(q-t-1)}$ are also polar syzygies coming from primitive cycles, for $1 \leq t \leq q - 3$.

Now, a primitive cycle is determined by any one of its 1-factors. Moreover, any nonzero (i, i') -coordinate of a polar syzygy corresponding to a cycle C of G is obtained by multiplying out all the “edges”, other than the (i, i') th edge, of the unique 1-factor of C containing the (i, i') th edge. It follows that distinct cycles $\mathbf{z}_s, \mathbf{z}_{s'}$ have distinct monomial parts on corresponding coordinates. Therefore, no cancellation of corresponding nonzero coordinates is possible on the left-hand side of (4); hence any nonzero coordinate on the left-hand side, say, η_1 , has to be cancelled against some nonzero coordinate of the same name on the right-hand side of (4). The latter is of the form $G_{k_t} F_{k_t}$, where F_{k_t} is the product of all edges (but one) of the 1-factor of a smaller primitive cycle. Since we are looking at corresponding coordinates, this is impossible unless the smaller cycle is obtained from \mathbf{z}_1 by means of a chord of the cycle corresponding to \mathbf{z}_1 . This gives a contradiction, since the latter is chordless by hypothesis. \square

2.2. Second Betti number and complete intersections. Given a connected graph G , let

$$0 \rightarrow R^{b_r} \rightarrow \dots \rightarrow R^{b_2} \rightarrow R^{b_1} \rightarrow R^{b_0} \rightarrow \mathfrak{J}(G) \rightarrow 0$$

be a minimal free resolution of its jacobian module over the polynomial ring $R = k[\mathbf{X}]$. The number $b_j = b_j(\mathfrak{J}(G))$ is known as the j th Betti number of the module $\mathfrak{J}(G)$. Clearly, $b_0 = n$ and b_1 are the vertex number and the edge number of G , respectively.

The number of primitive cycles of G will be denoted by $f \text{rank}(G)$. Note that $f \text{rank}(G) \geq \text{rank}(G)$, where $\text{rank}(G)$ stands for the usual graph theoretic rank of G (cf. [1]).

These numbers have a natural meaning in terms of the homology of $\mathfrak{J}(G)$ if G is bipartite. Let $k[G] \subset R = k[\mathbf{X}]$ denote the k -subalgebra generated by the generators of the edge-ideal $I(G)$. As previously, let $J \subset k[\mathbf{T}]$ denote the ideal of polynomial relations of $k[G]$.

The main result is the following.

Theorem 2.5. *Let G be a connected bipartite graph and let*

$$0 \rightarrow R^{b_r} \xrightarrow{\Theta_r} \dots \rightarrow R^{b_2} \xrightarrow{\Theta_2} R^{b_1} \xrightarrow{\Theta_1} R^n \rightarrow \mathfrak{J}(G) \rightarrow 0$$

be a minimal free resolution of its jacobian module over the polynomial ring $R = k[\mathbf{X}]$, with $\Theta_1 = {}^t\Theta(G)$. Then:

- (i) $b_2 = f \text{rank}(G)$.
- (ii) *The following conditions are equivalent:*
 - (a) $\mathfrak{J}(G)$ has homological dimension at most 2
 - (b) $f \text{rank}(G) = \text{rank}(G)$
 - (c) $k[G]$ is a complete intersection.

Proof. (i) By Theorem 2.3, one has $\mu(J) = \mu(\mathcal{Z})$, and by Lemma 2.4, $\mu(J) = f \operatorname{rank}(G)$. Therefore, $b_2 (= \mu(\mathcal{Z})) = f \operatorname{rank}(G)$, as required.

(ii) (a) \Rightarrow (b) If $\mathfrak{J}(G)$ has homological dimension at most 2 then, by Schanuel's lemma applied to the above resolution, $\ker \Theta_1$ is a free homogeneous submodule of R^{b_1} of rank $b_1 - \operatorname{rank}(\Theta_1) = b_1 - n + 1 = \operatorname{rank}(G)$ by Theorem 1.3 and [1]. By part (i), it follows that $b_2 = \operatorname{rank}(G)$.

(b) \Rightarrow (c) One has $f \operatorname{rank}(G) = b_2$ by (i). Therefore, it suffices to show that $\operatorname{rank}(\Theta_2) = \operatorname{ht} J$ (the codimension of the ideal J of polynomial relations of $k[G]$). But since G is bipartite, one has $\dim k[G] = \operatorname{rank} \Phi(G) = n - 1$. Therefore, $n - 1 = \#\mathbf{T} - \operatorname{ht} J = b_1 - \operatorname{ht} J$; hence $\operatorname{ht} J = b_1 - n + 1 = \operatorname{rank}(\Theta_2)$.

(c) \Rightarrow (a) The assumption means that $\mu(J) = \operatorname{ht}(J)$. From part (i) (and its proof), one knows that $b_2 = \mu(J)$ and $\operatorname{rank}(\Theta_2) = \operatorname{ht} J$. It follows that $b_2 = \operatorname{rank}(\Theta_2)$; hence Θ_2 yields an injective map. \square

REFERENCES

1. F. Harary, *Graph Theory*, Addison–Wesley Publishing Co., Reading, Mass., 1969. MR **41**:1566
2. L. R. Doering and T. Gunston, Algebras arising from bipartite planar graphs, *Comm. Algebra* **24** (1996), 3589–3598. CMP 96:17
3. A. Simis, W. V. Vasconcelos and R. Villarreal, On the ideal theory of graphs, *J. Algebra* **167**(2) (1994), 389–416. MR **95e**:13002
4. R. Villarreal, Cohen–Macaulay graphs, *Manuscripta Math.* **66** (1990), 277–293. MR **91b**:13031
5. R. Villarreal, Rees algebras of edge-ideals, *Comm. Algebra* **23** (1995), 3513–3524. MR **96e**:13005

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