VOLUME OF INTERSECTIONS AND SECTIONS OF THE UNIT BALL OF $\ell_p^n$

MICHAEL SCHMUCKENSCHLÄGER

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Abstract. An asymptotic formula for the volume of the intersection of a suitable multiple of the unit ball of $\ell_p^n$ and the cube $[-1/2, 1/2]^n$ will be proved. We also show that the isotropic constant of the unit ball of $\ell_p^n$, $1 \leq p \leq 2$, is bounded by $1/\sqrt{12}$.

1. Introduction and notation

Let $r(n, p)B_p^n$ be the multiple of the unit ball $B_p^n$ of $\ell_p^n$ such that
\[ \text{Vol}_n(r(n, p)B_p^n) = 1. \]

In [SS] the following theorem is proved:

Theorem 1.1. For all $0 < p \leq \infty$ and all $0 < q < \infty$ we have
\[ \lim_{n \to \infty} \text{Vol}_n(r(n, p)B_p^n \cap t.r(n, q)B_q^n) = \begin{cases} 0 & \text{if } tA(p, q) < 1, \\ 1 & \text{if } tA(p, q) > 1, \end{cases} \]

where
\[ A(p, q) = \begin{cases} \frac{e^{1/p} \Gamma(1 + \frac{1}{p})^{1+1/p}q^{1/q}}{e^{1/q} \Gamma(1 + \frac{1}{q})\Gamma(\frac{1}{p})^{1/q}p^{1/p}} & \text{if } p < \infty, \\ \frac{(1+q)^{1/q}}{e^{1/q} \Gamma(1 + \frac{1}{q})^{1/q}q^{1/q}} & \text{if } p = \infty. \end{cases} \]

Also, the following problem was posed: What is the asymptotic behavior of
\[ \text{Vol}_n(r(n, p)B_p^n \cap t.r(n, q)B_q^n) \]
for $t = A(p, q)^{-1}$? In section 2 it will be proved that in the case $p = \infty$ this limit equals $\frac{1}{2}$—the case $p = \infty$ and $q = 1$ has also been solved by B. Weißbach.

In section 3 we consider a different problem: Let $E$ be a subspace of $\mathbb{R}^n$ of codimension $k$. M. Meyer and A. Pajor (cf. [MeP]) proved that for all $p \geq 2$ and $p = 1$: $\text{Vol}_{n-k}(E \cap r(n, p)B_p^n) \geq 1$. We will prove this inequality in the case $k = 1$ and $1 < p < 2$. 

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2. INTERSECTION WITH THE CUBE

Let $0 < q < \infty$ and $t = A(\infty, q)^{-1}$. Define $a(n, q)$ by the equation

$$a(n, q) = \frac{1}{2(1 + q)^{1/q}} \frac{r(n, q)}{n^{1/q}A(\infty, q)}.$$ 

Using Stirling’s formula it is easily checked that

$$a(n, q) = 1 + \frac{q \log n}{2n} + O\left(\frac{1}{n}\right). \tag{1}$$

Now, let $X$ be uniformly distributed on the interval $[-\frac{1}{2}, \frac{1}{2}]$ and let $X_j, j = 1, \ldots, n$, be independent copies of $X$. Then

$$\text{Vol}_n \left( \left[ -\frac{1}{2}, \frac{1}{2} \right] \cap t r(n, p) B^n_p \right) = 1 - \mathbb{P} \left( \left( \frac{1}{n} \sum_{j=1}^n |X_j|^q \right)^{1/q} > \|X\|_q a(n, q) \right). \tag{2}$$

**Theorem 2.1.** For all $0 < q < \infty$ we have:

$$\lim_{n \to \infty} \text{Vol}_n \left( \left[ -\frac{1}{2}, \frac{1}{2} \right] \cap r(n, q) \frac{A(\infty, q)}{B^n_q} \right) = \frac{1}{2}.$$ 

**Proof.** By (1) and simple algebra we conclude that the probability in (2) is given by

$$\mathbb{P} \left( \left( \frac{1}{n} \sum_{j=1}^n |X_j|^q - \mathbb{E}|X|^q \right)^{1/q} > \sqrt{2} \frac{q^2 \log n}{2n} \right) \leq C \left( \frac{\log n}{\sqrt{n}} \right). \tag{3}$$

A version of the Berry-Esseen Theorem (cf. e.g. [C, p. 225]) states that if $Y_j, j = 1, \ldots, n$, is an i.i.d. sequence of random variables such that $\mathbb{E} Y = 0$ and $\|Y\|_3 < \infty$, then there exists an absolute constant $C$ such that for all $s \in \mathbb{R}$:

$$\left| \mathbb{P} \left( \left\|Y\right\|_2 \sqrt{n} \sum_{j=1}^n Y_j < s \right) - \frac{1}{\sqrt{2\pi}} \int_{-\infty}^s e^{-x^2/2} dx \right| \leq C \left( \frac{\|Y\|_3}{\|Y\|_2} \right)^2 \frac{1}{\sqrt{n}}.$$ 

