**-REPRESENTATIONS ON BANACH **-ALGEBRAS

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Abstract. We study notions of \(g\)-bounded linear functionals and representable functionals on Banach **-algebras. An equivalence between these two is established for general Banach **-algebras. In particular, we characterize \(g\)-bounded linear functionals on Banach **-algebras with approximate identity and isometric involution. In addition, we prove a result on representation of \(g\)-bounded positive linear functionals in terms of cyclic vectors for the corresponding **-representation.

1. Introduction

Let \(A\) be a complex Banach **-algebra. We assume neither the existence of an identity nor that the involution is continuous. We write \(S(A) = \{a \text{ in } A \text{ such that } a^* = a\}\) for the set of all self-adjoint elements of \(A\). A **-ideal of \(A\) is an ideal \(J\) of \(A\) where \(a \in J\) implies \(a^* \in J\).

A **-semi-norm on \(A\) is a function \(\eta : A \to \mathbb{R}\) such that for all \(a, b \in A\) and \(\alpha \in \mathbb{C}\)

\[
\begin{align*}
(1) \quad & \eta(a + b) \leq \eta(a) + \eta(b), \\
(2) \quad & \eta(\alpha a) = |\alpha| \eta(a), \\
(3) \quad & \eta(ab) \leq \eta(a) \cdot \eta(b), \\
(4) \quad & \eta(a^* a) = (\eta(a))^2.
\end{align*}
\]

\(P(A)\) denotes the set of all **-semi-norms on \(A\). For more on **-semi-norms, see [2, 3]. Suppose \(g(a) = \sup\{\eta(a) : \eta \in P(A)\}\). Then \(g\) defines a **-semi-norm on \(A\); in fact, \(g\) is the greatest **-semi-norm on \(A\) in the pointwise ordering.

A **-representation of \(A\) is a mapping \(\pi : A \to B(H)\), where \(B(H)\) denotes the algebra of all bounded linear operators on a Hilbert space \(H\), such that for all \(a, b \in A\) and \(\alpha \in \mathbb{C}\)

\[
\begin{align*}
(1) \quad & \pi(a + b) = \pi(a) + \pi(b), \\
(2) \quad & \pi(\alpha a) = \alpha \pi(a), \\
(3) \quad & \pi(ab) = \pi(a) \cdot \pi(b), \\
(4) \quad & \pi(a^*) = (\pi(a))^*.
\end{align*}
\]

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Thus \(|g| = \sup \{|f(a)| : g(a) \leq 1\}.

The set \(D(g)\) consists of all \(g\)-bounded positive linear functionals \(f\) on \(A\) with \(|f|_g \leq 1\). A positive \(g\)-bounded linear functional \(f\) on \(A\) will be called a state of \(A\) if \(|f|_g = 1\).

**Lemma 2.1.** Let \(A\) be a unital algebra and \(f\) be a positive \(g\)-bounded linear functional on \(A\). Then \(f\) is a state of \(A\).

**Proof.** Since \(f(1) \leq 1\), it follows that \(|f|_g \geq f(1)\), where 1 is the identity element in the algebra \(A\). But for all \(a \in A\),

\[|f(a)| \leq |f|_g g(a).\]

Hence for all \(a \in A\),

\[|f(a)|^2 \leq f(1)|f|_g g(a^*a) = f(1)|f|_g g(a)^2.\]

Thus \(|f|_g^2 \leq f(1)|f|_g\) and consequently \(f\) is a state of \(A\).

