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A CHARACTERIZATION FOR SPACES OF SECTIONS

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Abstract. The space of smooth sections of a bundle over a compact smooth manifold K can be equipped with a manifold structure, called an A-manifold, where A represents the Fréchet algebra of real valued smooth functions on K. We prove that the A-manifold structure characterizes the spaces of sections of bundles over K and its open subspaces. We also describe the $A^{(r)}$-maps between A-manifolds.

1. Introduction

The aim of this paper is to recognize among the infinite dimensional spaces those which are the spaces of smooth sections of bundles over a fixed compact connected manifold K. For this purpose, we use the concept of A-manifold structure, where A is the Fréchet algebra of all real valued smooth functions on K. The idea of A-manifold structure in terms of local charts is explained in [3], and in terms of sheaves in [2]. Roughly, an A-manifold is a Hausdorff topological space which is locally modeled on finitely generated projective A-modules through A-maps, where an A-map is a map whose linear approximations are A-linear. One needs to be careful in proving results about A-manifolds and A-maps, because the partition of unity by A-maps does not exist on an A-manifold. It should be interesting to see, as an analogy to the finite dimensional case, whether every A-manifold can be embedded in A^Λ for some index set Λ.

Let $\mathcal{M}$ be an A-manifold and $\Lambda = C^\infty_A(\mathcal{M})$ be the set of all A-maps from $\mathcal{M}$ to A. Unlike finite dimensional manifolds, A-manifolds as given in [2] and [3] do not have "bump" A-maps, and thus it still remains to be seen whether Λ separates the points of $\mathcal{M}$, i.e., for every pair of distinct points $m_1, m_2 \in \mathcal{M}$, whether there exists an A-map $F \in \Lambda$ such that $F(m_1) \neq F(m_2)$. If Λ separates the points of $\mathcal{M}$, then $\mathcal{M}$ can considered as a subset of A^Λ. In this case, we can define an A-manifold (Definition 2.4) similarly to the definition of n-manifold given in [5]. Our main result gives a concrete realization of these A-manifolds.

A bundle over K is a triple $M \xrightarrow{p} K$, where M is a finite dimensional manifold and p is a surjective submersion. One can verify that the space of smooth sections $\Gamma M = \{s : K \to M | s \circ p = id\}$ and its open subsets are equipped with A-manifold structure. Conversely, we prove that every A-manifold, as defined in

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Definition 2.4, can be embedded as an open subset in ΓM for some $M \xrightarrow{p} K$. Thus these A-manifolds characterize the spaces of sections of bundles over K and its open subspaces.

In the last section, we briefly describe $A^{(r)}$-maps between two A-manifolds as a generalization of differential operators of order r.

2. A-MANIFOLD STRUCTURE

Let us recall that there is a one-to-one correspondence between the category of smooth vector bundles over K with bundle morphisms as maps and the category of finitely generated projective A-modules with module morphisms as maps [6]. More explicitly, for every smooth vector bundle $E \to K$, the corresponding finitely generated projective A-module is the space ΓE of all smooth sections of $E \to K$. Hence every finitely generated projective A-module can be naturally equipped with a Fréchet space structure. An A-map between two finitely generated projective A-modules is a smooth map whose derivative at each point is A-linear.

It is shown in [3] that every fiber-preserving (not necessarily linear) map between two vector bundles induces an A-map between the corresponding spaces of sections, and, conversely, every A-map between two finitely generated projective A-modules is induced by a fiber-preserving map between the corresponding vector bundles over K.

We need the following discussion before giving the definition of A-manifold. Let $E \xrightarrow{p} K$ be a smooth vector bundle. Consider the evaluation map $ev : \Gamma E \times K \longrightarrow E$ defined by $ev(s,x) = s(x)$, which is a surjective smooth map.

