PROCEEDINGS OF THE
AMERICAN MATHEMATICAL SOCIETY
Volume 126, Number 4, April 1998, Pages 1115–1125
S 0002-9939(98)04240-3

FELL BUNDLES OVER GROUPOIDS

ALEX KUMJIAN

(Communicated by Palle E. T. Jorgensen)

Abstract. We study the C*-algebras associated to Fell bundles over groupoids and give a notion of equivalence for Fell bundles which guarantees that the associated C*-algebras are strongly Morita equivalent. As a corollary we show that any saturated Fell bundle is equivalent to a semi-direct product arising from the action of the groupoid on a C*-bundle.

A C*-algebraic bundle (see [F2, §11]) over a locally compact group may be thought of as a continuous version of a group grading in a C*-algebra; one may regard the associated C*-algebra as a fairly general sort of crossed product of the fiber algebra over the neutral element by the group (in [LPRS] it is shown that the C*-algebra is endowed with a coaction by the group). There is a natural extension of this definition to groupoids (see [Yg]) which when specialized to trivial groupoids (i.e. topological spaces) yields the more usual notion of C*-algebra bundle. This object is referred to below as a Fell bundle.

Closely related notions have appeared in the literature: in recent work [Yn], Yamanouchi studies the analogous notion in the von Neumann algebra setting under the name integrable coaction. Fell bundles are reminiscent of the C*-categories discussed in [GLR]. They are also presaged in [Re2, Def. 5.3] (the object is used to construct a strong Morita equivalence bimodule).

Since each fiber of a saturated Fell bundle may be regarded as a strong Morita equivalence bimodule, the theory of Fell bundles provides a natural locus for proving theorems related to strong Morita equivalence of the kind which appear in [MRW] and [Re2].

In §1 we fix notation and review some well-known facts concerning groupoids, Banach bundles, Hilbert modules, and equivalence bimodules. The notion of Fell bundle is defined in §2 and some examples are discussed, including the semi-direct product which results from a groupoid acting on a C*-algebra bundle fibered over the unit space. In §3 the associated C*-algebra is constructed in the case that the groupoid is r-discrete; the norm is defined by an analog of the left regular representation (the resulting C*-algebra may be regarded as the reduced C*-algebra associated to the Fell bundle). Finally, a Morita equivalence theorem of the sort discussed above is proved (Th. 4.2). This is used to show that there is an action of a groupoid on a C*-algebra bundle obtained from a saturated Fell bundle so that the C*-algebra associated to the Fell bundle is strongly Morita equivalent to that

---

Received by the editors September 23, 1996.
1991 *Mathematics Subject Classification.* Primary 46L55, 46L45; Secondary 46L05.

of the semi-direct product (Cor. 4.5). See [Kt, Th. 8], [PR, Cor. 3.7], [Q1, Th. 3.1], [Q2, Cor. 2.7], and [Yn, Th. 7.8] for related results.

1. PRELIMINARIES

1.1. Given a groupoid Γ, let Γ^0 denote the unit space and $r, s : \Gamma \to \Gamma^0$ denote the range and source maps (respectively); let Γ^2 denote the collection of composable pairs; write the inverse map $\gamma \mapsto \gamma^*$. We tacitly assume that all groupoids under discussion are locally compact and Hausdorff, that the structure maps are continuous, and that they admit left Haar systems (see [Re1]). Let Δ denote the transitive equivalence relation with unit space $\Delta^0 = \{0, 1\}$; write $\Delta = \{0, 1, \partial, \partial^*\}$ where $s(\partial) = 0$ and $r(\partial) = 1$. A groupoid Γ is said to be trivial if $\Gamma = \Gamma^0$.

1.2. Given a Banach bundle, $p : E \to X$, let $C_c(E)$ denote the collection of compactly supported continuous sections of E and $C_0(E)$ denote the collection of continuous sections of E vanishing at ∞. Note that $C_0(E)$ may be regarded as the completion of $C_c(E)$ in the supremum norm. For $x \in X$, let E_x denote the fiber over x, $p^{-1}(x)$. We shall tacitly assume that the total space, E, of any Banach bundle under consideration is second countable and that the base space, X, is locally compact and Hausdorff; it follows that the base space is second countable and that both the fiber, E_x, and the associated Banach space, $C_0(E)$, are separable (see [F2, Prop. 10.10]). By a result of Doady and dal Soglio-Hérault ([F2, appendix]), if X is locally compact, E has enough continuous sections; thus for every $e \in E$ there is $f \in C_c(E)$ such that $f(p(e)) = e$.

1.3. Let A be a C*-algebra and V be a right A-module; V is said to be a (right) pre-Hilbert A-module if it is equipped with an A valued inner product $\langle \cdot, \cdot \rangle$ which satisfies the following conditions:

i. $\langle u, \lambda v + w \rangle = \lambda \langle u, v \rangle + \langle u, w \rangle$,
ii. $\langle u, va \rangle = \langle u, v \rangle a$,
iii. $\langle v, u \rangle = \langle u, v \rangle^*$,
iv. $\langle v, v \rangle \geq 0$ and $\langle v, v \rangle = 0$ only if $v = 0$,

for all $u, v, w \in V$, $\lambda \in \mathbf{C}$, and $a \in A$. Say that V is a (right) Hilbert A-module if it is complete in the norm $\|v\| = \|\langle v, v \rangle\|^{1/2}$. There is an analogous definition for left Hilbert A-modules (the inner product in this case is linear in the first variable). Given a right Hilbert A-module V, there is a left Hilbert A-module V^* with a conjugate linear isometric isomorphism from V to V^*, written $v \mapsto v^*$, which is compatible with the module structure and inner product in the following way:

$$av^* = (va^*)^*,$$

$$\langle u^*, v^* \rangle = \langle u, v \rangle,$$

for all $u, v \in V$ and $a \in A$. The term, Hilbert A-module, will be understood to mean right Hilbert A-module. If the span of the values of the inner product of a Hilbert A-module V is dense in A, then V is said to be full (see [Ks, §2], [B, 13.1.1]). Note that A may be regarded as a Hilbert A-module when endowed with the inner product $\langle a, b \rangle = a^*b$; evidently, A is full.

