THE STABILITY RADIUS OF A QUASI-FREDHOLM OPERATOR

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Abstract. We extend the technique used by Kordula and M"uller to show that the stability radius of a quasi-Fredholm operator $T$ is the limit of $\gamma(T^n)^{1/n}$ as $n \to \infty$. If 0 is an isolated point of the Apostol spectrum $\sigma_\gamma(T)$, then the above limit is non-zero if and only if $T$ is quasi-Fredholm.

Let $L(X)$ be the set of all bounded linear operators on a complex Banach space $X$. For any $T \in L(X)$, we denote the null space and range of $T$ by $N(T)$ and $R(T)$ respectively. The Apostol spectrum of $T$ is defined to be the set

$$\sigma_\gamma(T) = \{ \mu \in \mathbb{C} : \lim_{\lambda \to \mu} \gamma(T - \lambda I) = 0 \},$$

where $\gamma(T)$ is the reduced minimum modulus of $T$, that is,

$$\gamma(T) = \begin{cases} \inf \{ \|Tx\| : x \in X, d(x, N(T)) = 1 \} & \text{if } T \neq 0, \\ \infty & \text{if } T = 0. \end{cases}$$

The Apostol spectrum was first defined in this form by Apostol in [1] for operators on a Hilbert space. Its complement in $\mathbb{C}$ is usually called the semi-regular region of $T$ and is denoted by $\rho_\gamma(T)$. $T$ is semi-regular (or s-regular) if $0 \in \rho_\gamma(T)$. Properties of the Apostol spectrum for operators on a Banach space can be found in [10, 11].

The stability radius of $T$ is defined as the distance

$$\delta(T) = d(0, \sigma_\gamma(T) \setminus \{0\}).$$

It is the radius of the largest punctured open disc centred at 0 in which $T - \lambda I$ is semi-regular. When $T$ is semi-regular, it is shown in [5] that

$$\delta(T) = \frac{\gamma(T^n)^{1/n}}{n \to \infty}.$$

In the special case when $T$ is bounded below or surjective, $\delta(T)$ is also the distance from 0 to the approximate point spectrum and to the surjectivity spectrum respectively. When 0 is an isolated point of $\sigma_\gamma(T)$, the formula (3) still applies to certain classes of operators. It includes the cases when $T$ is Fredholm [3], semi-Fredholm [14], essentially s-regular [5], or chain finite ($T$ has finite ascent and descent) [2].

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An operator $T$ is regular (or of Saphar type) if it is both relatively regular and semi-regular. The stability problem concerning regular operators is studied in [13].

**Definition.** An operator $T \in L(X)$ has topological uniform descent $d$ (where $d$ is a nonnegative integer) if $N(T^{n}) + R(T) = N(T^{d}) + R(T)$ is a closed subspace for all $n \geq d$ (Grabiner [4]). An operator $T$ is quasi-Fredholm if $T$ has topological uniform descent $d$ for some integer $d$ and $R(T^{n})$ is closed for all $n \geq d$.

Quasi-Fredholm operators were first defined on a Hilbert space by Labrousse [6] and on a Banach space by Mbekhta and Müller [9]. The definition used here is different from but equivalent to the one given in [9]. The class of quasi-Fredholm operators is well researched for the Hilbert space case [6, 7]. In [6], it was shown that an operator on a Hilbert space is quasi-Fredholm if and only if it has a Kato decomposition. A characterization of quasi-Fredholm operators in a Banach space is examined in [12]. Two pertinent properties of quasi-Fredholm operators are proved in Theorem 7 and Corollary 15 below.

In the present paper, the stability radius problem of a quasi-Fredholm operator is examined. It was shown in [1, Prop. 3.3] that the stability radius formula (3) holds for a Hilbert space operator $T$ if only if $T$ has a Kato decomposition described in [6], which is of course equivalent to $T$ being quasi-Fredholm. The main aim of this paper is to extend the result to Banach space operators. It turns out that the technique used by Kordula and Müller in [5] can be extended to solve the stability radius problem for quasi-Fredholm operators. The main result of this paper is the following theorem, which is a consequence of Theorem 10 and Corollary 15.

**Theorem.** If $T$ is a quasi-Fredholm operator, then the stability radius of $T$ is equal to $\lim_{n \to \infty} \gamma(T^{n})^{1/n}$. Moreover, if 0 is an isolated point of the Apostol spectrum of an arbitrary operator $T$, then $\lim_{n \to \infty} \gamma(T^{n})^{1/n}$ always exists. This limit is non-zero if and only if $T$ is quasi-Fredholm.

