COMMON OPERATOR PROPERTIES
OF THE LINEAR OPERATORS RS AND SR

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ABSTRACT. Let $S$ and $R$ be bounded linear operators defined on Banach spaces, $S : X \to Y$, $R : Y \to X$. When $\lambda \neq 0$, then the operators $\lambda - SR$ and $\lambda - RS$ have many basic operator properties in common. This situation is studied in this paper.

INTRODUCTION

It is a well-know and useful result that when $A$ and $B$ are elements of a Banach algebra, then

$$\sigma(AB) \setminus \{0\} = \sigma(BA) \setminus \{0\}$$

([BD, Prop. 6, p. 16], [R, Lemma (1.4.17)], [P, Prop. 2.1.8]). Here $\sigma(A)$ denotes the spectrum of $A$. The case where the Banach algebra is $B(X)$, the algebra of all bounded linear operators on a Banach space $X$, is of special interest.

More generally, let both $X$ and $Y$ be Banach spaces, and let $S : X \to Y$ and $R : Y \to X$ be bounded linear operators. Again, it is known that

$$\sigma(SR) \setminus \{0\} = \sigma(RS) \setminus \{0\}.$$ 

Here $RS \in B(X)$ and $SR \in B(Y)$. In this paper we study this situation, showing that, in fact, for $\lambda \neq 0$, $\lambda - SR$ and $\lambda - RS$ have many basic operator properties in common (for example: $\lambda - SR$ has closed range if and only if $\lambda - RS$ has closed range). Throughout we assume that $X$, $Y$, $S$, and $R$ are as stated above.

For $T \in B(X)$, let $\mathcal{N}(T)$ denote the null space of $T$, and let $\mathcal{R}(T)$ denote the range of $T$.

1. Spectrum

Let $A$ and $B$ be elements of a ring with unit $I$. We recall some notation: $A \circ B = A + B - AB$; $I - (A \circ B) = (I - A)(I - B)$. When $A \circ B = B \circ A = 0$, then $B$ is the unique element with this property. In this case we write $B = A^q$. Thus, $(I - A)(I - A^q) = (I - A^q)(I - A) = I$, and so $(I - A)^{-1} = I - A^q$.

We have the following known basic computation (which holds in a ring with unit). Of course, the computation holds with the roles of $R$ and $S$ reversed.
**Proposition 1** (The basic computation). For any $W \in B(X)$, let

$$V = S(W - I)R \in B(Y).$$

1. $(SR) \circ V = S((RS) \circ W)R$; and
2. $V \circ (SR) = S(W \circ (RS))R$.

We verify (1):


Suppose that $I - RS$ is invertible in $B(X)$, and set $W = (RS)^t \in B(X)$. Thus as remarked above, $(RS) \circ W = W \circ (RS) = 0$. Define $V \in B(Y)$ as in Proposition 1, so by that result, $(SR) \circ V = V \circ (SR) = 0$. Therefore $V = (SR)^t$, and $I - SR$ is invertible in $B(Y)$. From this it follows that:

$$\sigma(SR) \setminus \{0\} = \sigma(RS) \setminus \{0\}.$$

We will show later in this section that similar equalities hold for all the usual parts of the spectrum. Note that Proposition 1 also implies:

$I - SR$ has a left (right) inverse if and only if $I - RS$ has a left (right) inverse.

**Proposition 2.**

1. $S(N(I - RS)) = N(I - SR)$;
2. $N(S) \cap N(I - RS) = \{0\}$.

**Proof.** Statement (2) clearly holds. Assume $x \in N(I - RS)$, so $RSx = x$. Then $SRSx = Sx$, and thus, $S(N(I - RS)) \subseteq N(I - SR)$. To verify the opposite inclusion, suppose $y \in N(I - SR)$. Arguing as above, we have

$$R(N(I - SR)) \subseteq N(I - RS).$$

Therefore, $Ry \in N(I - RS)$. Then $y = SRy \in S(N(I - RS))$. This proves (1).

We use $\sigma_p$, $\sigma_ap$, $\sigma_r$, and $\sigma_c$ to denote the point, approximate point, residual, and continuous spectrum, respectively.

**Theorem 3.**

1. $\sigma(SR) \setminus \{0\} = \sigma(RS) \setminus \{0\}$;
2. $\sigma_p(RS) \setminus \{0\} = \sigma_p(SR) \setminus \{0\}$;
3. $\sigma_ap(RS) \setminus \{0\} = \sigma_ap(SR) \setminus \{0\}$;
4. $\sigma_r(RS) \setminus \{0\} = \sigma_r(SR) \setminus \{0\}$;
5. $\sigma_c(RS) \setminus \{0\} = \sigma_c(SR) \setminus \{0\}$.