1.4. Given Hilbert A-modules U and V, let $\mathcal{L}(V, U)$ denote the collection of bounded adjointable operators from V to U which commute with the right action of A. Put $\mathcal{L}(V) = \mathcal{L}(V, V)$; endowed with the operator norm, $\mathcal{L}(V)$ is a C*-algebra. For $u \in U$, $v \in V$, define $\theta_{u,v} \in \mathcal{L}(V, U)$ by $\theta_{u,v}(w) = u\langle v, w\rangle$ for $w \in V$. Note that $\theta_{u,v} = \theta^*_{v,u}$. The closure of the span of such operators is denoted $\mathcal{K}(V, U)$; write $\mathcal{K}(V) = \mathcal{K}(V, V)$. Note that $\mathcal{K}(V)$ is an (essential) ideal in $\mathcal{L}(V)$; in fact, $\mathcal{L}(V)$ may be identified with the multiplier algebra of $\mathcal{K}(V)$ (see [Ks, Th. 1]).

1.5. Given C*-algebras A and B, a B-A equivalence bimodule V is both a full right Hilbert A-module and a full left Hilbert B-module with compatible inner products (V^* may be viewed as an A-B equivalence bimodule); note that equivalence bimodules are also known as imprimitivity bimodules (see [Ri1, Def. 6.10]). If such a bimodule exists the two C*-algebras are said to be strongly Morita equivalent (see [Ri2]). If V is a full Hilbert A-module, then V is a $\mathcal{K}(V)$-A equivalence bimodule with $\mathcal{K}(V)$ valued inner product: $\langle u, v\rangle = \theta_{u,v}$. Indeed, every equivalence bimodule is of this form. Moreover, if U and V are full Hilbert A-modules then $\mathcal{K}(V, U)$ may be regarded as a $\mathcal{K}(U)$-$\mathcal{K}(V)$ equivalence bimodule.

1.6. Given C*-algebras A, B, and C together with a B-A equivalence bimodule V and a C-B equivalence bimodule U, one may form the C-A equivalence bimodule: $U \otimes_B V$ (see [Ri1, Th. 5.9]). Note that if V is a B-A equivalence bimodule and W is a C-A equivalence bimodule then one may identify

$$W \otimes_A V^* = \mathcal{K}(V, W)$$

(as C-B equivalence bimodules) via the map $w \otimes v^* \mapsto \theta_{u,v}$. By this identification and the associativity of the tensor product one obtains $\mathcal{K}(U \otimes_B V) \cong \mathcal{K}(U)$:

$$\begin{aligned}\mathcal{K}(U \otimes_B V) &= (U \otimes_B V) \otimes_A (U \otimes_B V)^* \cong (U \otimes_B (V \otimes_A V^*)) \otimes_B U^* \\ &\cong (U \otimes_B B) \otimes_B U^* \cong U \otimes_B U^* = \mathcal{K}(U).\end{aligned}$$

Note that this gives an isomorphism of C*-algebras; in fact, $K(U \otimes_B V) \cong C \cong \mathcal{K}(U)$.

1.7. Given a C*-algebra bundle A over a space X, a Banach bundle V over X is said to be a Hilbert A-module bundle if each fiber V_x is a Hilbert A_x-module with continuous module action and inner product. Equipped with the natural inner product, $C_0(V)$ is a Hilbert $C_0(A)$-module, and it is full if and only if V_x is full for every $x \in X$ (in this case V is said to be full). Associated to V one obtains a C*-algebra bundle $\mathcal{K}(V)$ where $\mathcal{K}(V)_x = \mathcal{K}(V_x)$. One has $C_0(\mathcal{K}(V)) \cong \mathcal{K}(C_0(V))$.

2. Fell bundles

We define below the natural analog of Fell's C*-algebraic bundles (cf. [F2], [Yg]) for groupoids. This notion is a generalization of both C*-algebraic bundles (over groups) and C*-algebra bundles (over spaces).

2.1. Let Γ be a groupoid and $p : E \to \Gamma$ a Banach bundle; set

$$E^2 = \{(e_1, e_2) \in E \times E : (p(e_1), p(e_2)) \in \Gamma^2\}.$$

Definition. A *multiplication* on E is a continuous map, $E^2 \to E$ (write $(e_1, e_2) \mapsto e_1e_2$) which satisfies:

i. $p(e_1e_2) = p(e_1)p(e_2)$ for all $(e_1, e_2) \in E^2$,
ii. the induced map, $E_{\gamma_1} \times E_{\gamma_2} \to E_{\gamma_1\gamma_2}$, is bilinear for all $(\gamma_1, \gamma_2) \in \Gamma^2$,

iii. $(e_1e_2)e_3 = e_1(e_2e_3)$ whenever the multiplication is defined,
iv. $\|e_1e_2\| \le \|e_1\|\,\|e_2\|$ for all $(e_1, e_2) \in E^2$.