If $M, N$ are closed $T$-invariant subspaces with $N \subseteq M$, then we denote the map induced by $T$ on the quotient $M/N$ by $T_{M/N}$. More precisely, $T_{M/N}$ is the map $x + N \mapsto Tx + N$. We also denote the restriction of $T$ to $M$ by $T_{M}$. The ascent and descent of $T$ will be denoted by $\text{asc}(T)$ and $\text{des}(T)$ respectively. The hyperkernel $\bigcup_{n=1}^{\infty} N(T^{n})$ and hyperrange $\bigcap_{n=1}^{\infty} R(T^{n})$ of $T$ are denoted by $N(T^{\infty})$ and $R(T^{\infty})$ respectively. We first prove some properties of the reduced minimum modulus.

**Lemma 1.** Let $M$ be a closed $T$-invariant subspace.

(i) If $T_{X/M}$ is injective, then $\gamma(T_{M}) \geq \gamma(T)$.

(ii) If $T_{M}$ has dense range, then $\gamma(T_{X/M}) \geq \gamma(T)$.

**Proof.** (i) The hypothesis shows that $T^{-1}M \subseteq M$. In particular, $N(T) \subseteq M$. Hence $N(T_{M}) = N(T) \cap M = N(T)$.

It is clear from the definition of the minimum modulus that $\gamma(T_{M}) \geq \gamma(T)$.

(ii) If $\gamma(T_{X/M}) = \infty$, the result is trivial. If $\gamma > \gamma(T_{X/M})$, we can find $x \in X$ such that $d(x + M, N(T_{X/M})) = 1$ and $\|T_{X/M}(x + M)\| < \gamma$. So

$$d(x, T^{-1}M) = 1 \quad \text{and} \quad d(Tx, M) < \gamma.$$ 

Since $TM$ is dense in $M$, we can find $m \in M$ such that $\|T(x + m)\| < \gamma$. Note that $N(T) \subseteq T^{-1}M$. As $TM \subseteq M$, we also have $M \subseteq T^{-1}M$. Hence

$$\gamma(T) \leq \frac{\|T(x + m)\|}{d(x + m, N(T))} \leq \frac{\|T(x + m)\|}{d(x + m, T^{-1}M)} = \frac{\|T(x + m)\|}{d(x, T^{-1}M)} < \gamma.$$
Considering all \( \gamma > \gamma(T_{X/M}) \), we conclude that \( \gamma(T_{X/M}) \geq \gamma(T) \).

\[ \blacktriangleleft \]

**Lemma 2.** Suppose \( \text{asc}(T) = d \), i.e. \( N(T^d) = N(T^n) \) for all \( n \geq d \). Let \( \hat{T} \) be the map induced by \( T \) on the quotient \( X/N(T^d) \); then

\[
\begin{align*}
\liminf_{n \to \infty} \gamma(T^n)^{1/n} &= \liminf_{n \to \infty} (\hat{T}^n)^{1/n}, \\
\limsup_{n \to \infty} \gamma(T^n)^{1/n} &= \limsup_{n \to \infty} (\hat{T}^n)^{1/n}.
\end{align*}
\]

Moreover, \( \limsup_{n \to \infty} \gamma(T^n)^{1/n} > 0 \iff \hat{T} \) is bounded below \( \iff \lim_{n \to \infty} \gamma(T^n)^{1/n} > 0 \).

**Proof.** Let us first assume that \( T^d \neq 0 \). Since \( \text{asc}(T) = d \), we have \( T^{-1}N(T^d) = N(T^d) \) and \( \hat{T} \) is injective. Taking any \( x \in X \), we have

\[
\|T^n x\| \geq d(T^n x, N(T^d)) \geq \|T^n\|^{-1}\|T^{n+d} x\|.
\]

The first inequality is obvious. The second follows from the relation

\[
\|T^{n+d} x\| = \|T^d(T^n x + z)\| \leq \|T^d\|\|T^n x + z\|,
\]

true for all \( z \in N(T^d) \). Write \( x + N(T^d) \) as \( \hat{x} \); then for \( n \geq d \), we have

\[
\|\hat{T}^n \hat{x}\| = d(T^n x, N(T^d)), \quad \|\hat{x}\| = d(x, N(T^d)) = d(x, N(T^n)) = d(x, N(T^{n+d})).
\]