**Proposition 2.1.** Let \(u\) and \(v\) be positive \(g\)-bounded linear functionals on \(A\). Then for all \(a \in A\)

\[(i) \ |u(a)|^2 \leq |u|_g^2 |u(a^*a)|,
(ii) \ |u + v|_g = |u|_g + |v|_g,
(iii) \ |u|_g = \sup \{|u(a^*a) : g(a) \leq 1\},
(iv) \ u \ and \ v \ are \ hermitian \ functionals.\]

**Proof.** Suppose that \(A\) has no unit element. If \(J_g = \{a \in A : g(a) = 0\}\), then \(J_g\) is a closed two-sided \(*\)-ideal of \(A\). In that case \(A/J_g\) becomes a quotient \(*\)-algebra. Let \(a \rightarrow \lambda_a\) denote the canonical mapping of \(A\) onto \(A/J_g\). We define a \(B^*\)-norm \(\pi\) on \(A/J_g\) as follows: For all \(a \in A/J_g\), \(\pi(\lambda_a) = g(a)\). The completion \(A_g\) of \(A/J_g\) with respect to this norm is a \(B^*\)-algebra. On \(A/J_g\), define \(\pi(\lambda_a) = \overline{u}(a), \overline{v}(\lambda_a) = v(a)\), for all \(\lambda_a\) in \(A/J_g\). Then \(\overline{u}\) and \(\overline{v}\) are well defined positive \(\overline{g}\)-bounded linear functionals on \(A/J_g\). Furthermore, \(|\overline{u}|_{\pi|_\overline{g}} = |u|_g\) and \(|\overline{v}|_{\pi|_\overline{g}} = |v|_g\). Hence \(\overline{u}\) and \(\overline{v}\) have a unique norm preserving extension to the \(B^*\)-algebra \(A_g\). Denote these extensions by \(U\) and \(V\), respectively.

Thus \(U\) and \(V\) are positive linear functionals on \(A_g\), hence (i), (ii), and (iv) follow from 2.1.5 and 2.1.6 in [1]. To prove (iii) we proceed as follows. Let \(\sup\{|u(a^*a) : g(a) \leq 1\} = \alpha\). Then \(\alpha \leq |u|_g\), since if \(g(a) \leq 1\), \(u(a^*a) \leq |u|_g g(a^*a) \leq |u|_g\). By the definition of \(|u|_g\) there exists a sequence \(\{a_k\}\) of elements of \(A\) with \(g(a_k) \leq 1\) and \(|u|_g = \lim_{k \rightarrow \infty} |u(a_k)|\).
It follows from (i) that
\[ |u(a_k)|^2 \leq |u|_g u(a_k^*a_k) \leq |u|_g^2 g(a_k) \leq |u|_g^2. \]
Hence \( \lim_{k \to \infty} u(a_k^*a_k) = |u|_g \). The case where \( A \) is unital follows from Lemma 2.1 above.

**Remark 2.1.** Proposition 2.1 is also true for any \( B^* \)-semi-norm. The following theorem gives a necessary and a sufficient condition for a positive functional on an algebra to be \( g \)-bounded.

**Theorem 2.1.** A positive linear functional \( f \) on \( A \) is \( g \)-bounded if and only if there exists a positive constant \( M \), which depends only on \( f \), such that for all \( a \) in \( A \), \(|f(a)| \leq Mf(a^*a)\).

Moreover if \( f \) is \( g \)-bounded, then
\[ |f|_g = \sup_{a \in A} \left\{ \frac{|f(a)|^2}{f(a^*a)} \right\}. \]

**Proof.** Let \( f \) be a \( g \)-bounded positive functional. Then by Proposition 2.1(i) it follows that for all \( a \) in \( A \), \(|f(a)|^2 \leq Mf(a^*a)\).