An *involution* on E is a continuous map, $E \to E$ (write $e \mapsto e^*$) which satisfies:

v. $p(e^*) = p(e)^*$ for all $e \in E$,
vi. the induced map, $E_\gamma \to E_{\gamma^*}$, is conjugate linear for all $\gamma \in \Gamma$,
vii. $e^{**} = e$ for all $e \in E$.

Finally, the bundle E together with the structure maps is said to be a *Fell bundle* if in addition the following conditions hold:

viii. $(e_1e_2)^* = e_2^*e_1^*$ for all $(e_1, e_2) \in E^2$,
ix. $\|e^*e\| = \|e\|^2$ for all $e \in E$,
x. $e^*e \ge 0$ for all $e \in E$.

Note that if $x \in \Gamma^0$ then E_x is a C*-algebra (with norm, multiplication, and involution induced from the bundle); if $e \in E_\gamma$ then $e^*e \in E_{s(\gamma)}$, hence it makes sense to require that e^*e be positive. Yamagami refers to such a bundle as a C*-algebra over a groupoid (see [Yg]).

2.2. E is said to be nondegenerate if $E_\gamma \ne 0$ for all $\gamma \in \Gamma$; note that E_γ is a right Hilbert $E_{s(\gamma)}$-module with inner product $\langle e_1, e_2\rangle = e_1^*e_2$ and a left Hilbert $E_{r(\gamma)}$-module with inner product $\langle e_1, e_2\rangle = e_1e_2^*$. Note also that $E_\gamma^* \cong E_{\gamma^*}$.

2.3. Given a Fell bundle E over Γ, let E^0 denote the restriction $E|_{\Gamma^0}$; clearly, E^0 is a C*-algebra bundle and $C_0(E^0)$ is a C*-algebra (with pointwise operations).

2.4. The Fell bundle E is said to be saturated if $E_{\gamma_1} \cdot E_{\gamma_2}$ is total in $E_{\gamma_1\gamma_2}$ for all $(\gamma_1, \gamma_2) \in \Gamma^2$; note that if E is saturated then E_γ may be regarded as an $E_{r(\gamma)}$-$E_{s(\gamma)}$ equivalence bimodule (with inner products as above). For $(\gamma_1, \gamma_2) \in \Gamma^2$, one has $E_{\gamma_1} \otimes_{E_x} E_{\gamma_2} \cong E_{\gamma_1\gamma_2}$ where $x = s(\gamma_1) = r(\gamma_2)$ (via the map $e_1 \otimes e_2 \mapsto e_1e_2$).

2.5. **Examples.**

i. Let E be a C*-algebra bundle over a space X. If we regard X as a trivial groupoid (so $X = X^0$), then E is seen to satisfy the above definition.
ii. Let E be a C*-algebraic bundle over a locally compact group G (in the sense of Fell). If we regard G as a groupoid (so $G^0 = 1_G$) then E is easily seen to satisfy the above definition (it is essentially the same definition).
iii. Let A and B be C*-algebras and let C be a B-A equivalence bimodule. Form a Fell bundle E over the groupoid Δ as follows: set $E_0 = A$, $E_1 = B$, $E_\partial = C$, $E_{\partial^*} = C^*$; since Δ is discrete the Banach bundle structure is trivial. One defines multiplication and involution in the obvious way and checks that the above conditions are satisfied. Note that E is saturated.
iv. Let Σ be a proper $\mathbf{T}$-groupoid over Γ (see [Ku, Def. 2.2]). Form the associated line bundle: $E = \Sigma *_{\mathbf{T}} \mathbf{C} = (\Sigma \times \mathbf{C})/\mathbf{T}$ (where $t(\sigma, z) = (ts, t^{-1}z)$). One defines multiplication and involution as follows:

$$(\sigma_1, z_1)(\sigma_2, z_2) = (\sigma_1\sigma_2, z_1z_2),$$

$$(\sigma, z)^* = (\sigma^*, \overline{z});$$

one must also check that both are well-defined (cf. [F2, §12]).

v. An action of Γ on a C*-algebra bundle, $q : A \to \Gamma^0$, is a continuous map (see [Re2], [M]): $\alpha : \Gamma * A \to A$ (where $\Gamma * A = \{(\gamma, a) \in \Gamma \times A : s(\gamma) = q(a)\}$) (write $\alpha(\gamma, a) = \alpha_\gamma(a)$) which satisfies the following conditions:

a. $q(\alpha_\gamma(a)) = r(\gamma)$ for all $\gamma \in \Gamma$ and $a \in A_{s(\gamma)}$,
b. $\alpha_\gamma : A_{s(\gamma)} \to A_{r(\gamma)}$ is a *-isomorphism for all $\gamma \in \Gamma$,
c. $\alpha_x(a) = a$ for all $x \in \Gamma^0$ and $a \in A_x$,
d. $\alpha_{\gamma_1\gamma_2}(a) = \alpha_{\gamma_1}(\alpha_{\gamma_2}(a))$ for all $(\gamma_1, \gamma_2) \in \Gamma_2$ and $a \in A_{s(\gamma_2)}$.