**Proof.** As noted previously, (1) follows from Proposition 1. Also, (2) is an immediate corollary of Proposition 2 (1).

Now assume $\lambda \in \sigma_ap(RS) \setminus \{0\}$. This means there exists $\{x_n\} \subseteq X$, $\|x_n\| = 1$ for all $n$, and $\|(\lambda - RS)x_n\| \to 0$. Therefore, $\|(\lambda - SR)(Sx_n)\| = \|S(\lambda - RS)x_n\| \to 0$. Also, $\|Sx_n\|$, $n \geq 1$, is bounded away from zero, for if not, $\|Sx_n\| \to 0$ for some subsequence $\{x_{n_k}\}$ of $\{x_n\}$. But then,

$$|\lambda| = |\lambda||x_n| \leq \|(\lambda - RS)x_n\| + \|RSx_n\| \to 0,$$

a contradiction. This proves $\lambda \in \sigma_ap(SR)$.

(4): $\lambda \in \sigma_r(RS) \setminus \{0\}$ means exactly that $\lambda \neq 0$, $\lambda \notin \sigma_p(RS)$, and $\mathcal{R}(\lambda - RS)^- \neq X$. We use $T'$ to denote the adjoint of an operator $T$. We have $\mathcal{N}(\lambda - S'T') = \mathcal{N}(\lambda - (SR)^t) \neq \{0\}$. By Proposition 2 (1), $\mathcal{N}(\lambda - (SR)^t) = \mathcal{N}(\lambda - R'S') \neq \{0\}$. Therefore $\mathcal{N}(\lambda - SR)^- \neq Y$, and so $\lambda \in \sigma_r(SR)$. 

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Note here that \( \operatorname{LRw} \) is well-defined.

2. Closed range

Recall that \( I - RS \) has pseudoinverse (or generalized inverse) \( I - W \) means that
\[
\]
The existence of a pseudoinverse for \( I - RS \) implies that \( \mathcal{R}(I - RS) \) is closed (and more) [TL, Theorem 12.9, p. 251].

**Theorem 4.** \( I - RS \) has a pseudoinverse if and only if \( I - SR \) has a pseudoinverse.

**Proof.** Assume \( I - RS \) has pseudoinverse \( I - W \) (as above). Let \( V = S(W - I)R \).
From Proposition 1 we have
\[
(I - SR)(I - V) = I - (SR) \circ V = I + S[-(RS) \circ W]R
\]
Thus,
\[
(I - SR)(I - V)(I - SR)
= I + S[(I - RS)(I - W)]R - SR - SR - S[(I - RS)(I - W)]RSR + SRSR
= I - 2SR + SRSR + S[(I - RS)(I - W)(I - RS)]R
= I - 2SR + SRSR + S[I - RS]R = I - SR.
\]

Note that the argument in the proof of Theorem 4 is completely algebraic, so the result holds in any ring with unit.

Let \( T \in B(X) \). Equip \( X/\mathcal{N}(T) \) with the usual quotient norm. A standard condition which is equivalent to \( \mathcal{R}(T) \) being closed is:
there exists a bounded linear operator \( M: \mathcal{R}(T) \to X/\mathcal{N}(T) \) such that
\[
MTx = x + \mathcal{N}(T) \quad \text{for all } x \in X.
\]

**Theorem 5.** \( \mathcal{R}(I - RS) \) is closed if and only if \( \mathcal{R}(I - SR) \) is closed.

**Proof.** Assume \( I - RS \) has closed range \( Z \). Write \( (I - RS)^{\sim}: X/\mathcal{N}(I - RS) \to Z \) where \( (I - RS)^{\sim}(x + \mathcal{N}(I - RS)) = (I - RS)x \). There exists a bounded linear operator
\[
L: Z \to X/\mathcal{N}(I - RS)
\]
such that for all \( x \in X \),
\[
L(I - RS)^{\sim}(x + \mathcal{N}(I - RS)) = x + \mathcal{N}(I - RS).
\]
Define \( \bar{S}: X/\mathcal{N}(I - RS) \to Y/\mathcal{N}(I - SR) \) by
\[
\bar{S}(x + \mathcal{N}(I - RS)) = Sx + \mathcal{N}(I - SR).
\]
Note that \( \bar{S} \) is well-defined by Proposition 2. Also, it is straightforward to check that \( \bar{S} \) is continuous.

