A CHARACTERIZATION FOR SPACES OF SECTIONS

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Abstract. The space of smooth sections of a bundle over a compact smooth manifold \( K \) can be equipped with a manifold structure, called an \( A \)-manifold, where \( A \) represents the Fréchet algebra of real valued smooth functions on \( K \). We prove that the \( A \)-manifold structure characterizes the spaces of sections of bundles over \( K \) and its open subspaces. We also describe the \( A^{(r)} \)-maps between \( A \)-manifolds.

1. Introduction

The aim of this paper is to recognize among the infinite dimensional spaces those which are the spaces of smooth sections of bundles over a fixed compact connected manifold \( K \). For this purpose, we use the concept of \( A \)-manifold structure, where \( A \) is the Fréchet algebra of all real valued smooth functions on \( K \). The idea of \( A \)-manifold structure in terms of local charts is explained in [3], and in terms of sheaves in [2]. Roughly, an \( A \)-manifold is a Hausdorff topological space which is locally modeled on finitely generated projective \( A \)-modules through \( A \)-maps, where an \( A \)-map is a map whose linear approximations are \( A \)-linear. One needs to be careful in proving results about \( A \)-manifolds and \( A \)-maps, because the partition of unity by \( A \)-maps does not exist on an \( A \)-manifold. It should be interesting to see, as an analogy to the finite dimensional case, whether every \( A \)-manifold can be embedded in \( A^\Lambda \) for some index set \( \Lambda \).

Let \( \mathcal{M} \) be an \( A \)-manifold and \( \Lambda = C^\infty_A(\mathcal{M}) \) be the set of all \( A \)-maps from \( \mathcal{M} \) to \( A \). Unlike finite dimensional manifolds, \( A \)-manifolds as given in [2] and [3] do not have “bump” \( A \)-maps, and thus it still remains to be seen whether \( \Lambda \) separates the points of \( \mathcal{M} \), i.e., for every pair of distinct points \( m_1, m_2 \in \mathcal{M} \), whether there exists an \( A \)-map \( F \in \Lambda \) such that \( F(m_1) \neq F(m_2) \). If \( \Lambda \) separates the points of \( \mathcal{M} \), then \( \mathcal{M} \) can considered as a subset of \( A^\Lambda \). In this case, we can define an \( A \)-manifold (Definition 2.4) similarly to the definition of \( n \)-manifold given in [5]. Our main result gives a concrete realization of these \( A \)-manifolds.

A bundle over \( K \) is a triple \( M \xrightarrow{p} K \), where \( M \) is a finite dimensional manifold and \( p \) is a surjective submersion. One can verify that the space of smooth sections \( \Gamma M = \{ s : K \to M | s \circ p = id \} \) and its open subsets are equipped with \( A \)-manifold structure. Conversely, we prove that every \( A \)-manifold, as defined in

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Definition 2.4, can be embedded as an open subset in $\Gamma M$ for some $M \xrightarrow{p} K$. Thus these $A$-manifolds characterize the spaces of sections of bundles over $K$ and its open subspaces.

In the last section, we briefly describe $A(r)$-maps between two $A$-manifolds as a generalization of differential operators of order $r$.

2. $A$-MANIFOLD STRUCTURE

Let us recall that there is a one-to-one correspondence between the category of smooth vector bundles over $K$ with bundle morphisms as maps and the category of finitely generated projective $A$-modules with module morphisms as maps [6]. More explicitly, for every smooth vector bundle $E \rightarrow K$, the corresponding finitely generated projective $A$-module is the space $\Gamma E$ of all smooth sections of $E \rightarrow K$. Hence every finitely generated projective $A$-module can be naturally equipped with a Fréchet space structure. An $A$-map between two finitely generated projective $A$-modules is a smooth map whose derivative at each point is $A$-linear.

It is shown in [3] that every fiber-preserving (not necessarily linear) map between two vector bundles induces an $A$-map between the corresponding spaces of sections, and, conversely, every $A$-map between two finitely generated projective $A$-modules is induced by a fiber-preserving map between the corresponding vector bundles over $K$.