Form the semi-direct product (cf. [F2, §12]) $\Gamma \times_\alpha A = \Gamma * A$, with multiplication and involution given by the formulas

$$(\gamma_1, a_1)(\gamma_2, a_2) = (\gamma_1\gamma_2, \alpha_{\gamma_2^*}(a_1)a_2),$$

$$(\gamma, a)^* = (\gamma^*, \alpha_\gamma(a^*)).$$

Note that as a Banach bundle, $\Gamma \times_\alpha A$ is the pull-back of A by s. It is routine to check that with this norm and the above operations, the semi-direct product, $\Gamma \times_\alpha A$, is a Fell bundle over Γ.

vi. Let H be a Hilbert bundle over a locally compact space X with an inner product $\langle \cdot, \cdot \rangle$ which is conjugate linear in the first variable. Let Γ denote the transitive groupoid $X \times X$ (with obvious structure maps). There is a natural Fell bundle over Γ associated to H. Set $E_{(x,y)} = \mathcal{K}(H_y, H_x)$. The topology of E is prescribed by giving a linear space of norm continuous sections which is dense in every fiber (see [F2, 10.4]); for each pair of continuous sections, ξ, η, of H define a continuous section $\theta_{\xi,\eta}$ of E by the formula

$$\theta_{\xi,\eta}(x, y)\zeta = \xi(x)\langle \eta(y), \zeta \rangle$$

for all $(x, y) \in \Gamma, \zeta \in H_y$. The span of such sections determines a bundle topology for E. Multiplication is given by composition, and involution by the usual adjoint.

2.6. *Remark.* Let $j : \Omega \to \Gamma$ be a continuous groupoid morphism and $p : E \to \Gamma$ be a Fell bundle. The pull-back of E by j, $j^*(E) = \Omega * E = \{(\omega, e) \in \Omega \times E : j(\omega) = p(e)\}$, may be regarded as a Fell bundle over Ω in the obvious way.

3. CONSTRUCTION OF THE ASSOCIATED C*-ALGEBRA

3.1. Assume that Γ is an r-discrete groupoid (see [Re1]). Let $p : E \to \Gamma$ be a Fell bundle; we construct the analog of the reduced C*-algebra $C_r^*(E)$ as in [Ku, §2] (in [Yg] the full C*-algebra is constructed). Given $f, g \in C_c(E)$, define multiplication and involution by means of the formulas

$$fg(\gamma) = \sum_{\gamma=\alpha\beta} f(\alpha)g(\beta),$$

$$f^*(\gamma) = f(\gamma^*)^*.$$

With these operations $C_c(E)$ forms a *-algebra. Let $P : C_c(E) \to C_c(E^0)$ be the restriction map. Define a $C_c(E^0)$-valued inner product on $C_c(E)$ by $\langle f, g \rangle = P(f^*g)$.

3.2. Proposition. *With this inner product, $C_c(E)$ is a pre-Hilbert $C_0(E^0)$-module.*

Proof. We verify that $\langle f, f \rangle$ is positive as an element of the C*-algebra $C_0(E^0)$ for every $f \in C_c(E)$:

$$\langle f, f \rangle(x) = f^*f(x) = \sum_{x=\alpha\beta} f^*(\alpha)f(\beta) = \sum_{x=s(\gamma)} f(\gamma)^*f(\gamma) \geq 0$$

for all $x \in \Gamma^0$, and $f \in C_c(E)$; the same computation shows that if $\langle f, f\rangle = 0$ then $f = 0$. The remaining properties are left for the reader to verify. $\square$

For $f \in C_c(E)$, put $\|f\|_2 = \|\langle f, f\rangle\|^{1/2}$ and denote the completion of $C_c(E)$ in this norm by $L^2(E)$ (which is now a Hilbert $C_0(E^0)$-module).

3.3. We show that $L^2(E)$ is the Hilbert module associated to a bundle of Hilbert modules over Γ^0 (see 1.7). For each $x \in \Gamma^0$ consider the Hilbert E_x-module $V_x = \bigoplus_{x=s(\gamma)} E_\gamma$, with inner product

$$\left\langle \sum_{x=s(\gamma)} c_\gamma, \sum_{x=s(\gamma)} d_\gamma \right\rangle = \sum_{x=s(\gamma)} {c_\gamma}^* d_\gamma .$$

One may obtain a bundle topology on the union of the fibers by using elements of $C_c(E)$ to provide continuous sections in the obvious way (given $f \in C_c(E)$, one obtains the section $x \mapsto \sum_{x=s(\gamma)} f(\gamma) \in V_x$); let V denote the bundle obtained in this way. Then V is a Hilbert E^0-module bundle; one has $C_0(V) \cong L^2(E)$ as Hilbert $C_0(E^0)$-modules. Since $E^0 \subset V$, as Hilbert E^0-module bundles, V is full. If E is saturated, then for all $\beta \in \Gamma$ one has

$$V_{r(\beta)} \otimes_{E_{r(\beta)}} E_\beta \cong V_{s(\beta)}$$

via the identification

$$\left(\sum_{s(\gamma)=r(\beta)} c_\gamma \right) \otimes e \mapsto \sum_{s(\gamma)=r(\beta)} c_\gamma e$$

where $e \in E_\beta$, $c_\gamma \in E_\gamma$, and

$$\sum_{s(\gamma)=r(\beta)} c_\gamma \in \bigoplus_{s(\gamma)=r(\beta)} E_\gamma = V_{r(\beta)} .$$