Now define \( M: \mathcal{R}(I - SR) \to Y/\mathcal{N}(I - SR) \) by \( M(w) = w + \mathcal{N}(I - SR) + \bar{S}LRw \).
Note here that \( w = y - SRy \) for some \( y \), so \( Rw = (I - RS)y \in \mathcal{R}(I - RS) \). Thus, \( LRw \) is well-defined.
Also, since by definition $M$ is an algebraic combination of continuous maps, $M$ is continuous. Finally,
\[
M(y - SRy) = y - SRy + N(I - SR) + \tilde{S}LR(y - SRy)
\]
\[
= y - SRy + N(I - SR) + \tilde{S}Ry = y + N(I - SR).
\]
Therefore $R(I - SR)$ is closed.

3. Fredholm properties

An operator $T \in B(X)$ is semi-Fredholm if $R(T)$ is closed and either $\text{nul}(T) = \dim(V(T))$ or $\text{nul}(T')$ is finite (as before, $T'$ is the adjoint of $T$); see [CPY, 1.3 and Chapter 4]. We use the notation:
\[
\Phi^+ = \{ T \in B(X) : R(T) \text{ is closed and } \text{nul}(T) < \infty \};
\]
\[
\Phi^- = \{ T \in B(X) : R(T) \text{ is closed and } \text{nul}(T') < \infty \};
\]
\[
\Phi = \Phi^+ \cap \Phi^-.
\]
Recall, when $T$ is semi-Fredholm, then $\text{ind}(T) = \text{nul}(T) - \text{nul}(T')$.

**Theorem 6.** $I - RS \in \Phi (\Phi^+, \Phi^-)$ if and only if $I - SR \in \Phi (\Phi^+, \Phi^-)$. Furthermore, when $I - RS \in \Phi$, then $\text{ind}(I - RS) = \text{ind}(I - SR)$.

**Proof.** The first statement follows directly from Theorem 5 and Proposition 2. Also, when $I - RS \in \Phi$, Proposition 2 implies that $\text{nul}(I - RS) = \text{nul}(I - SR)$ and $\text{nul}(I - RS)' = \text{nul}(I - S'R') = \text{nul}(I - R'S') = \text{nul}(I - (RS)')$. Thus, $\text{ind}(I - RS) = \text{ind}(I - SR)$.

4. Functional calculus

In this section we derive a useful relationship between the holomorphic functional calculi of $RS$ and $SR$.

**Theorem 7.** Let $g(\lambda)$ be a holomorphic function on some open set $U$ such that $\sigma(SR) \cup \{0\} \subseteq U$. Let $f(\lambda) = \lambda g(\lambda)$. Then $f(SR) = Sg(RS)R$.

**Proof.** Set $\Gamma = \sigma(SR) \cup \{0\}$, and let $\gamma$ be a cycle which is contained in $U \setminus \Gamma$ with $\text{ind}_\gamma(z) = 1$ for all $z \in \Gamma$, and $\text{ind}_\gamma(z) = 0$ for all $z \notin U$. For $\lambda \neq 0$, we have
\[
\lambda^{-1}[I - (\lambda^{-1}SR)^q] = (\lambda - SR)^{-1}.
\]
Also, by Proposition 1 with $W = (\lambda^{-1}RS)^q$,
\[
(\lambda^{-1}SR)^q = -\lambda^{-1}S[I - (\lambda^{-1}RS)^q]R.
\]
Therefore,
\[ f(SR) = (2\pi i)^{-1} \int f(\lambda)(\lambda - SR)^{-1} d\lambda \]
\[ = (2\pi i)^{-1} \left[ \int g(\lambda)\lambda^{-1}S[I - (\lambda^{-1}RS)^q]R d\lambda \right] \]
\[ = (2\pi i)^{-1} \left[ 0 + \int g(\lambda)\lambda^{-1}S[I - (\lambda^{-1}RS)^q]R d\lambda \right] \]
\[ = S \left[ (2\pi i)^{-1} \int g(\lambda)(\lambda - RS)^{-1} d\lambda \right] R = Sg(RS)R \quad \text{(by (1))}. \]

**Corollary 8.** Let \( f \) and \( g \) be as above. Set \( R_1 = g(RS)R \). Then \( f(SR) = SR_1 \), and since \( f(\lambda) = g(\lambda)\lambda \), \( f(RS) = g(RS)RS = R_1S \).

Therefore the results of this paper apply to \( f(SR) \) and \( f(RS) \).