Suppose conversely that for all \( a \) in \( A \), and for every positive linear functional \( f \) on \( A \), there exists a positive constant \( M \) such that \(|f(a)|^2 \leq Mf(a^*a)\). Then for all \( x \) and \( a \) in \( A \)
\[ \beta_f = \sup_{f(x^*x) \leq 1} \left\{ \sqrt{f(x^*a^*ax)} \right\} \geq \sup_{f(x^*x) \leq 1} \left\{ \frac{|f(ax)|}{\sqrt{M}} \right\}. \]
Let \( f(a^*a) > 0 \) and \( a/\sqrt{f(a^*a)} = x \). Then \( f(x^*x) = 1 \). Now \( \beta_f(a) \geq |f(a^2)|/\sqrt{Mf(a^*a)} \). If \( a \) is in \( S(A) \), then we have \( \sqrt{|f(a)|^2/M} \leq \sqrt{f(a^*a)/M} \leq \beta_f(a) \); that is, \( |f(a)| \leq M\beta_f(a) \). Hence, \(|f(a)|^2 \leq Mf(a^*a) \leq M^2\beta_f(a)^2 \) so that \(|f(a)| \leq M\beta_f(a) \). This proves that \( f \) is \( g \)-bounded. Furthermore, for all \( a \) in \( A \) and for any positive linear functional \( f \) on \( A \),
\[ \sup_{a \in A} \left\{ \frac{|f(a)|^2}{f(a^*a)} \right\} \leq |f|_g \leq M, \]
where \( M \) is a positive constant (depending only on \( f \)). Therefore we may suppose that \( M = \sup_{a \in A} |f(a)|/f(a^*a) \) so that we obtain \(|f|_g = \sup_{a \in A} |f(a)|^2/f(a^*a) \).

**Remark 2.2.** If \( A \) has an identity \( 1 \), then every positive linear functional \( f \) is \( g \)-bounded. This follows directly by Theorem 2.1 and the Cauchy-Schwarz inequality. Further, if \( A \) is unital with isometric involution, then by Lemma 2.1 it follows that \( \|f\| = |f|_g = f(1) \).

We use the fact of Remark 2.2 and Theorem 2.1 to give a characterization of \( g \)-bounded linear functionals when the given algebra has approximate identity and isometric involution.

**Theorem 2.2.** If \( A \) has isometric involution and approximate identity, a positive linear functional \( f \) is \( g \)-bounded if and only if it is continuous.

**Proof.** If \( f \) is \( g \)-bounded, then for all \( a \) in \( A \), \(|f(a)| \leq |f|_g g(a) \leq |f|_g \|a\| \) and consequently \( f \) is continuous. Suppose conversely that \( f \) is continuous, then for all \( a \) in \( A \), \(|f(a)|^2 \leq \|f\|f(a^*a) \) by 2.1.5 in [1]. Hence by Theorem 2.1 \( f \) is \( g \)-bounded.

**Corollary 2.1.** \(|f|_g = \|f\| \), for all \( f \) as in Theorem 2.2.
Proof. Since \( f \) is \( g \)-bounded we have \(|f|_g \leq \|f\|\). Also the involution on \( A \) is isometric. Therefore \(|f(a)| \leq |f|_g g(a) \leq |f|_g \|a\|\). This implies that \( \|f\| \leq |f|_g \) and hence \(|f|_g = \|f\|\).

3. \( g \)-BOUNDED FUNCTIONALS IN TERMS OF REPRESENTABLE FUNCTIONALS

In the following theorem we establish a relationship between \( g \)-bounded and representable functionals. Here it is shown that the representable functionals are the positive \( g \)-bounded linear functionals and these are precisely the functionals generated by cyclic \(*\)-representations of the algebra.

**Theorem 3.1.** A positive linear functional \( f \) on \( A \) is representable if and only if it is \( g \)-bounded.

**Proof.** Suppose that \( f \) is representable. Then by the definition of representable functionals there exists \(*\)-representation \( \pi \) of \( A \) on \( H \) and a vector \( x \) in \( H \) such that for all \( a \) in \( A \), \( |f(a)| \leq |\pi(a)| \) and \( ||x||^2 \leq g(a)||x||^2 \). Thus \( f \) is \( g \)-bounded.