Moreover, one has

$${V_x}^* = \bigoplus_{x=r(\gamma)} E_\gamma$$

(via the involution map) and

$$E_\beta \otimes_{E_{s(\beta)}} {V_{s(\beta)}}^* \cong {V_{r(\beta)}}^* .$$

3.4. Left multiplication by an element in $C_c(E)$ is a bounded operator with respect to the norm, $\|\cdot\|_2$, and hence extends to the completion. One checks that

$$\langle fg, h\rangle = \langle g, f^* h\rangle$$

for all $f, g, h \in C_c(E)$; hence, left multiplication by an element in $C_c(E)$ is adjointable and one obtains a *-monomorphism

$$C_c(E) \to \mathcal{L}(L^2(E)).$$

Let $C_r^*(E)$ denote the completion of $C_c(E)$ with respect to the operator norm; since $C_r^*(E)$ is a closed *-subalgebra of $\mathcal{L}(L^2(E))$, it is a C*-algebra. Moreover, for each $x \in \Gamma^0$ one has a representation

$$\pi_x : C_r^*(E) \to \mathcal{L}(V_x),$$

so that for each $a \in C_r^*(E)$, $\|a\| = \sup \|\pi_x(a)\|$. Note: Every bounded continous complex-valued function g on Γ^0 may be identified with an element of the multiplier algebra, $M(C_r^*(E))$, as follows: for $f \in C_c(E)$ put $gf(\gamma) = g(r(\gamma))f(\gamma)$; one checks that this defines an element of $\mathcal{L}(L^2(E))$ which centralizes $C_r^*(E)$.

3.5. Examples. i. If E is a Fell bundle over a trivial groupoid X (so $X = X^0$), then E is a C*-algebra bundle and $C_r^*(E) = C_0(E)$.

ii. Refer to example 2.5iii above; if C is a B-A equivalence bimodule and E is the associated Fell bundle over Δ, then $C_r^*(E)$ is the linking algebra associated to C (cf. [BGR, Th. 1.1]).

iii. Refer to example 2.5iv and assume that Γ is a principal r-discrete groupoid. Then $C_r^*(E)$ has a diagonal subalgebra (see [Ku, §2]) isomorphic to $C_0(G^0)$; the twist invariant for the diagonal pair $(C_r^*(E), C_0(G^0))$ is the inverse of $[\Sigma]$.

iv. Let Γ be a transitive equivalence relation on a countable set (with the discrete topology) and E be a saturated Fell bundle over Γ. Let V be the Hilbert E^0-module bundle over G^0 described above (3.3); for every $x \in \Gamma^0$, V_x is full, so V_x is a $\mathcal{K}(V_x)$-E_x equivalence bimodule (see 1.5). Now, since E is saturated, it follows that for any $\gamma \in \Gamma$ with $r(\gamma) = x$ and $s(\gamma) = y$, one has $V_x \otimes_{E_x} E_\gamma \cong V_y$ (see 3.3). This induces a *-isomorphism $\mathcal{K}(V_y) \cong \mathcal{K}(V_x)$ (see 1.6). Moreover, for each $x \in G^0$, π induces a *-isomorphism:

$$C_r^*(E) \cong \mathcal{K}(V_x).$$

3.6. Proposition. *The restriction map, $P : C_c(E) \to C_c(E^0)$, extends to a conditional expectation $P : C_r^*(E) \to C_0(E^0)$.*

Proof. This follows from Tomiyama's characterization of a conditional expection as a projection of norm one onto a subalgebra (see [T]). One checks that, for $f \in C_c(E)$,

$$\langle P(f), P(f)\rangle(x) = f(x)^* f(x) \le \sum_{x=s(\gamma)} f(\gamma)^* f(\gamma) = \langle f, f\rangle(x)$$

for each $x \in \Gamma^0$; hence, $\|P(f)\|^2 \le \|f\|^2$. Thus, P extends to a projection $q \in \mathcal{L}(L^2(E))$; note that q is a projection onto a Hilbert submodule isomorphic to $C_0(E^0)$. If $f \in C_c(E)$ is regarded as a left multiplication operator, then $\|P(f)\| = \|qfq\| \le \|f\|$. Hence, P extends uniquely to a linear map, also denoted P, from $C_r^*(E)$ to $C_0(E^0)$ which restricts to the identity on $C_0(E^0)$, and P is a projection of norm one. $\square$

3.7. Corollary. *For all $f \in C_c(E)$, $\|f\|_2 \le \|f\|$; thus, the inclusion, $C_c(E) \subset L^2(E)$, extends to a continuous map,*

$$\iota : C_r^*(E) \to L^2(E).$$

Proof. For all $f \in C_c(E)$, $(\|f\|_2)^2 = \|P(f^*f)\| \le \|f^*f\| = \|f\|^2$. $\square$

3.8. Definition. An open subset $U \subset \Gamma$ is said to be an *open Γ-set* if the restrictions, $r|_U$ and $s|_U$, are one-to-one. An element $f \in C_c(E)$ is said to be a *normalizer* if $\operatorname{supp} f$ is contained in an open Γ-set; let $\mathcal{N}(E)$ denote the collection of all normalizers. Note: Since $C_c(E) = \operatorname{span}\mathcal{N}(E)$, $\mathcal{N}(E)$ is total in $C_r^*(E)$ and $L^2(E)$.