Lemma 2.1. *The evaluation map ev has a local right inverse at every point $e \in E$.*

Proof. We prove this lemma by using the Nash-Moser-Hamilton inverse function theorem [1]. Notice that $ev : \Gamma E \times K \longrightarrow E$ is a smooth tame map. Let $x = p(e)$ and W be a neighborhood of x which is diffeomorphic to an open subset of R^k such that $p^{-1}(W)$ is trivial. Locally on W, every section s can be considered as a bounded smooth map with values in R^n. Consider $ev : \mathcal{U} \times W \longrightarrow R^n$, where $\mathcal{U}$ is an open subset of $C^\infty_B(W, R^n)$, the space of bounded smooth R^n-valued maps on W. Then, locally, the derivative

$$D(ev) : (\mathcal{U} \times W) \times (C^\infty_B(W, R^n) \times R^k) \longrightarrow R^n$$

is given by

$$D(ev)(s,x)(\alpha, v) = \alpha(x) + \frac{ds}{dx}(x)(v),$$

where $(s,x) \in \mathcal{U} \times W$ and $(\alpha, v) \in C^\infty_B(W, R^n) \times R^k$. Since R^n is finite dimensional, $D(ev)$ is a surjective tame map. By considering every element of R^n as a constant map on W, one can assume that $R^n \subset C^\infty_B(W, R^n)$. Define

$$(Vev) : (\mathcal{U} \times W) \times TE \longrightarrow R^n \times R^k \subset C^\infty_B(W, R^n) \times R^k$$

by

$$(Vev)(s,x)(u) = (u_v, dp(u_s)),$$

where u_v is the vertical component of u and u_s is the $\frac{ds}{dx}$ component of $u \in T_{s(x)}E$.

Since $\mathrm{Im}(Vev)$ is finite dimensional, (Vev) is tame. Thus (Vev) is a smooth tame family of right inverses for $D(ev)$. Then Theorem 1.1.3 on p.172 of [1] implies that ev is locally surjective and has a local right inverse at every point of E. $\square$

Corollary 2.2. *ev is an open map.*

Proposition 2.3. *Let $\mathcal{U}$ be a convex open subset of ΓE and $ev(\mathcal{U} \times K) = U$, which is open in E by the above corollary. For any given A-map $F : \mathcal{U} \to A$, there exists a unique smooth map $f : U \to R$ such that $F(s) = f \circ s$ for every $s \in \mathcal{U}$.*

Proof. For any $e \in U$ and $x = p(e)$, choose a section $s \in \mathcal{U}$ such that $s(x) = e$. Define $f(e) = F(s)(x)$. By a similar proof as in Lemma 2.4 of [3], one can verify that f is well-defined. We need to verify that f is smooth. Let $\mu : O \to \mathcal{U} \times K$ be a local smooth right inverse of ev at a neighborhood O of e, and let $\pi_1 : \mathcal{U} \times K \to \mathcal{U}$ and $\pi_2 : \mathcal{U} \times K \to K$ be the projection maps. Then

$$F(\pi_1 \mu e)(\pi_2 \mu e) = f(\pi_1 \mu e(\pi_2 \mu e)) = f(ev \circ \mu(e)) = f(e).$$

Hence f is smooth. ☐

We now we give the definition of A-manifolds.

Definition 2.4. For any index set Λ, let A^Λ be equipped with the product topology. A subset $\mathcal{M} \subset A^\Lambda$ is an A-manifold if for each $s \in \mathcal{M}$ there exists a smooth map $H : \mathcal{U} \to A^\Lambda$, defined on an open convex subset $\mathcal{U}$ of ΓE, where $E \xrightarrow{p} K$ is a smooth vector bundle of rank n, such that

1. H is an A-map, or in other words, the composition $\mathcal{U} \overset{H}{\to} A^\Lambda \overset{pr_\lambda}{\to} A$ is an A-map for each projection pr_λ for all $\lambda \in \Lambda$.
2. H maps $\mathcal{U}$ homeomorphically onto a neighborhood $\mathcal{V}$ of $s \in \mathcal{M}$.
3. For each $t \in \mathcal{U}$, $DH(t)$ is injective.
4. By the previous proposition, H is induced by a unique map $h : U \to R^\Lambda$. For each $x \in K$, h_x maps $U_x = p^{-1}(x) \cap U$ homeomorphically onto a neighborhood V_x of $s(x) \in M_x$, where $M_x = ev(\mathcal{M} \times x)$.