5. **Poles**

Let \( \lambda_0 \neq 0 \) be an isolated point of \( \sigma(RS) \). We adopt the notation and terminology in [TL, pp. 328–331]. In particular, for \( n \neq 1 \), let
\[ f_{-n}(\lambda) = \begin{cases} (\lambda - \lambda_0)^{-n} & \text{if } |\lambda - \lambda_0| < r; \\ 0 & \text{if } |\lambda - \lambda_0| > 2r. \end{cases} \]
(Here \( r > 0 \) is chosen so that \( (\sigma(RS) \cup \{0\}) \setminus \{\lambda_0\} \subseteq \{\lambda: |\lambda - \lambda_0| > 2r\}. \)
Let \( B_0(RS) = f_{-1}(RS) \), and note that \( B_1(RS) \) is the spectral projection corresponding to the spectral set \( \{\lambda_0\} \). We use the same notation relative to \( SR; B_n(SR) = f_{-n}(SR) \). Define
\[ h(\lambda) = \begin{cases} \lambda^{-1} & \text{if } |\lambda - \lambda_0| < r; \\ 0 & \text{if } |\lambda - \lambda_0| > 2r. \end{cases} \]

We have \( f_{-n}(\lambda) = (\lambda f_{-n}(\lambda)h(\lambda)) \), which gives:
1. \( B_0(SR) = SRB_0(SR)h(SR) \); and
2. \( B_1(SR) = S[B_1(RS)h(RS)]R \).
((2) follows by applying Theorem 7.)

By definition \( \lambda_0 \) is a pole of order \( p \) of the resolvent of \( RS \) if \( B_p(RS) \neq 0 \), and \( B_n(RS) = 0 \) for all \( n > p \) [TL, p. 330].

**Theorem 9.** An isolated point \( \lambda_0 \neq 0 \) of \( \sigma(RS) \) is a pole of order \( p \) of the resolvent of \( RS \) if and only if \( \lambda_0 \) is a pole of order \( p \) of the resolvent of \( SR \). Furthermore, \( \mathcal{R}(B_1(RS)) \) is finite dimensional if and only if \( \mathcal{R}(B_1(SR)) \) is finite dimensional.

**Proof.** We use the notation introduced above. Assume \( B_n(RS) = 0 \) for some \( n > 1 \). By Theorem 7 with \( g(\lambda) = f_{-n}(\lambda), f(\lambda) = \lambda g(\lambda) \), it follows that \( SRB_n(SR) = S[B_n(RS)]R = 0 \). By (1) it follows that \( B_n(SR) = 0 \). This argument establishes that \( B_n(RS) = 0 \) if and only if \( B_n(SR) = 0 \). The statement of the theorem concerning poles follows from this.

The second statement of the theorem follows from (2), since if \( \mathcal{R}(B_1(RS)) \) is finite dimensional, then \( B_1(SR) = S[B_1(RS)h(RS)]R \) has finite dimensional range.
Recall that the smallest integer \( p \geq 0 \) such that \( \mathcal{N}(T^p) = \mathcal{N}(T^{p+1}) \) is called the ascent of the operator \( T \) (the ascent of \( T \) is infinite if \( \mathcal{N}(T^n) \neq \mathcal{N}(T^{n+1}) \) for all \( n \geq 0 \) [TL, Section V6]. The property that \( \lambda_0 - T \) has finite ascent is closely connected to \( \lambda_0 \) being a pole of the resolvent of \( T \); see [TL, Section V10].

Let \( n \geq 0 \) be an integer. There exists \( U_n \) such that
\[
(I - SR)^{n+1} = I - SU_n; \quad (I - RS)^{n+1} = I - U_nS.
\]

In fact, by direct computation, \( U_n = \sum_{k=1}^{n+1} (-1)^{k-1} \binom{n+1}{k} R(SR)^{k-1} \) works.

**Proposition 10.** \( I - RS \) has finite ascent \( p \) if and only if \( I - SR \) has finite ascent \( p \).

**Proof.** Suppose, for some \( n \geq 0 \), \( \mathcal{N}((I - RS)^n) = \mathcal{N}((I - RS)^{n+1}) \). By the existence of \( U_k \) as indicated above, Proposition 2 applies to \( (I - SR)^{k+1} \) for all \( k \geq 0 \). Thus,
\[
\mathcal{N}((I - SR)^{n+1}) = \mathcal{N}(I - SU_n) = S(\mathcal{N}(I - U_nS)) = S(\mathcal{N}((I - RS)^{n+1}))
\]
\[
= S(\mathcal{N}((I - RS)^n)) = \mathcal{N}((I - SR)^n).
\]

This implies that \( I - RS \) has ascent \( p \).