3.9. Fact. If $g \in C_c(E^0)$ and $f \in \mathcal{N}(E)$, then $f^*gf \in C_c(E^0)$.

Proof. For each $x \in \Gamma^0$ there is at most one $\gamma \in \Gamma$ so that $s(\gamma) = x$ and $f(\gamma) \neq 0$. For $\beta \in \Gamma$ with $s(\beta) = x$, if $\beta = x$ and there is such a γ, one has

$$f^*gf(\beta) = \sum_{\beta=\gamma_1\gamma_2\gamma_3} f^*(\gamma_1)g(\gamma_2)f(\gamma_3) = f(\gamma)^*g(r(\gamma))f(\gamma);$$

if $\beta \neq x$ or there is no such γ, each term in the above sum is zero and one has $f^*gf(\beta) = 0$. Thus, $f^*gf \in C_c(E^0)$. ☐

3.10. Proposition. *$P : C_r^*(E) \to C_0(E^0)$ is faithful; thus, $\iota : C_r^*(E) \to L^2(E)$ is injective.*

Proof. We will show that $P(a^*a) \neq 0$ for every $a \in C_r^*(E)$, $a \neq 0$. For all $b \in C_r^*(E)$ and $f \in \mathcal{N}(E)$ we have

$$f^*P(b)f = P(f^*bf) = \langle bf, f \rangle.$$

It suffices to check this for $b \in C_c(E)$ where it follows by a calculation similar to that in the above fact. Since $\mathcal{N}(E)$ is total in $L^2(E)$, there is $f \in \mathcal{N}(E)$ so that $af \neq 0$; by the above with $b = a^*a$,

$$f^*P(a^*a)f = \langle a^*af, f \rangle = \langle af, af \rangle \neq 0.$$

Hence $P(a^*a) \neq 0$ and P is faithful. ☐

3.11. Fact. The norm on $C_c(E)$ is the unique C*-norm extending the supremum norm on $C_c(E^0)$ for which P extends to the completion as a faithful conditional expectation.

Proof. Let A denote the completion of $C_c(E)$ in such a norm. P extends to a conditional expectation, so left multiplication in $C_c(E)$ extends to a continuous *-homomorphism, $A \to \mathcal{L}(L^2(E))$; since P is faithful the map must be injective. ☐

3.12. Let Ω be an open subgroupoid of Γ; denote the inclusion map $j : \Omega \to \Gamma$ and put $D = j^*(E)$.

Proposition. *The inclusion $C_c(D) \subset C_c(E)$ is isometric and thus extends to an inclusion $C_r^*(D) \subset C_r^*(E)$.*

Proof. That the inclusion extends to a *-homomorphism is immediate. The norm on $C_c(D)$ may be characterized as the C*-norm which extends the supremum norm on $C_c(D^0)$ for which P is faithful on the completion. Since P commutes with the inclusion $C_c(D) \subset C_c(E)$ and P is faithful on the closure of $C_c(D)$ in $C_r^*(E)$, the inclusion is isometric. ☐

3.13. Observe that $\|f\|_\infty \leq \|f\|_2 \leq \|f\|$ for every $f \in C_c(E)$.

4. Morita equivalence

We continue to restrict attention to principal r-discrete groupoids. Let E be a saturated Fell bundle over a groupoid, Γ, and V be the associated Hilbert E^0-module bundle over Γ^0 (see 3.3), we show below that there is an action σ of Γ on the C*-bundle $\mathcal{K}(V)$ so that $C_r^*(\Gamma \times_\sigma \mathcal{K}(V))$ and $C_r^*(E)$ are strongly Morita equivalent.

4.1. Definition. A groupoid morphism, $\varphi : \Gamma \to \Delta$, is said to be full if for every $x \in \Gamma^0$ there is $\gamma \in \Gamma$ with $\varphi(\gamma) \notin \Delta^0$ such that $s(\gamma) = x$. For $i = 0, 1$ set $\Gamma_i = \varphi^{-1}(i)$ and note that Γ_i is an open subgroupoid of Γ; note further that Γ_0 and Γ_1 are equivalent (see [MRW]). Let $j_i : \Gamma_i \to \Gamma$ denote the embeddings; if E is a Fell bundle over Γ, put $E_i = j_i^*(E)$.

4.2. Theorem. *Let Γ be a groupoid, $\varphi : \Gamma \to \Delta$ be a full groupoid morphism, and E a saturated Fell bundle over Γ. Then, with notation as above, $C_r^*(E_0)$ and $C_r^*(E_1)$ are strongly Morita equivalent (cf.* [Re2, Cor. 5.4]*).*

Proof. By Prop. 3.12 we may regard $C_r^*(E_0)$ and $C_r^*(E_1)$ as subalgebras of $C_r^*(E)$. We show below that $C_r^*(E_0)$ and $C_r^*(E_1)$ are complementary full corners in $C_r^*(E)$; it will then follow (see [BGR, Th. 1.1]) that they are strongly Morita equivalent. It is clear that these subalgebras are complementary corners (as in 3.4 the characteristic functions on the unit spaces of Γ_0 and Γ_1 may be identified with complementary projections in $M(C_r^*(E))$), and so it remains to show that they are full. By symmetry we need only show that $C_r^*(E_0)$ is contained in the ideal generated by $C_r^*(E_1)$. It suffices to show that $C_c((E_0)^0)$ is contained in this ideal. Since φ is full, for every $x \in (\Gamma_0)^0$ there is $\gamma \in \Gamma$ with $\varphi(\gamma) = \partial$ and $s(\gamma) = x$. Choose an open Γ-set U containing γ so that $U \subset \varphi^{-1}(\partial)$; every $g \in C_c(E)$ with $\operatorname{supp} g \subset \varphi^{-1}(\partial)$ is in the ideal generated by $C_r^*(E_1)$, since $\operatorname{supp} gg^* \subset \varphi^{-1}(\partial\partial^*) = \Gamma_1$. Since E is saturated we may regard the restriction of E to U as a full Hilbert module bundle over the restriction of $(E_0)^0$ to $s(U)$. Hence, given $f \in C_c((E_0)^0)$ with $\operatorname{supp} f \in s(U)$ and $\epsilon > 0$, there are $g_k, h_k \in C_c(E)$ with $\operatorname{supp} g_k$, $\operatorname{supp} h_k \subset U$ for $k = 1, \dots, n$ such that

$$\|f - \sum_{1 \le k \le n} g_k^* h_k\| < \epsilon.$$

Hence f is in the ideal generated by $C_r^*(E_1)$. Since each element in $C_c((E_0)^0)$ can be written as the finite sum of such elements, $C_c((E_0)^0)$ is contained in the ideal as desired. □