Suppose conversely that \( f \) is \( g \)-bounded. If the norm on \( A^+ \), the unitization of \( A \), is given by \(||(a, \lambda)|| = \|a\| + |\lambda|\) for all \( a \) in \( A \) and \( \lambda \) in \( \mathbb{C} \), then \( A \) is isometrically and \(*\)-isomorphically embedded in the unital Banach \(*\)-algebra \( A^+ \). Since \( f \) is \( g \)-bounded it follows from Proposition 2.1(i) that for all \( a \) in \( A \), \( |f(a)|^2 \leq |f|_g f(a^*a) \).

Let \( f^+ \) be defined on \( A^+ \) by \( f^+((a, \lambda)) = f(a) + \lambda|f|_g \), where \((a, \lambda)\) is in \( A^+ \). Then \( f^+ \) is a linear functional on \( A^+ \), which also extends \( f \) on \( A \) and

\[
\begin{align*}
 f^+((a, \lambda)^*(a, \lambda)) &= f^+(a^*a + \lambda a + \lambda^*a + \lambda^*) \\
 &= f(a^*a) + \overline{\lambda}f(a) + |\lambda|^2|f|_g.
\end{align*}
\]

Thus,

\[
 f^+((a, \lambda)^*(a, \lambda)) \geq f(a^*a) - 2|\lambda|(|f|_g f(a^*a))^{1/2} + |\lambda|^2|f|_g
\]

\[
= (f(a^*a))^{1/2} - |\lambda| |f|_g^{1/2})^2 \geq 0.
\]

Hence \( f^+ \) is a positive linear functional on \( A^+ \).

Let \( \mathcal{L}_f = \{a \in A^+: f^+(ba) = 0 \text{ for all } b \in A^+\} \). Then on the quotient space \( A^+/\mathcal{L}_f \) we define \((x_a, x_b)_f = f^+(b^*a)\), \( a \) in \( x_a, b \) in \( x_b \). If \( H_f \) is the completion of \( A^+/\mathcal{L}_f \), then

\[
||x_f||^2 = \langle x_f, x_f \rangle_f = |f^+(0, 1)^*(0, 1)| = |f|_g.
\]

For each \( a \) in \( A^+ \) let \( \pi(a) \) be defined by \( \pi(a)x_b = x_{ab}, x_b \in A^+/\mathcal{L}_f \). Then \( \pi(a) \) is a well-defined linear operator on \( A^+/\mathcal{L}_f \) and for \( a \) in \( A^+
\]

\[
||\pi(a)x_b||_f^2 = \langle x_{ab}, x_{ab} \rangle_f = f^+(b^*a^*ab).
\]

It is easy to see that \( \pi(a) \) is a bounded linear operator on \( A^+/\mathcal{L}_f \) and so it has a unique extension to a bounded linear operator \( \pi^+(a) \) on \( H_f \). Also the mapping \( a \to \pi^+(a) \) is an algebra homomorphism from \( A^+ \) into \( B(H_f) \) and for all \( a, b \) and \( c \) in \( A^+
\]

\[
(\pi(a)x_b, x_c)_f = (x_b, \pi(a^*)x_c)_f.
\]

Hence for all \( x, y \) in \( H_f \), \((\pi(a)x, y)_f = (y, \pi(a^*)y)_f \) so that for all \( a \) in \( A^+ \),

\[
(\pi^+(a))^* = \pi^+(a^*) \text{.}
\]

Thus \( \pi^+ \) is a \(*\)-representation of \( A^+ \) on \( H_f \) and hence the restriction map \( \pi_f = \pi^+|_A \) is a \(*\)-representation of \( A \) on \( H_f \). Next, for all \( a \) in \( A \)

\[
(\pi(a)x, y)_f = (x_{(a,0)}, x_{(0,1)})_f = f^+((a,0)) = f(a).
\]
Thus $f$ is a positive linear functional represented by the pair $(\pi_f, x_f)$, and the proof is complete.

The following proposition shows that our definition of representability of positive linear functionals is equivalent to the definition given by Rickart ([5], 4.5.5). The proof uses the construction of the proof of Theorem 3.1.