We deduce from (5) that \( \gamma(T^n) \geq \gamma(\hat{T}^n) \geq \|T^n\|^{-1}\gamma(T^{n+d}) \) by taking infima over all \( x \) with \( \|\hat{x}\| = 1 \); (4) then follows by taking limits. When \( T^d = 0, \hat{T} = 0 \) and all limits in (4) are infinite. To establish the last statement, we proceed as follows:

(a) If \( \limsup_{n \to \infty} \gamma(T^n)^{1/n} > 0 \), then it follows from (4) that \( \gamma(\hat{T}^k) > 0 \) for some \( k \). Therefore \( R(\hat{T}^k) \) is a closed subspace. Since \( \hat{T} \) is injective, so is \( \hat{T}^k \). Thus \( \hat{T}^k \) is injective and has closed range. So \( \hat{T}^k \) is bounded below. Hence the Banach space adjoints of \( \hat{T}^k \) and \( \hat{T} \) are surjective. So \( \hat{T} \) is bounded below.

(b) If \( \hat{T} \) is bounded below, then the limit \( \lim_{n \to \infty} \gamma(\hat{T}^n)^{1/n} \) exists and is positive \cite{8}. By (4), the same is true for \( \lim_{n \to \infty} \gamma(T^n)^{1/n} \).

(c) If \( \lim_{n \to \infty} \gamma(T^n)^{1/n} > 0 \), then obviously \( \limsup_{n \to \infty} \gamma(T^n)^{1/n} > 0 \).

\[ \blacktriangleleft \]

By applying the lemma to \( T^* \) instead of \( T \) and using Banach space duality, one can readily verify the following lemma.

**Lemma 3.** Suppose \( \overline{R(T^n)} = \overline{R(T^d)} \) for all \( n \geq d \). Let \( \hat{T} \) be the restriction of \( T \) to \( \overline{R(T^d)} \); then

\[
\begin{align*}
\liminf_{n \to \infty} \gamma(T^n)^{1/n} &= \liminf_{n \to \infty} (\hat{T}^n)^{1/n}, \\
\limsup_{n \to \infty} \gamma(T^n)^{1/n} &= \limsup_{n \to \infty} (\hat{T}^n)^{1/n}.
\end{align*}
\]

Moreover, \( \limsup_{n \to \infty} \gamma(T^n)^{1/n} > 0 \iff \hat{T} \) is surjective \( \iff \lim_{n \to \infty} \gamma(T^n)^{1/n} > 0 \).

The following lemma is a refinement of Kordula and Müller \cite[Lemma 2]{5}. The technique used in the proof is adapted from that paper.

**Lemma 4.** Let \( N \subseteq M \) be closed \( T \)-invariant subspaces of \( X \) such that \( T_{X/M} \) is bounded below, \( T_N \) is surjective and \( T^0 M \subseteq N \). Then:

(i) \( N = R(T^n) \cap M = T^n M \) for each \( n \geq d \).

(ii) \( M = N(T^n) + N = T^{-n}N \) for each \( n \geq d \).
(iii) \( \text{des}(T_M) \leq d \), \( R(T^d_M) = N \) and
\[
\lim_{n \to \infty} \gamma(T^n_M)^{1/n} = \lim_{n \to \infty} \gamma(T^n_N)^{1/n}.
\]
(iv) \( \text{asc}(T_{X/N}) \leq d \), \( N(T^d_{X/N}) = M/N \) and
\[
\lim_{n \to \infty} \gamma(T^n_{X/N})^{1/n} = \lim_{n \to \infty} \gamma(T^n_{X/M})^{1/n}.
\]
(v) \( \lim_{n \to \infty} \gamma(T^n)^{1/n} = \min \{ \lim_{n \to \infty} \gamma(T^n_M)^{1/n}, \lim_{n \to \infty} \gamma(T^n_N)^{1/n} \} \).

Proof. Consider any integer \( n \geq d \).

(i) Since \( T_{X/M} \) is injective, we have \( T^{-1}M = M \). So \( T^nM = T^nT^{-n}M = M \cap R(T^n) \). From the fact that \( T_N \) is surjective, \( N \subseteq M \) and \( T^dM \subseteq N \), we have the inclusions \( N \subseteq T^nN \subseteq T^nM \subseteq N \). This establishes (i).

(ii) Since \( T_N \) is surjective, we have \( T^{-n}N = T^{-n}T^nN = N(T^n) + N \). From the fact that \( T^nM \subseteq N \), \( N \subseteq M \) and \( T_{X/M} \) is injective, we have the inclusions \( M \subseteq T^{-n}N \subseteq T^{-n}M \subseteq M \). This shows (ii).