4.3. Let E be a saturated Fell bundle over Γ and V be the associated Hilbert E_0-module bundle over Γ_0; we construct another Fell bundle F using V; for $\gamma \in \Gamma$ set

$$F_\gamma = V_{r(\gamma)} \otimes_{E_{r(\gamma)}} E_\gamma \otimes_{E_{s(\gamma)}} V^*_{s(\gamma)};$$

note that $F_x = V_x \otimes_{E_x} V_x^* = \mathcal{K}(V_x)$. Involution is defined in the obvious way:

$$u \otimes e \otimes v^* \mapsto v \otimes e^* \otimes u^*;$$

given $(\alpha, \beta) \in \Gamma^2$, if $t \otimes d \otimes u^* \in F_\alpha$ and $v \otimes e \otimes w^* \in F_\beta$, define multiplication by the formula

$$(t \otimes d \otimes u^*)(v \otimes e \otimes w^*) = t \otimes d\langle u, v\rangle e \otimes w^* \in F_{\alpha\beta}.$$

The verification of the Fell bundle properties is straightforward (associativity follows from the associativity of the tensor product of equivalence bimodules — see [Ri1, Prop. 6.21]). Since E is saturated and V is full, F is saturated.

4.4. The following proposition is analogous to [Yn, Th. 5.3], in which the existence of an action, which is then defined to be the dual action of a given coaction (see [Yn, Def. 5.4]), is established.

Proposition. *Let* F *be as above. There is an action,* $\sigma : \Gamma * \mathcal{K}(V) \to \mathcal{K}(V)$*, so that* $F \cong \Gamma \times_\sigma \mathcal{K}(V)$.

Proof. First we identify F with the pull-back bundle, $\Gamma * \mathcal{K}(V)$, by means of the following (see 3.3):

$$F_\gamma = (V_{r(\gamma)} \otimes_{E_{r(\gamma)}} E_\gamma) \otimes_{E_{s(\gamma)}} V^*_{s(\gamma)} \cong V_{s(\gamma)} \otimes_{E_{s(\gamma)}} V^*_{s(\gamma)} \cong \mathcal{K}(V_{s(\gamma)});$$

the action $\sigma_\gamma : \mathcal{K}(V_{s(\gamma)}) \to \mathcal{K}(V_{s(\gamma)})$ is likewise defined by the isomorphism (see also 3.5iv):

$$\mathcal{K}(V_{s(\gamma)}) \cong F_\gamma = V_{r(\gamma)} \otimes_{E_{r(\gamma)}} (E_\gamma \otimes_{E_{s(\gamma)}} V^*_{s(\gamma)}) \cong V_{r(\gamma)} \otimes_{E_{r(\gamma)}} V^*_{r(\gamma)} \cong \mathcal{K}(V_{r(\gamma)}).$$

On elementary tensors of the form $ac \otimes b^* \in \mathcal{K}(V_{s(\gamma)})$, where $a \in E_\alpha$, $b \in E_\beta$, $c \in E_\gamma$, and $s(\alpha) = r(\gamma)$, $s(\beta) = s(\gamma)$ (note that $ac \in E_{\alpha\gamma} \subset V_{s(\gamma)}$ and $b \in E_\beta \subset V_{s(\gamma)}$, and that elements of the form, $ac \otimes b^*$ span a dense subset of $\mathcal{K}(V_{s(\gamma)})$), σ_γ is given by

$$\sigma_\gamma(ac \otimes b^*) = a \otimes cb^*.$$

It is a routine matter to verify that σ defines an action of Γ on $\mathcal{K}(V)$ (for example, $\sigma_x = \mathrm{id}_{\mathcal{K}(V_x)}$ for $x \in \Gamma^0$ follows from the fact that $\mathcal{K}(V_x) = V_x \otimes_{E_x} V^*_x$ is a balanced tensor product) and that $F \cong \Gamma \times_\sigma \mathcal{K}(V)$. □

4.5. Corollary. *With notation as above,* $C^*_r(E)$ *and* $C^*_r(\Gamma \times_\sigma \mathcal{K}(V))$ *are strongly Morita equivalent.*