As usual, one may call the pair $(\mathcal{U}, H)$ a chart for $\mathcal{M}$, and a collection of charts which cover $\mathcal{M}$ an atlas.

Remark. If $K = \{pt\}$, then $A = R$, and the above definition is simply the definition of n-dimensional manifolds as given in [5]. It may be interesting to see whether condition 4 in the above definition is independent of the first three conditions.

Example 2.5. If $M \to K$ is a bundle, then the space of all sections ΓM is an A-manifold.

Proof. Let $\Lambda = C^\infty(M)$. ΓM can be considered as a subset of A^Λ by defining $i : \Gamma M \longrightarrow A^\Lambda$ by $i(\gamma)_\lambda = \lambda \circ \gamma$ for each $\gamma \in \Gamma M$ and $\lambda \in \Lambda$. ΓM is locally modeled near $\gamma \in \Gamma M$ by $\Gamma(\gamma^* T_v M)$, the sections of the pull-back of the vertical tangent bundle. Indeed, one can find an explicit construction of a homeomorphism $\Phi : \mathcal{U} \to \Phi(\mathcal{U}) \subset \Gamma M$ in Proposition 3.5 of [3], where $\mathcal{U}$ is a convex open neighborhood of the zero section of $\gamma^* T_v M \to K$. For each such $(\mathcal{U}, \Phi)$, simply define $H : \mathcal{U} \to A^\Lambda$ by $H = i \circ \Phi$ and see that H satisfies the conditions of the above definition. ☐

Example 2.6. As an open subset of an A-manifold, every open subset of ΓM is itself an A-manifold.

3. Embedding of A-manifolds

In this section we show that the space of sections of the bundles and its open subsets are the only A-manifolds as defined in 2.4.

Let $\mathcal{M}$ be an A-manifold as defined in 2.4. For every chart $(\mathcal{U}, H)$ of $\mathcal{M}$, Proposition 2.3 implies that H induces a unique smooth map $h : U \to R^{\Lambda}$ and thus a unique map $\bar{h} : U \to R^{\Lambda} \times K$ defined by $\bar{h}(e) = (h(e), p(e))$, where $p : U \to K$ is the restriction of the bundle projection $E \xrightarrow{p} K$.

Proposition 3.1. *The map* $\bar{h} : U \to R^{\Lambda} \times K$ *is such that* $d\bar{h}(e)$ *is injective for every* $e \in U$ *and* $\bar{h}$ *maps* U *homeomorphically onto* $\bar{h}(U)$.

Proof. Let $e \in U$ and $v \in T_eU$, where T_eU is the tangent space of U at e. Suppose that $d\bar{h}(e)(v) = (dh(e)(v), dp(e)(v)) = 0$. $dp(e)(v) = 0$ implies that v is a vertical tangent vector in T_eU. One can choose $s \in \mathcal{U}$ and $t \in \Gamma E$ such that $s(x) = e$ and $t(x) = v$. $dh(e)(v) = 0$ implies that $DH(s)(t)_x = 0$. Let us say that $\{s_i\}$ is a local base near x, and $t = \sum_i \alpha_i s_i$ in this local coordinate system. Then $0 = DH(s)(t)_x = DH(s)(\sum_i \alpha_i s_i)_x = \sum_i \alpha_i(x) DH(s)(s_i)_x$, since $DH(s)$ is A-linear. But the injectivity of $DH(s)$ implies that $\{DH(s)(s_i)_x\}_{i=1}^{n}$ are linearly independent. Therefore $\alpha_i(x) = 0$ for all i, which shows that $v = t(x) = 0$. Thus $d\bar{h}(e)$ is injective for each $e \in U$.