We need the following discussion before giving the definition of $A$-manifold. Let $E \xrightarrow{p} K$ be a smooth vector bundle. Consider the evaluation map $ev : \Gamma E \times K \rightarrow E$ defined by $ev(s, x) = s(x)$, which is a surjective smooth map.

**Lemma 2.1.** The evaluation map $ev$ has a local right inverse at every point $e \in E$.

**Proof.** We prove this lemma by using the Nash-Moser-Hamilton inverse function theorem [1]. Notice that $ev : \Gamma E \times K \rightarrow E$ is a smooth tame map. Let $x = p(e)$ and $W$ be a neighborhood of $x$ which is diffeomorphic to an open subset of $R^k$ such that $p^{-1}(W)$ is trivial. Locally on $W$, every section $s$ can be considered as a bounded smooth map with values in $R^n$. Consider $ev : U \times W \rightarrow R^n$, where $U$ is an open subset of $C_C^\infty(W, R^n)$, the space of bounded smooth $R^n$-valued maps on $W$. Then, locally, the derivative

$$D(ev) : (U \times W) \times (C_C^\infty(W, R^n) \times R^k) \rightarrow R^n$$

is given by

$$D(ev)(s, x)(\alpha, v) = \alpha(x) + \frac{ds}{dx}(x)(v),$$

where $(s, x) \in U \times W$ and $(\alpha, v) \in C_C^\infty(W, R^n) \times R^k$. Since $R^n$ is finite dimensional, $D(ev)$ is a surjective tame map. By considering every element of $R^n$ as a constant map on $W$, one can assume that $R^n \subset C_C^\infty(W, R^n)$. Define

$$(Vev) : (U \times W) \times TE \rightarrow R^n \times R^k \subset C_C^\infty(W, R^n) \times R^k$$

by

$$(Vev)(s, x)(u) = (u_v, dp(u_s)),$$

where $u_v$ is the vertical component of $u$ and $u_s$ is the $\frac{du}{dx}$ component of $u \in T_s(x)E$.

Since $\text{Im}(Vev)$ is finite dimensional, $(Vev)$ is tame. Thus $(Vev)$ is a smooth tame family of right inverses for $D(ev)$. Then Theorem 1.1.3 on p.172 of [1] implies that $ev$ is locally surjective and has a local right inverse at every point of $E$. 

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Corollary 2.2. $ev$ is an open map.

Proposition 2.3. Let $\mathcal{U}$ be a convex open subset of $\Gamma E$ and $ev(\mathcal{U} \times K) = U$, which is open in $E$ by the above corollary. For any given $A$-map $F : \mathcal{U} \rightarrow A$, there exists a unique smooth map $f : U \rightarrow R$ such that $F(s) = f \circ s$ for every $s \in \mathcal{U}$.

Proof. For any $e \in U$ and $x = p(e)$, choose a section $s \in \mathcal{U}$ such that $s(x) = e$. Define $f(e) = F(s)(x)$. By a similar proof as in Lemma 2.4 of [3], one can verify that $f$ is well-defined. We need to verify that $f$ is smooth. Let $\mu : O \rightarrow \mathcal{U} \times K$ be a local smooth right inverse of $ev$ at a neighborhood $O$ of $e$, and let $\pi_1 : \mathcal{U} \times K \rightarrow \mathcal{U}$ and $\pi_2 : \mathcal{U} \times K \rightarrow K$ be the projection maps. Then

$$F(\pi_1\mu)(\pi_2\mu) = f(\pi_1\mu(\pi_2\mu)) = f(ev \circ \mu) = f(e).$$

Hence $f$ is smooth. \hfill $\square$

We now give the definition of $A$-manifolds.