**Proposition 3.1.** If $f$ is a $g$-bounded positive linear functional on $A$ then $f$ can be represented by a pair $(\pi, x)$, where $x$ is a cyclic vector for the $*$-representation $\pi$, and moreover $|f|_g = \|x_f\|^2_f$.

**Proof.** By the definition of cyclic vector and Theorem 3.1 it is obvious that $x_f$ is a cyclic vector for the $*$-representation $\pi^+$ of $A^+$ on $H_f$. We claim that $x_f$ is a cyclic vector for the $*$-representation $\pi_f$ of $A$.

Since $f$ is $g$-bounded, there exists a sequence $\{a_k\}$ of elements of $A$ with $g(a_k) \leq 1$ such that $|f|_g = \lim_{k \to \infty} f(a_k^* a_k)$. Let $b_k = a_k^* a_k$. Then $b_k$ is in $S(A)$ and $g(b_k) \leq 1$, and since $|f(b_k)|^2 \leq |f|_g |b_k|^2 \leq |f|_g^2$, it follows that $\lim_{k \to \infty} f(b_k^* b_k) = |f|_g$.

Consider $\|x_{b_k} - x_f\|_f$. Then $\|x_{b_k} - x_f\|_f^2 = f(b_k^* b_k) - 2f(b_k) + |f|_g \to 0$ as $k \to \infty$. Hence for any $a$ in $A^+$ we have $\|ax_{b_k} - xa\|_f = \|\pi^+(a)(xb_k - xf)\|_f \to 0$ as $k \to \infty$. However, $ab_k$ is in $A$ and hence $\pi_f(A)x_f$ is dense in $A^+L_f$. It follows that it is also dense in $H_f$. Thus $x_f$ is a cyclic vector for $\pi_f$. The equality $|f|_g = \|x_f\|^2_f$ follows from the proof of Theorem 3.1.

4.

In this section we question why $g$-bounded functionals may be interesting and supply an answer to this question as well. During the course of this research, we observed that in the case of $B^*$-algebras, $g$-bounded functionals coincide with the original norm. A $g$-bounded functional is more general in defining the state space of $A$ since the set $D(g)$ of $g$-bounded functionals is a subspace of $A^*$, the dual space of $A$.

Proposition 2.1 holds for any $B^*$-semi-norm. A $g$-bounded functional can be represented by a pair of cyclic vectors for the $*$-representations.

Let $\pi$ be a $*$-representation of $A$ on the Hilbert space $H$ and let $x$ be in $H$. Then any positive linear functional $f$, which is represented by $(\pi, x)$, defines a $g$-bounded positive functional $f_T$ on $A$ with $f_T \leq f$, where $T$ is a self-adjoint operator on $H$ such that $\pi(A) = \pi(A)T$ and $0 \leq T \leq I_H$ (see Lemma 1.1 in [4]). Lemma 1.1 in [4] is an extension of a result by Dixmier ([1], 2.5.1).

Another application of $g$-bounded functionals to the representation theory can be seen in characterizing representable functionals which can be represented by a topologically irreducible representation.

A positive linear functional $f$ is a pure state of $A$ (see Definition 2.1, [4]) if it is non-zero and $g$-bounded and if any $g$-bounded positive linear functional dominated by $f$ is of the form $\beta f$ with $\beta$ in $[0, 1]$. The following result, which has been submitted to another journal, gives an application of $g$-bounded functionals.

**Theorem 4.1 ([4]).** Let $(\pi, x)$ be a cyclic representation of a positive linear functional $f$. Then $\pi$ is topologically irreducible and non-zero if and only if $f$ is a pure state of $A$.

An intriguing development in the representation theory of $g$-bounded functionals is Theorem 3.1 in [4] which states that the extreme points of $D(g)$ are the zero functional and the pure states of $A$. 
In closing, we would like to express our appreciation to the referee for his or her valuable suggestions which improved the clarity of our presentation.

References


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