(iii) It follows from (i) that \( \text{des}(T_M) \leq d \), \( R(T^d_M) = N \). Using the notation in Lemma 3, we have \( (T_M)^{\sim} = T_N \), a surjective operator by hypothesis. The rest of (iii) follows from an application of Lemma 3 to the operator \( T_M \).

(iv) It follows from (ii) that \( N(T^n_{X/N}) = T^{-n}N/N = M/N \) for all \( n \geq d \). So \( \text{asc}(T_{X/N}) \leq d \) and \( N(T^n_{X/N}) = M/N \). Using the notation in Lemma 2, we can identify \( (T_{X/N})^{\sim} \) with \( T_{X/M} \), which is bounded below. The rest of (iv) follows from an application of Lemma 2 to the operator \( T_{X/N} \).

(v) From (iii), (iv) and Lemma 1,
\[
\limsup_{n \to \infty} \gamma(T^n)^{1/n} \leq \min \{ \lim_{n \to \infty} \gamma(T^n_M)^{1/n}, \lim_{n \to \infty} \gamma(T^n_N)^{1/n} \}.
\]

To prove the reverse inequality, we adopt the approach of [5, Lemma 2]. We assume that \( M \neq X \) and \( N \neq 0 \); otherwise, (v) follows from (iii) and (iv) directly. This means that \( T_N, T_{X/M} \) are non-zero operators which are either surjective or bounded below. Hence, both \( \gamma(T_N^n), \gamma(T_{X/M}^n) \) are finite and positive for all \( n \). Since (v) holds if and only if it holds for some non-zero multiple of \( T \), we can also assume without loss of generality that \( \|T\| = 1 \). For each \( i \geq d \), let \( \gamma^{-1}_i \) be the maximum of \( \gamma(T_N^n)^{-1} \) and \( \gamma(T_{X/M}^n)^{-1} \). We also let \( t > 1 \) and \( n \geq d \), and we let \( x \) be an arbitrary unit vector in \( R(T^n) \). For each \( i = d, \ldots, n \), it is possible to pick \( x_i \in T^{-i}[x + M] \) such that \( \|x_i\| \leq td(x_i, M) \). Since \( d(x_i, M) \leq \gamma(T_{X/M}^i)^{-1}d(x, M) \),
\[
\|x_i\| \leq t\gamma(T_{X/M}^i)^{-1}\|x\| \leq t\gamma^{-1}_i.
\]

Let \( m_i = Tx_{i+1} - x_i \) for \( i = d, \ldots, n - 1 \); then
\[
\|m_i\| \leq \|Tx_{i+1}\| + \|x_i\| \leq \|x_{i+1}\| + \|x_i\| \leq t(\gamma^{-1}_{i+1} + \gamma^{-1}_i),
\]
and \( \sum_{i=d}^{n-1} T^i m_i + T^d x_d - x = T^nx_n - x \). It is clear from the definition of \( x_i \) that \( T^nm_i \in M \). So \( m_i \in T^{-M} \subseteq M \). If \( i \geq d \), then \( T^d m_i \in R(T^d) \cap M = N \) by (i). As \( T_N \) is surjective, there exists \( u_i \in N \) such that \( T^{d-i} m_i = T^{n-i}u_i \) and \( \|u_i\| \leq td(u_i, N(T^{n-i}M)) \). Therefore,
\[
\|u_i\| \leq t\gamma(T_N^{n-i})^{-1}\|T^{d-i} m_i\| \leq t\gamma^{-1}_n \|m_i\| \leq t^2\gamma^{-1}_{n-i} (\gamma^{-1}_i + \gamma^{-1}_i).
\]
It is easy to verify that \( T' m_i = T^n u_i \). Let \( m = T^d x_d - x \). Since \( x \in R(T^n) \), we have \( m \in R(T^d) \cap M = N \). We can pick \( u \in N \) with \( m = T^n u \), \( \|u\| \leq t d(u, N(T^n)) \),

\[
\|u\| \leq t \gamma(T^n)^{-1} \|m\| \leq t \gamma^{-1} \|m\| \leq t^2 \gamma^{-1}(\gamma_d^{-1} + 1).
\]