Definition 2.4. For any index set $\Lambda$, let $A^{\Lambda}$ be equipped with the product topology. A subset $\mathcal{M} \subset A^{\Lambda}$ is an $A$-manifold if for each $s \in \mathcal{M}$ there exists a smooth map $H : \mathcal{U} \rightarrow A^{\Lambda}$, defined on an open convex subset $\mathcal{U}$ of $\Gamma E$, where $E \xrightarrow{p} K$ is a smooth vector bundle of rank $n$, such that

1. $H$ is an $A$-map, or in other words, the composition $\mathcal{U} \xrightarrow{H} A^{\Lambda} \xrightarrow{pr_{\lambda}} A$ is an $A$-map for each projection $pr_{\lambda}$ for all $\lambda \in \Lambda$.
2. $H$ maps $\mathcal{U}$ homeomorphically onto a neighborhood $\mathcal{V}$ of $s \in \mathcal{M}$.
3. For each $t \in \mathcal{U}$, $DH(t)$ is injective.
4. By the previous proposition, $H$ is induced by a unique map $h : U \rightarrow R^{\Lambda}$. For each $x \in K$, $h_x$ maps $U_x = p^{-1}(x) \cap \mathcal{U}$ homeomorphically onto a neighborhood $\mathcal{V}_x$ of $s(x) \in M_x$, where $M_x = ev(\mathcal{M} \times x)$.

As usual, one may call the pair $(\mathcal{U}, H)$ a chart for $\mathcal{M}$, and a collection of charts which cover $\mathcal{M}$ an atlas.

Remark. If $K = \{pt\}$, then $A = R$, and the above definition is simply the definition of $n$-dimensional manifolds as given in [5]. It may be interesting to see whether condition 4 in the above definition is independent of the first three conditions.

Example 2.5. If $M \rightarrow K$ is a bundle, then the space of all sections $\Gamma M$ is an $A$-manifold.

Proof. Let $\Lambda = C^\infty(M)$. $\Gamma M$ can be considered as a subset of $A^{\Lambda}$ by defining $i : \Gamma M \rightarrow A^{\Lambda}$ by $i(\gamma)_{\lambda} = \lambda \circ \gamma$ for each $\gamma \in \Gamma M$ and $\lambda \in \Lambda$. $\Gamma M$ is locally modeled near $\gamma \in \Gamma M$ by $\Gamma(\gamma^*T_e,M)$, the sections of the pull-back of the vertical tangent bundle. Indeed, one can find an explicit construction of a homeomorphism $\Phi : \mathcal{U} \rightarrow \Phi(\mathcal{U}) \subset \Gamma M$ in Proposition 3.5 of [3], where $\mathcal{U}$ is a convex open neighborhood of the zero section of $\gamma^*T_eM \rightarrow K$. For each such $(\mathcal{U}, \Phi)$, simply define $H : \mathcal{U} \rightarrow A^{\Lambda}$ by $H = i \circ \Phi$ and see that $H$ satisfies the conditions of the above definition. \hfill $\square$

Example 2.6. As an open subset of an $A$-manifold, every open subset of $\Gamma M$ is itself an $A$-manifold.
3. Embedding of $A$-manifolds

In this section we show that the space of sections of the bundles and its open subsets are the only $A$-manifolds as defined in 2.4.

Let $\mathcal{M}$ be an $A$-manifold as defined in 2.4. For every chart $(\mathcal{U}, H)$ of $\mathcal{M}$, Proposition 2.3 implies that $H$ induces a unique smooth map $h : U \rightarrow R^A$ and thus a unique map $\bar{h} : U \rightarrow R^A \times K$ defined by $\bar{h}(e) = (h(e), p(e))$, where $p : U \rightarrow K$ is the restriction of the bundle projection $E \xrightarrow{p} K$.

**Proposition 3.1.** The map $\bar{h} : U \rightarrow R^A \times K$ is such that $dh(e)$ is injective for every $e \in U$ and $\bar{h}$ maps $U$ homeomorphically onto $\bar{h}(U)$.

**Proof.** Let $e \in U$ and $v \in T_eU$, where $T_eU$ is the tangent space of $U$ at $e$. Suppose that $d\bar{h}(e)(v) = (dh(e)(v), dp(e)(v)) = 0$. $dp(e)(v) = 0$ implies that $v$ is a vertical tangent vector in $T_eU$. One can choose $s \in \mathcal{U}$ and $t \in \Gamma E$ such that $s(x) = e$ and $t(x) = v$. $dh(e)(v) = 0$ implies that $DH(s)(t)x = 0$. Let us say that $\{s_i\}$ is a local base near $x$, and $t = \sum_i \alpha_is_i$ in this local coordinate system. Then $0 = DH(s)(t)x = DH(s)(\sum_i \alpha_is_i)x = \sum_i \alpha_i(x)DH(s)(s_i)x$, since $DH(s)$ is $A$-linear. But the injectivity of $DH(s)$ implies that $\{DH(s)(s_i)x\}_{i=1}^n$ are linearly independent. Therefore $\alpha_i(x) = 0$ for all $i$, which shows that $v = t(x) = 0$. Thus $dh(e)$ is injective for each $e \in U$.