Applying this theorem to $Y = |X|^q - \mathbb{E}|X|^q$ and

$$s = \frac{\|X\|_3^{q}}{\|Y\|_2} \left( \frac{q^2 \log n}{2 \sqrt{n}} + O \left( \frac{1}{\sqrt{n}} \right) \right) = O \left( \frac{\log n}{\sqrt{n}} \right)$$

we get:

$$\left| \mathbb{P} \left( \left( \frac{1}{n} \sum_{j=1}^n |X_j|^q \right)^{1/q} > \|X\|_q a(n, q) \right) - \frac{1}{2} \right| \leq C \frac{\log n}{\sqrt{n}} + C \frac{1}{\sqrt{n}},$$

which proves the theorem. \qed
3. Central sections of $B^n_p$ 

Suppose $B$ is a convex symmetric body in $\mathbb{R}^n$ with $\text{Vol}_n(B) = 1$. It is well-known that there exists an affine image $\tilde{B}$ of $B$ such that the function 

$$x \mapsto \int_{\tilde{B}} \langle x, y \rangle^2 dy$$

is constant on $S^{n-1}$. This constant is called the isotropic constant of $B$ and is denoted by $L_B$. We also say that $B$ is in isotropic position. It is easy to see that if the standard basis of $\mathbb{R}^n$ is a 1-symmetric basis of $B$, then $B$ is in isotropic position.

Let $1 \leq p \leq 2$. Then

$$L^2_B = \int_{r(n,p)B^n_p} x^2 dx$$

where as above $r(n,p) = \text{Vol}_n(B^n_p)^{-1/n}$. A direct computation yields:

$$L^2_B = \frac{\Gamma(1 + \frac{n}{p})^2}{12 \Gamma(1 + \frac{n+2}{p}) \Gamma(1 + \frac{1}{p})^3}.$$ 

Let $H$ be a hyperplane containing the origin. A well known result (cf. e.g. [B1]) states that:

$$\text{Vol}_{n-1}(r(n,p)B^n_p \cap H)L^2_B \geq \frac{1}{\sqrt{12}}.$$ 

In order to prove $\text{Vol}_{n-1}(\alpha B^n_p \cap H) \geq 1$ it is enough to prove the following inequality:

$$\frac{\Gamma(1 + \frac{n}{p})^2}{12 \Gamma(1 + \frac{n+2}{p}) \Gamma(1 + \frac{1}{p})^3} \leq \frac{\Gamma(1 + \frac{1}{p})^3}{\Gamma(1 + \frac{2}{p})}.$$ 

By Stirling’s formula we have:

$$\Gamma(1 + z) = \sqrt{2\pi}(1 + z)^{z+\frac{1}{2}}e^{-1-z}\exp(\gamma(z))$$

where $\gamma$ is a decreasing function on the interval $[0, \infty)$ satisfying $0 < \gamma(z) < (12(z + 1))^{-1}$. Putting $x = p^{-1}$ the inequality (4) can be written equivalently:

$$(2\pi(1 + nx))^{1/n} \frac{1 + nx}{1 + 2x}^{1/2 + (n+2)x} e^{1/n + (1+2/n)\gamma(nx) - \gamma((n+2)x)} \leq 2\pi(1 + x)^{1/2 + 3x} e^{1 + 3\gamma(x) - \gamma(3x)}.$$ 

Putting $z_n = \frac{2x}{1 + (n+2)x}$ we get

$$\log \left( \frac{1 + nx}{1 + (n + 2)x} \right)^{1/2 + (n+2)x} - \log \left( \frac{1 + x}{1 + 3x} \right)^{1/2 + 3x} = \sum_{k=1}^{\infty} \left( \frac{1}{k} - \frac{2x}{k + 1} \right) (z_n^k - z_1^k) \leq (2x - 1) \log \left( \frac{1}{1 - z_1} \right).$$

On the other hand we have for all $n \geq 2$ and all $x > 0$:

$$(1 + nx)^{1/n} \leq 1 + x \quad \text{and} \quad 3\gamma(x) - \gamma(3x) - \left( 1 + \frac{2}{n} \right) \gamma(nx) + \gamma((n+2)x) \geq 0.$$
Therefore it suffices to prove that for all $\frac{1}{2} \leq x \leq 1$:

$$
\left( \frac{1 + 3x}{1 + x} \right)^{2x-1} \leq \sqrt{2\pi e}
$$

which follows from the fact that the left hand side is bounded by 2. With some modifications the proof given above also yields for all $n \geq 3$:

$$
\frac{\Gamma(1 + \frac{n}{p})^{1+2/n}}{\Gamma(1 + \frac{n+2}{p})} \leq \frac{\Gamma(1 + \frac{2}{p})^2}{\Gamma(1 + \frac{4}{p})}.
$$

Hence we have the following

**Proposition 3.1.** For all hyperplanes $H$ containing the origin, all $1 < p < 2$ and all $n \geq 2$ we have:

$$
\text{Vol}_{n-1}(r(n,p)B^n_p \cap H) \geq \sqrt{\frac{\Gamma(1 + \frac{4}{p})\Gamma(1 + \frac{1}{p})^3}{\Gamma(1 + \frac{2}{p})^2\Gamma(1 + \frac{3}{p})}} \geq 1.
$$

**References**


**Weizmann Institute of Science, Rehovot, Israel**

**Mathematisches Seminar, Universität Kiel, Germany**

**Institut für Mathematik, Universität Linz, Austria**

*E-mail address:* schmucki@caddo.bayou.uni-linz.ac.at

*Current address:* Institut für Mathematik, J. Kepler Universität, A-4040 Linz, Austria