Let \( z = x_n - \sum_{i=1}^{n-1} u_i - u \). We now have

\[
T^n z = T^n x_n - \sum_{i=1}^{n-1} T^i m_i - (T^d x_d - x) = x,
\]

\[
d(z, N(T^n)) \leq \|z\| \leq C \left[ \gamma_n^{-1} + \sum_{i=d}^{n-1} \gamma_{n-i+d}(\gamma_{i+1}^{-1} + \gamma_i^{-1}) + \gamma_n^{-1}(\gamma_d^{-1} + 1) \right],
\]

where \( C \) is the constant \( \max\{t, t^2\} \), which is independent of \( n \) and \( x \). Since \( x \) is an arbitrary unit vector in \( R(T^n) \), we have

\[
\gamma(T^n)^{-1} \leq C \left[ \gamma_n^{-1} + \sum_{i=d}^{n-1} \gamma_{n-i+d}(\gamma_{i+1}^{-1} + \gamma_i^{-1}) + \gamma_n^{-1}(\gamma_d^{-1} + 1) \right].
\]

Suppose \( 0 < \gamma < \min\{ \lim_{n \to \infty} \gamma(T^n)^{1/n}, \lim_{n \to \infty} \gamma(T^n)^{1/n} \} \); then for large enough \( i \), say \( i \geq n_0 \), we have \( \gamma_i^{-1} \geq \gamma^{-1} \). Let \( K = 1 + \max_{i \leq n_0} \gamma_i^{-1} \); then \( \gamma_i^{-1} \leq K \gamma^{-i} \) for all \( i \). It is a routine calculation that

\[
\gamma(T^n)^{-1} \leq CK^2 \left[ \gamma^{-n} + \sum_{i=d}^{n-1} (\gamma^{-n-d-1} + \gamma^{-n-d} + \gamma^{-n-d} + \gamma^{-n}) \right]
\]

\[
\leq \gamma^{-n} CK^2(3 + 2n - 2d) \max\{1, \gamma^{-d} \gamma^{-d-1}\}.
\]

Taking limits, we have \( \lim_{n \to \infty} \gamma(T^n)^{1/n} \geq \gamma \). By considering all possible \( \gamma \), we have

\[
\lim_{n \to \infty} \gamma(T^n)^{1/n} \geq \min\{ \lim_{n \to \infty} \gamma(T^n)^{1/n}, \lim_{n \to \infty} \gamma(T^n)^{1/n} \}.
\]

Using the lemma, the stability radius formula can be proved via the Apostol representation for quasi-Fredholm operator [12]. For the sake of completeness, we give an independent proof of the result.

**Lemma 5** ([5, Lemma 1]). \( T \) is \( s \)-regular if and only if there exists a closed subspace \( M \) with \( TM = M \) and \( T_{X/M} \) bounded below. We may choose \( M \) to be \( R(T^\infty) \).

**Lemma 6.** Let \( T \) be quasi-Fredholm; then \( \delta(T) > 0 \). If \( \Omega \) is the component of \( \rho_\gamma(T) \) containing \( \{ \lambda : 0 < |\lambda| < \delta(T) \} \) and \( d \) is the uniform descent of \( T \), then

\[
R[(T - \lambda I)^\infty] = R(T^\infty) + N(T^\infty) = R(T^\infty) + N(T^d)
\]

for all \( \lambda \in \Omega \).

**Proof.** See [4, Theorem 4.7], and note that \( T - \lambda I \) has closed range and uniform descent for \( n \geq 0 \) if and only if \( \lambda \in \rho_\gamma(T) \) [10, Corollaire 4.2 (iii)].

**Theorem 7.** \( T \) is a quasi-Fredholm operator if and only if there exist closed \( T \)-invariant subspaces \( M, N \) with \( N \subseteq M \) such that \( T_{X/M} \) is bounded below, \( T_N \) is surjective and \( T^d M \subseteq N \) for some nonnegative integer \( d \). We may take \( N = R(T^\infty) \) and \( M = N(T^d) + R(T^\infty) \), where \( d \) is the uniform descent of \( T \).
Proof. Suppose there are subspaces $M, N$ with the required properties; then the requirements for Lemma 4 are satisfied. In particular, $N = R(T^n) \cap M$ and $M = N(T^n) + N$ for each $n \geq d$. As $T_{X/M}$ is bounded below, $R(T_{X/M})$ is closed. So is $R(T) + M$. Since
\[ R(T) + M = R(T) + N(T^n) + N = R(T) + N(T^n) \] for all $n \geq d$,
$T$ has topological uniform descent for $n \geq d$.

It remains to show that $R(T^n)$ is closed for all $n \geq d$. For any $n \geq d$,
\[ M + R(T^n) = N(T^d) + N + R(T^n) = N(T^d) + R(T^n), \]
which is a closed subspace by [4, Theorem 3.2]. Clearly $M \cap R(T^n) = N$ is also a closed subspace. Now both $M$ and $R(T^n)$ are paracompact [6, Prop. 2.1.4]. Using the Neubauer Lemma [6, Prop. 2.1.2], we deduce that $R(T^n)$ is closed. Hence $T$ is quasi-Fredholm.