Next we verify that $\bar{h}$ is injective. Suppose that $e_1, e_2 \in U$ and $\bar{h}(e_1) = \bar{h}(e_2)$. $p(e_1) = p(e_2)$ implies that e_1 and e_2 are in the same fiber, say in U_x. Since $h(e_1) = h(e_2)$ and h_x is injective on U_x, we have $e_1 = e_2$.

Since $d\bar{h}(e)$ is injective for each $e \in U$, $\bar{h}$ is one-to-one, and h_x maps U_x homeomorphically onto its image, it follows that $\bar{h}$ maps U homeomorphically onto $\bar{h}(U)$. □

Theorem 3.2. *Every* A*-manifold* $\mathcal{M}$ *can be embedded as an open subset in* $\Gamma(M \to K)$ *for some bundle* $M \to K$.

Proof. Let $\mathcal{M} \subset A^{\Lambda}$ be an A-manifold. We will construct an $(n+k)$-dimensional manifold $M \subset R^{\Lambda} \times K$ such that M is a bundle over K, and show that $\mathcal{M}$ is embedded as an open subset in ΓM.

Each chart $(\mathcal{U}, H)$ of $\mathcal{M}$ induces $\bar{h} : U \to R^{\Lambda} \times K$ such that $d\bar{h}(e)$ is injective for each $e \in U$ and maps U homeomorphically onto $\bar{h}(U) \subset R^{\Lambda} \times K$ by the previous proposition. Let $\{(\mathcal{U}_\alpha, H_\alpha)\}_\alpha$ be an atlas for $\mathcal{M}$. Let $M = \bigcup_\alpha \bar{h}_\alpha(U_\alpha)$. One can see that M is a manifold of dimension $n+k$ with atlas $\{(U_\alpha, \bar{h}_\alpha\}_\alpha$. Each $\bar{h}_\alpha$ is fiber-preserving, which implies that there exists a projection $p : M \to K$ which is a surjective submersion.

Since each $\mathcal{U}_\alpha$ is an open subset of ΓU_α and $\bar{h}_\alpha(U_\alpha)$ is open in M, there exists $i_\alpha : H_\alpha(\mathcal{U}_\alpha) \hookrightarrow \Gamma M$, embedded as an open subset. Each i_α and i_β agree on $H_\alpha(\mathcal{U}_\alpha) \bigcap H_\beta(\mathcal{U}_\beta)$, which implies that there exists a unique map $i : \mathcal{M} \hookrightarrow \Gamma M$ such that $i(\mathcal{M}) = \bigcup_\alpha i_\alpha(H_\alpha(\mathcal{U}_\alpha))$ is open in ΓM. □

Remark. If $K = \{pt\}$, the above theorem simply states that every finite dimensional manifold M can be considered as the space of sections of the bundle $M \to \{pt\}$.

Let $M_1 \to K$ and $M_2 \to K$ be any two bundles. It is shown in [3] that every fiber-preserving map $f : M_1 \to M_2$ induces the A-map $\Gamma f : \Gamma M_1 \to \Gamma M_2$, and, conversely, every A-map $\Phi : \Gamma M_1 \to \Gamma M_2$ is of the form Γf for some f.