Next we verify that $\bar{h}$ is injective. Suppose that $e_1, e_2 \in U$ and $\bar{h}(e_1) = \bar{h}(e_2)$. $p(e_1) = p(e_2)$ implies that $e_1$ and $e_2$ are in the same fiber, say in $U_x$. Since $h(e_1) = h(e_2)$ and $h_x$ is injective on $U_x$, we have $e_1 = e_2$.

Since $dh(e)$ is injective for each $e \in U$, $h$ is one-to-one, and $h_x$ maps $U_x$ homeomorphically onto its image, it follows that $h$ maps $U$ homeomorphically onto $\bar{h}(U)$.

**Theorem 3.2.** Every $A$-manifold $\mathcal{M}$ can be embedded as an open subset in $\Gamma(M \rightarrow K)$ for some bundle $M \rightarrow K$.

**Proof.** Let $\mathcal{M} \subset A^A$ be an $A$-manifold. We will construct an $(n + k)$-dimensional manifold $M \subset R^A \times K$ such that $M$ is a bundle over $K$, and show that $\mathcal{M}$ is embedded as an open subset in $\Gamma M$.

Each chart $(\mathcal{U}, H)$ of $\mathcal{M}$ induces $h : U \rightarrow R^A \times K$ such that $d\bar{h}(e)$ is injective for each $e \in U$ and maps $U$ homeomorphically onto $\bar{h}(U) \subset R^A \times K$ by the previous proposition. Let $\{(\mathcal{U}_{\alpha}, H_{\alpha})\}_{\alpha}$ be an atlas for $\mathcal{M}$. Let $M = \bigcup_{\alpha} h_{\alpha}(U_{\alpha})$. One can see that $M$ is a manifold of dimension $n + k$ with atlas $\{(U_{\alpha}, h_{\alpha})\}_{\alpha}$. Each $h_{\alpha}$ is fiber-preserving, which implies that there exists a projection $p : M \rightarrow K$ which is a surjective submersion.

Since each $U_{\alpha}$ is an open subset of $\Gamma U_{\alpha}$ and $\bar{h}_{\alpha}(U_{\alpha})$ is open in $M$, there exists $i_{\alpha} : H_{\alpha}(U_{\alpha}) \hookrightarrow \Gamma M$, embedded as an open subset. Each $i_{\alpha}$ and $i_{\beta}$ agree on $H_{\alpha}(U_{\alpha}) \cap H_{\beta}(U_{\beta})$, which implies that there exists a unique map $i : \mathcal{M} \hookrightarrow \Gamma M$ such that $i(\mathcal{M}) = \bigcup_{\alpha=1}^n i_{\alpha}(H_{\alpha}(U_{\alpha}))$ is open in $\Gamma M$.

**Remark.** If $K = \{pt\}$, the above theorem simply states that every finite dimensional manifold $M$ can be considered as the space of sections of the bundle $M \rightarrow \{pt\}$.

Let $M_1 \rightarrow K$ and $M_2 \rightarrow K$ be any two bundles. It is shown in [3] that every fiber-preserving map $f : M_1 \rightarrow M_2$ induces the $A$-map $\Gamma f : \Gamma M_1 \rightarrow \Gamma M_2$, and, conversely, every $A$-map $\Phi : \Gamma M_1 \rightarrow \Gamma M_2$ is of the form $\Gamma f$ for some $f$. 

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Proposition 3.3. Let $\mathcal{M}$ be embedded as an open subset in $\Gamma \mathcal{M}$ as in the above theorem. Then every $A$-map $F : \mathcal{M} \to A$ can be uniquely extended to an $A$-map $\Gamma F : \Gamma \mathcal{M} \to A$.