Conversely, assume $T$ is quasi-Fredholm with uniform descent $d$. Take $N = R(T^\infty)$ and $M = N(T^d) + R(T^\infty)$. It is clear that $M$ and $N$ are $T$-invariant subspaces. It is also clear that $N \subseteq M$ and $T^d M \subseteq N$. Since $R(T^n)$ is closed for $n \geq d$ and $N = R(T^\infty)$, $N$ is closed. Moreover, we have $T N = N$ [4, Theorem 3.4]. So $T_N$ is surjective and $T^{-1} N = N + N(T^n)$ for all $n$. By Lemma 6, we have $M = N + N(T^n) = T^{-1} N$ for all $n \geq d$. It follows that $M$ is closed and $T^{-1} M = M$. Hence, $T_{X/M}$ is injective. Also,
\[ R(T) + M = R(T) + N + N(T^d) = R(T) + N(T^d), \]
which is a closed subspace by the definition of topological uniform descent. We conclude that $R(T_{X/M})$ is closed and hence $T_{X/M}$ is bounded below. \qed

Theorem 8. If $\text{des}(T) = d$ and $R(T^d)$ is closed, then the stability radius of $T$ is given by $\delta(T) = \lim_{n \to \infty} \gamma(T^n)^{1/n}$.

Remark. We readily deduce from [4, Corollary 4.8 (c)] that $T - \lambda I$ is surjective for every $\lambda \in \Omega$, where $\Omega$ is the component of $\rho_s(T)$ defined in Lemma 6.

Proof. We assume that $T^d \neq 0$; otherwise both $\delta(T)$ and $\lim_{n \to \infty} \gamma(T^n)^{1/n}$ are infinite. Let $M = R(T^d) = R(T^\infty)$. It is then easy to verify that $T_M$ is surjective and $T_{X/M}$ is nilpotent. By Lemma 3, there is positive real number $\delta$ such that
\[ \delta = \lim_{n \to \infty} \gamma(T^n)^{1/n} = \lim_{n \to \infty} \gamma(T_M^n)^{1/n}. \]

We know from [8] that $\delta$ is the surjectivity radius of $T_M$. We proceed to show that $\delta = \delta(T)$. If $0 < |\lambda| < \delta$, then $(T - \lambda I)_M$ is surjective. Since $T_{X/M}$ is nilpotent, $(T - \lambda I)_{X/M}$ is invertible and hence bounded below. Using Lemma 5, we deduce that $T - \lambda I$ is $s$-regular. Hence, $\delta \leq \delta(T)$.

Conversely, assume $0 < |\lambda| < \delta(T)$. Since $T_{X/M}$ is nilpotent, $(T - \lambda I)_{X/M}$ is invertible and $(T - \lambda I)^{-1} M = M$. It follows that
\[ (T - \lambda I) M = (T - \lambda I)(T - \lambda I)^{-1} M = M \cap R(T - \lambda I). \]

It is routine to verify that $T$ is quasi-Fredholm. Thus, $M = R(T^\infty) \subseteq R(T - \lambda I)$ by Lemma 6. So we have $(T - \lambda I) M = M$. This shows that the surjectivity radius of $T_M$ is no less than $\delta(T)$. Hence, $\delta(T) \leq \delta$. \qed

A dual to the above theorem is the following.
**Theorem 9.** If $\text{asc}(T) = d$ and $R(T^n)$ is closed for all $n \geq d$, then the stability radius of $T$ is given by $\delta(T) = \lim_{n \to \infty} \gamma(T^n)^{1/n}$.

**Theorem 10.** Let $T$ be a quasi-Fredholm operator; then the stability radius of $T$ is given by $\delta(T) = \lim_{n \to \infty} \gamma(T^n)^{1/n}$.

**Proof.** Let $M = R(T^\infty) + N(T^d)$, $N = R(T^\infty)$. It follows from Theorem 7 that the subspaces $M$ and $N$ satisfy the requirement for Lemma 4. We now have:

(a) $T_{X/M}$ is bounded below and $\delta(T_{X/M}) = \lim_{n \to \infty} \gamma(T_{X/M}^n)^{1/n}$.