Proposition 3.3. *Let $\mathcal{M}$ be embedded as an open subset in ΓM as in the above theorem. Then every A-map $F : \mathcal{M} \to A$ can be uniquely extended to an A-map $\Gamma f : \Gamma M \to A$.*

Proof. Let $\{(\mathcal{U}_\alpha, H_\alpha)\}_\alpha$ be an atlas for $\mathcal{M}$ and let $F \colon \mathcal{M} \to A$ be an A-map. For each α, $F \circ H_\alpha : \mathcal{U}_\alpha \to A$ is an A-map. This implies that there exists a unique $f_\alpha : U_\alpha \to R$ such that $\Gamma f_\alpha = F \circ H_\alpha$ by Proposition 2.3, and hence there exists a unique $\tilde{f}_\alpha : h_\alpha(U_\alpha) \to R$. By uniqueness, $\tilde{f}_\alpha$ and $\tilde{f}_\beta$ agree on $h_\alpha(U_\alpha) \cap h_\beta(U_\beta)$ and hence induce a unique map $f : M \to R$, which yields $\Gamma f : \Gamma M \to A$. □

4. Differential operators

In the language of category theory, we characterized A-manifolds in the last section which are the objects of our category. The natural choice for the maps of our category are A-maps. A smooth map $\Phi : \mathcal{M} \to \mathcal{N}$ between two A-manifolds $\mathcal{M}$ and $\mathcal{N}$ is an A-map if $D_s\Phi : T_s\mathcal{M} \to T_{\Phi(s)}\mathcal{N}$, the derivative of Φ at each $s \in \mathcal{M}$, is A-linear, where $T_s\mathcal{M}$ is the tangent space of $\mathcal{M}$ at s. Unfortunately, the collection of A-maps is too 'small'. For example, if $K = S^1$ then the first order differential operator $\Phi : C^\infty(S^1) \to C^\infty(S^1)$ defined by $\Phi(f) = f'$ is not an A-map. We can 'enlarge' the class of maps in our category by including $A^{(r)}$-maps, which generalize A-maps.

Definition 4.1. A smooth map $\Phi : \mathcal{M} \to \mathcal{N}$ is called an $A^{(r)}$-map if

$$(D_s\Phi)(\mathbf{m}^{r+1}T_s\mathcal{M}) \subset \mathbf{m}T_{\Phi(s)}\mathcal{N}$$

for every maximal ideal $\mathbf{m}$ of A and for every $s \in \mathcal{M}$.

Let $E_1, E_2 \to K$ be any two smooth vector bundles and $j^rE_1 \to K$ be the r-jet bundle of $E_1 \to K$. A non-linear differential operator of order r between ΓE_1 and ΓE_2 is a smooth map $\Phi : \Gamma E_1 \to \Gamma E_2$ defined by $\Phi(s) = \phi \circ j^r(s)$ for some fiber-preserving map $\phi : j^rE_1 \to E_2$. It is verified in [4] that a smooth map $\Phi : \Gamma E_1 \to \Gamma E_2$ is a non-linear differential operator of order r if and only it is an $A^{(r)}$-map.

Of course, when $\mathcal{M} = \Gamma M$ and $\mathcal{N} = \Gamma N$, the above definition includes the standard non-linear differential operators.

Example 4.2. Let $M, N \to K$ be any two bundles and let $j^rM \to K$ be the r-jet bundle of $M \to K$. If $\phi : j^rM \to N$ is a fiber-preserving smooth map, then $\Phi : \Gamma M \to \Gamma N$, defined as $\Phi(s) = \phi \circ j^r(s)$, is an $A^{(r)}$-map.

Proof. Let $s \in \Gamma M$. Choose a chart $\Gamma U \subset \Gamma E_1$ at s and a chart $\Gamma V \subset \Gamma E_2$ at $\Phi(s)$. We wish to show that $(D_s\Phi)(\mathbf{m}^{r+1}\Gamma E_1) \subset \mathbf{m}\Gamma E_2$.

Now $D_s\Phi = (D_{j^rs}\Gamma\phi) \circ D_sj^r$. Since $\Gamma\phi$ is an A-map, it is enough to show that $(D_sj^r)(\mathbf{m}^{r+1}\Gamma E_1) \subset \mathbf{m}(j^r\Gamma E_1)$, which immediately follows from Lemma 2.4 of [4], because $(D_sj^r)(h) = j^rh$. □

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