Proof. Let $\{(U_\alpha,H_\alpha)\}_\alpha$ be an atlas for $\mathcal{M}$ and let $F : \mathcal{M} \to A$ be an $A$-map. For each $\alpha$, $F \circ H_\alpha : U_\alpha \to A$ is an $A$-map. This implies that there exists a unique $f_\alpha : U_\alpha \to R$ such that $\Gamma f_\alpha = F \circ H_\alpha$ by Proposition 2.3, and hence there exists a unique $\tilde{f}_\alpha : h_\alpha(U_\alpha) \to R$. By uniqueness, $\tilde{f}_\alpha$ and $\tilde{f}_\beta$ agree on $h_\alpha(U_\alpha) \cap h_\beta(U_\beta)$ and hence induce a unique map $f : \mathcal{M} \to R$, which yields $\Gamma f : \Gamma \mathcal{M} \to A$.

4. Differential Operators

In the language of category theory, we characterized $A$-manifolds in the last section which are the objects of our category. The natural choice for the maps of our category are $A$-maps. A smooth map $\Phi : \mathcal{M} \to \mathcal{N}$ between two $A$-manifolds $\mathcal{M}$ and $\mathcal{N}$ is an $A$-map if $D_s \Phi : T_s \mathcal{M} \to T_{\Phi(s)} \mathcal{N}$, the derivative of $\Phi$ at each $s \in \mathcal{M}$, is $A$-linear, where $T_s \mathcal{M}$ is the tangent space of $\mathcal{M}$ at $s$. Unfortunately, the collection of $A$-maps is too ‘small’. For example, if $K = S^1$ then the first order differential operator $\Phi : C^\infty(S^1) \to C^\infty(S^1)$ defined by $\Phi(f) = f'$ is not an $A$-map. We can ‘enlarge’ the class of maps in our category by including $A^{(r)}$-maps, which generalize $A$-maps.

Definition 4.1. A smooth map $\Phi : \mathcal{M} \to \mathcal{N}$ is called an $A^{(r)}$-map if

$$(D_s \Phi)(m^{r+1} T_s \mathcal{M}) \subset m T_{\Phi(s)} \mathcal{N}$$

for every maximal ideal $m$ of $A$ and for every $s \in \mathcal{M}$.

Let $E_1, E_2 \to K$ be any two smooth vector bundles and $j^r E_1 \to K$ be the $r$-jet bundle of $E_1 \to K$. A non-linear differential operator of order $r$ between $\Gamma E_1$ and $\Gamma E_2$ is a smooth map $\Phi : \Gamma E_1 \to \Gamma E_2$ defined by $\Phi(s) = \phi \circ j^r(s)$ for some fiber-preserving map $\phi : j^r E_1 \to E_2$. It is verified in [4] that a smooth map $\Phi : \Gamma E_1 \to \Gamma E_2$ is a non-linear differential operator of order $r$ if and only it is an $A^{(r)}$-map.

Of course, when $\mathcal{M} = \Gamma \mathcal{M}$ and $\mathcal{N} = \Gamma \mathcal{N}$, the above definition includes the standard non-linear differential operators.

Example 4.2. Let $M, N \to K$ be any two bundles and let $j^r M \to K$ be the $r$-jet bundle of $M \to K$. If $\phi : j^r M \to N$ is a fiber-preserving smooth map, then $\Phi : \Gamma M \to \Gamma N$, defined as $\Phi(s) = \phi \circ j^r(s)$, is an $A^{(r)}$-map.

Proof. Let $s \in \Gamma M$. Choose a chart $U \subset \Gamma E_1$ at $s$ and a chart $V \subset \Gamma E_2$ at $\Phi(s)$. We wish to show that $(D_s \Phi)(m^{r+1} \Gamma E_1) \subset m \Gamma E_2$.

Now $D_s \Phi = (D j^r \phi) \circ D_s j^r$. Since $\Phi$ is an $A$-map, it is enough to show that $(D_s j^r)(m^{r+1} \Gamma E_1) \subset m (j^r \Gamma E_1)$, which immediately follows from Lemma 2.4 of [4], because $(D_s j^r)(h) = j^r h$.

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