(b) By Lemma 4 (iii), $T_M$ satisfies the requirement for Theorem 8 and

$$\delta(T_M) = \lim_{n \to \infty} \gamma(T_M^n)^{1/n} = \lim_{n \to \infty} \gamma(T_N^n)^{1/n}.$$ 

(c) By (a), (b) and Lemma 4 (v), the limit $\delta = \lim_{n \to \infty} \gamma(T^n)^{1/n}$ exists and

$$\delta = \min\{\delta(T_{X/M}), \delta(T_M)\}.$$ 

If $0 < |\lambda| < \delta$, then $|\lambda|$ is less than both $\delta(T_{X/M})$ and $\delta(T_M)$. Thus $(T - \lambda I)_{X/M}$ is bounded below [8] and $(T - \lambda I)_M$ is surjective (see the remark for Theorem 8). Lemma 5 shows that $T - \lambda I$ is $s$-regular. Hence, $\delta \leq \delta(T)$.

Conversely, let $0 < |\lambda| < \delta(T)$. By Theorem 6, $M = R[(T - \lambda I)^\infty]$. Since $T - \lambda I$ is $s$-regular, it follows from Theorem 5 that $(T - \lambda I)_{X/M}$ is bounded below and $(T - \lambda I)_M$ is surjective. Therefore,

$$|\lambda| < \min\{\delta(T_{X/M}), \delta(T_M)\} = \delta.$$ 

This shows that $\delta(T) \leq \delta$. \hfill \Box

So far, the results obtained are independent of the Apostol representation of operators given in [12]. For the rest of this paper, we assume the following results from [12]. Let

$$\mathcal{N} = \bigvee_{\lambda \in \rho_s(T)} N(T - \lambda I) \quad \text{and} \quad \mathcal{R} = \bigcap_{\lambda \in \rho_s(T)} R(T - \lambda I).$$ 

So $\mathcal{N}$ is the closed subspace generated by $N(T - \lambda I)$ and $\mathcal{R}$ is the intersection of $R(T - \lambda I)$ over all $\lambda \in \rho_s(T)$. Then $\mathcal{N}, \mathcal{R}$ are hyper-invariant subspaces of $T$ with $\mathcal{N} \subseteq \mathcal{R}$. If $T_\delta, T_0, T_\pi$ are the maps induced by $T$ on the spaces $\mathcal{N}, \mathcal{R}/\mathcal{N}, X/\mathcal{R}$ respectively, then the following properties hold.

(i) $T_\delta$ has dense range, $T_\pi$ is injective.

(ii) $T$ is $s$-regular if and only if $T_\delta$ is surjective, $T_0$ is invertible and $T_\pi$ is bounded below.

These two facts are required for the proofs of the following theorems. In particular, we need to know that $\sigma(T_0) \subseteq \sigma_s(T)$, which is a consequence of (ii). It is also known that $T$ is quasi-Fredholm if and only if $T_\delta$ is surjective, $T_0$ is chain finite and $T_\pi$ is bounded below. However, we will not use this fact.

**Lemma 11.** Let $N \subseteq M$ be closed $T$-invariant subspaces of $X$. Suppose that $T_{X/M}$ is injective, $T^d M \subseteq N$ for some $d$ and $T_N$ has dense range. If the limit $\limsup_{n \to \infty} \gamma(T^n)^{1/n}$ is non-zero, then $T$ is quasi-Fredholm.

**Proof.** In the light of Theorem 7, it suffices to prove that $T_{X/M}$ is bounded below and $T_N$ is surjective. It is clear from Lemma 1 that both $\limsup_{n \to \infty} \gamma(T_{X/N}^n)^{1/n}$ and $\limsup_{n \to \infty} \gamma(T_M^n)^{1/n}$ are positive.
Since $T^dM \subseteq N$ and $T_{X/M}$ is injective, we have the inclusions

$$M \subseteq T^{-n}N \subseteq T^{-n}M \subseteq M$$

for each $n \geq d$. Hence $M = T^{-n}N$ and $N(T_{X/N}^n) = M/N$ for $n \geq d$. Applying Lemma 2 to the operator $T_{X/N}$, we deduce that $(T_{X/N})^\wedge$ and hence $T_{X/M}$ is bounded below.

Since $T_N$ has dense range, so has $T_N^n$ for each $n \geq d$. Therefore

$$N \subseteq T^nN \subseteq T^nM \subseteq N.$$ 

Hence, $\overline{R(T^n_M)} = \overline{R(T^n_N)} = N$ for $n \geq d$. Applying Lemma 3 to the operator $T_M$, we deduce that $(T_M)^\sim = T_N$ is surjective.