### 6. Examples, applications

In this section we look at several situations where the results of the previous sections apply.

**Example 11.** Assume that the Banach space \( X \) is continuously embedded as a subspace of a Banach space \( Y \). Assume that \( T \in B(X) \) has an extension \( \mathcal{T} \in B(Y) \). In [B1] operator properties of \( T \) and \( \mathcal{T} \) are studied with the hypothesis that \( \mathcal{T}(Y) \subseteq X \). All of the main results of [B1, §2] (and more!) can be derived from results in this paper. For let \( S: X \to Y \) be the continuous embedding, \( S(x) = x \). Let \( R: Y \to X \) be the bounded operator, \( R(y) = \mathcal{T}(y) \in X \). Then \( T = RS \) and \( \mathcal{T} = SR \). Therefore in this situation the results of the previous sections apply to \( T \) and \( \mathcal{T} \).

**Example 12.** Let \( H \) be a Hilbert space. Assume \( S: X \to H \) and \( R: H \to X \) have the special property that \( SR \) is selfadjoint. Then \( T = RS \) has many of the operator properties of a selfadjoint operator. Exactly this situation is studied in [B2].

In particular, suppose \( X = H \), \( R \geq 0 \), and \( S = S^* \). Then an operator of the form \( SR \) is called symmetrizable. The operator \( SR \) has many operator properties in common with the selfadjoint operator \( R^2SR^2 \).

**Example 13.** Let \((\Omega, \mu)\) be a \( \sigma \)-finite measure space. Let \( K(x, t) \) be a kernel on \( \Omega \times \Omega \) with the property
\[
k(x) = \operatorname{ess sup}_{t \in \Omega} |K(x, t)| \in L^1(\mu).
\]
The linear integral operator
\[
T_K(f)(x) = \int_{\Omega} K(x, t)f(t) \, d\mu(t), \quad f \in L^1(\mu),
\]
is an operator in \( B(L^1(\mu)) \). In this case \( T_K \) is a Hille-Tamarkin operator, \( T_K \in H_{11} \); see [J, Sections 11.3 and 11.5].
We may assume that \( k(x) \) is everywhere defined and nonnegative. Define
\[
J(x, t) = \begin{cases} k(x)^{-\frac{1}{2}}K(x, t) & \text{if } k(x) > 0; \\ 0 & \text{if } k(x) = 0. \end{cases}
\]
Since
\[
\text{ess sup}_{t \in \Omega} |J(x, t)|^2 = k(x) \in L^1(\mu), 
\]
the integral operator, \( T_J: L^1 \rightarrow L^2 \), is in the Hille-Tamarkin class \( H_{21} \). Also, define
\[
H(x, t) = J(x, t)k(t)^{\frac{1}{2}}.
\]
Since \( |H(x, t)|^2 \leq k(x)k(t) \), it follows that \( T_H \) is a Hilbert-Schmidt operator on \( L^2(\mu) \).

Now consider the operators \( S: L^2 \rightarrow L^1 \) and \( R: L^1 \rightarrow L^2 \) given by
\[
S(f) = k^{\frac{1}{2}}f \quad (f \in L^2); \quad R(g) = T_J(g) \quad (g \in L^1).
\]
Then \( SR = T_K \) and \( RS = T_H \). We summarize:

**Theorem 14.** Let \( T_K: L^1 \rightarrow L^1 \) be a Hille-Tamarkin operator in class \( H_{11} \). Then there exist bounded operators \( S: L^2 \rightarrow L^1 \) and \( R: L^1 \rightarrow L^2 \) such that \( T_K = SR \) and \( RS \) is a Hilbert-Schmidt operator.

**Corollary 15.** Let \( T_K: L^1 \rightarrow L^1 \) be a Hille-Tamarkin operator in class \( H_{11} \). Then \( T_K^2 \) is compact, and the nonzero eigenvalues of \( T_K \) (counted according to multiplicities) form a square summable sequence.

**Proof.** Let \( T_K = SR \) with \( RS \) Hilbert-Schmidt. Then \( T_K^2 = S(RS)R \), so \( T_K^2 \) is compact. Also, the sequence of nonzero eigenvalues (counted according to multiplicities) of \( T_K \) and \( RS \) are the same by Proposition 2 and Theorem 3. This sequence is square summable by [Rg, Corollary 2.3.6, p. 89].

**References**


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