Proof. To apply Th. 4.2, we construct a Fell bundle D over $\Gamma \times \Delta$ which restricts to E on $\Gamma \times 0$ and to F on $\Gamma \times 1$ (by Prop. 4.4, $F \cong \Gamma \times_\sigma \mathcal{K}(V)$). For $\gamma \in \Gamma$, define $D_{(\gamma,0)} = E_\gamma$, $D_{(\gamma,1)} = F_\gamma$, $D_{(\gamma,\partial)} = V_{r(\gamma)} \otimes_{E_{r(\gamma)}} E_\gamma$, and $D_{(\gamma,\partial^*)} = E_\gamma \otimes_{E_{s(\gamma)}} V^*_{s(\gamma)}$. One defines multiplication and involution in the natural way. For example, if $(\alpha, \beta) \in \Gamma^2$ the map

$$D_{(\alpha,\partial^*)} \times D_{(\beta,\partial)} \to D_{(\alpha\beta,0)} = E_{\alpha\beta}$$

is given by the formula

$$(d \otimes u^*)(v \otimes e) = d\langle u, v\rangle e;$$

or the map

$$D_{(\alpha,\partial)} \times D_{(\beta,\partial^*)} \to D_{(\alpha\beta,1)} = F_{\alpha\beta} = V_{r(\alpha)} \otimes_{E_{r(\alpha)}} E_{\alpha\beta} \otimes_{E_{s(\beta)}} V^*_{s(\beta)},$$

is given by the formula

$$(v \otimes e)(d \otimes u^*) = v \otimes ed \otimes u^*.$$

Note that D is saturated. Thus, the theorem applies (the map, $\Gamma \times \Delta \to \Delta$, is given by projection onto the second factor). □

4.6. *Remark.* If the groupoid is a (discrete) group this result may be obtained by combining [Q2, Cor. 2.7] and [Kt, Th. 8]; I wish to thank Quigg for bringing this to my attention.

REFERENCES

[B] B. Blackadar, *K-theory for operator algebras,* MSRI Publication Series 5, Springer-Verlag (1986). MR **88g:**46082

[BGR] L. G. Brown, P. Green and M. A. Rieffel, *Stable isomorphism and strong Morita Equivalence of C*-algebras,* Pacific J. Math. **21** (1977) 349-363. MR **57:**3866

[GLR] P. Ghez, R. Lima and J. E. Roberts, *W*-categories,* Pacific J. Math. **120** (1985) 79-109. MR **87g:**46091

[F1] J. M. G. Fell, *An extension of Mackey's method to Banach *-algebraic bundles,* Mem. Amer. Math. Soc. **90** (1969). MR **41:**4255

[F2] J. M. G. Fell, *Induced representations and Banach *-algebraic bundles,* Springer Lecture Notes in Mathematics **582** (1977). MR **56:**15825

[FD] J. M. G. Fell and R. S. Doran, *Representations of *-algebras, locally compact groups, and Banach *-algebraic bundles,* Vol. 2, Academic Press (1988). MR **90c:**46002

[Ks] G. G. Kasparov, *Hilbert C*-modules: theorems of Stinespring and Voiculescu,* J. Operator Theory **4** (1980) 133-150. MR **82b:**46074

[Kt] Y. Katayama, *Takesaki's duality for a nondegenerate co-action,* Math. Scand. **55** (1984) 141-151. MR **86b:**46112

[Ku] A. Kumjian, *On C*-diagonals,* Can. J. Math. **38** (1986) 969-1008. MR **88a:**46060

[LPRS] M. B. Landstad, J. Phillips, I. Raeburn and C. E. Sutherland, *Representations of crossed products by coactions and principal bundles,* Trans. Amer. Math. Soc. **299** (1987) 747-784. MR **88f:**46127

[M] T. Masuda, *Groupoid dynamical systems and crossed product. II: the case of C*-systems,* Publ. Res. Inst. Math. Sci.**20** (1984) 959-970. MR **86g:**46100

[MRW] P. S. Muhly, J. N. Renault and D. P. Williams, *Equivalence and isomorphism for groupoid C*-algebras,* J. Operator Theory **17** (1987) 3-22. MR **88h:**46123

[PR] J. A. Packer and I. Raeburn, *Twisted crossed products of C*-algebras,* Math. Proc. Cambridge Philos. Soc. **106** (1989) 293-311. MR **90g:**46097

[Q1] J. C. Quigg, *Duality for reduced twisted crossed products of C*-algebras,* Indiana Univ. Math. J. **35** (1986) 549-571. MR **88d:**46122

[Q2] J. C. Quigg, *Discrete C*-coactions and C*-algebraic bundles,* J. Austral. Math. Soc. (Ser. A) **60** (1996), 204–221. MR **97c:**46086

[Re1] J. Renault, *A groupoid approach to C*-algebras,* Springer Lecture Notes in Mathematics **793** (1980). MR **82h:**46075

[Re2] J. Renault, *Représentations des produits croisés d'algèbres de gropoïdes,* J. Operator Theory **18** (1987) 67-97. MR **89g:**46108

[Ri1] M. A. Rieffel, *Induced Representations of C*-algebras,* Advances Math. **13** (1974) 176-257. MR **50:**5489

[Ri2] M. A. Rieffel, *Morita equivalence for operator algebras,* Operator algebras and applications (R. Kadison, ed.), Proc. Symp. Pure Math. Vol. 38, Part 1, Amer. Math. Soc., 1982) 285-298. MR **84k:**46045

[T] J. Tomiyama, *On the projection of norm one in W*-algebras,* Proc. Japan Acad. **33** (1957) 608-612. MR **20:**2635

[Yg] S. Yamagami, *On primitive ideal spaces of C*-algebras over certain locally compact groupoids,* Mappings of operator algebras (H. Araki and R. Kadison, eds.), Progress in Math. Vol. 84, Birkhäuser, Boston, (1991) 199-204. MR **92j:**46110

[Yn] T. Yamanouchi, *Duality for actions and coactions of measured groupoids on von Neumann Algebras,* Mem. Amer. Math. Soc. **484** (1993). MR **94b:**46095

Department of Mathematics, University of Nevada, Reno, Nevada 89557
E-mail address: `alex@unr.edu`