The proofs of the following two lemmas are elementary, so we omit them.

**Lemma 12.** Let $N \subseteq M$ be closed $T$-invariant subspaces of $X$. If both $T_{X/N}$ and $T_{M/N}$ are injective, then $T_{X/N}$ is also injective.

**Lemma 13.** Let $M_1, M_2$ be $T$-invariant closed subspaces of $X$ such that $M_1 + M_2$ is closed. Let

$$X_1 = \frac{M_1}{M_1 \cap M_2}, \quad X_2 = \frac{M_2}{M_1 \cap M_2}, \quad Y_1 = \frac{M_1 + M_2}{M_1}, \quad Y_2 = \frac{M_1 + M_2}{M_2}.$$ 

Then the diagrams

$$\begin{array}{ccc}
X_1 & \xrightarrow{\varphi} & Y_2 \\
T_{X_1} \downarrow & & \downarrow T_{Y_2} \\
X_1 & \xrightarrow{\varphi} & Y_2 \\
X_2 & \xrightarrow{\psi} & Y_1 \\
T_{X_2} \downarrow & & \downarrow T_{Y_1} \\
X_2 & \xrightarrow{\psi} & Y_1
\end{array}$$

are commutative, and $\varphi$ and $\psi$ are linear homeomorphisms induced by the identity on $X$.

We now give a generalization of [1, Prop. 3.3].

**Theorem 14.** Let $\sigma$ be a closed and open subset of $\sigma_\gamma(T)$. If $0 \in \sigma$ and

$$\limsup_{n \to \infty} \gamma(T_n^{1/n}) = r > \sup_{\lambda \in \sigma} |\lambda|,$$

then $T$ is quasi-Fredholm.

**Proof.** Since $T_\delta$ has dense range and $T_{\sigma}$ is injective, we can apply both part (i) and (ii) of Lemma 1 to show that $\gamma(T_0^n) \geq \gamma(T^n)$ for all $n$. Let $\sigma_1 = \sigma \cap \sigma(T_0)$, $\sigma_2 = \sigma(T) \setminus \sigma(T_0)$; then both $\sigma_1$ and $\sigma_2$ are closed. Since $\sigma(T_0) \subseteq \sigma_\gamma(T)$, we have $\sigma_1 \cup \sigma_2 = \sigma$. Hence $\sigma_1$ and $\sigma_2$ are spectral sets of $T_0$. The spectral sets induce a decomposition

$$\mathcal{R}/\mathcal{N} = X_1 \oplus X_2, \quad T_0 = T_1 \oplus T_2 \quad \text{with} \quad \sigma(T_1) = \sigma_1, \quad \sigma(T_2) = \sigma_2.$$ 

Since $0 \not\in \sigma_2$, $T_2$ is invertible. We claim that $T_1$ is nilpotent. Let us assume that $\sigma_1 \neq \emptyset$; otherwise, $X_1 = 0$ and $T_1$ is trivially nilpotent. For each $x_1 \in X_1$, $x_2 \in X_2$, $T_0(x_1 + x_2) \in X_1 \Rightarrow T_2x_2 \in X_1 \Rightarrow T_2x_2 \in X_1 \cap X_2 = \{0\} \Rightarrow x_2 = 0.$

Therefore, $T_0^{-1}X_1 \subseteq X_1$. Thus $T_0$ and $X_1$ satisfy part (i) of Lemma 1. Hence, $\gamma(T_n^0) \geq \gamma(T_0^n) \geq \gamma(T^n)$ for all $n$. Taking limits,

$$r = \limsup_{n \to \infty} \gamma(T_n^{1/n}) \leq \limsup_{n \to \infty} \gamma(T_0^n)^{1/n}.$$
Lemma 12. Moreover, \( \psi_T \) is always injective. Thus, both \( \eta \) is the stability radius of \( T \). Therefore, the limit \( \lim_{n \to \infty} \gamma(T^n)^{1/n} \) always exists and equals \( \eta \). If \( \eta \neq 0 \), then case (b) must be true and \( T \) is quasi-Fredholm. Conversely, if \( T \) is quasi-Fredholm, then \( \eta \) is the stability radius of \( T \). From Lemma 6, we know that the stability radius of a quasi-Fredholm operator is always non-zero. This completes our proof.

The significance of the corollary is that the class of quasi-Fredholm operators is the most general class of operators for which the limit \( \lim_{n \to \infty} \gamma(T^n)^{1/n} \) is equal to the stability radius \( \delta(T) \) with respect to the Apostol spectrum.

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References


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