THE DETERMINATION OF THE PAIRS OF TWO-BRIDGE KNOTS OR LINKS WITH GORDIAN DISTANCE ONE

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ABSTRACT. We thoroughly determine the pairs of two-bridge knots or links with Gordian distance one. In addition, we examine the Gordian distance between a Montesinos knot (or link) and a two-bridge knot (or link).

1. Introduction

For any two knots or links $K$, $K'$ in $S^3$, we can define the Gordian distance from $K$ to $K'$, denoted by $d_G(K,K')$, to be the minimal number of crossing changes needed to deform a diagram of $K$ into that of $K'$, where the minimum is taken over all diagrams of $K$ from which one can obtain a diagram of $K'$.

Then $d_G$ defines a metric on the space of the equivalence classes of knots or links. If $O$ is a trivial knot or link, then $d_G(K,O)$ is the unknotting or unlinking number of $K$, denoted by $u(K)$ (see [15]).

In this paper we determine the pairs of two-bridge knots or links with Gordian distance one. This result can be thought of as a generalization of those of Kanenobu-Murakami [7] and Kohn [8].

After having done this work, the author heard that J. Berge and I. Dacey-D. W. Sumners had independently obtained a result similar to the main theorem, respectively (see [4]).

Throughout this paper we say that $K$ and $K'$ are equivalent, denoted by $K = K'$, if and only if there exists an orientation preserving homeomorphism of $S^3$ which maps $K$ to $K'$.

2. Main theorem

Let $S(p,q)$ be the two-bridge knot or link whose two-fold branched cover is the lens space $L(p,q)$, where $p$ and $q$ are relatively prime. When $p$ is even, $S(p,q)$ is a two-component link, for $p$ odd, $S(p,q)$ is a knot.

$S(p,q)$ and $S(p',q')$ are equivalent if and only if $p = p'$ and (I) $q \equiv q' \pmod{p}$ or (II) $qq' \equiv 1 \pmod{p}$ [2, Theorem 12.6 (b)].

Our main theorem is then the following.

Theorem 1. Let $S(p,q)$ and $S(r,s)$ be two-bridge knots or links. Then the following conditions are equivalent:

(i) $d_G(S(p,q), S(r,s)) = 1$.

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There exist pairs of relatively prime integers \((m, n)\) and \((a, b)\) such that \(n \neq 0, rm + an \neq 0, rb - sa = 1\) and \(S(p, q)\) is equivalent to
\[
S(2an^2 + r(2mn \pm 1), 2bn^2 + s(2mn \pm 1)).
\]

There exist rational numbers \(r_1\) and \(r_2\) such that \(S(p, q) = S(r, s)\) where \(S\) is a rational tangle of slope \(r_i\) (for the definition of a rational tangle, see [2, Chapter 12]).

Remark 2. (i) \(S(1, 0)\) is a trivial knot. So the condition (ii) of Theorem 1 says that \(d_G(S(p, q), S(1, 0)) = 1\) if and only if
\[
S(p, q) = S(2an^2 + 2mn \pm 1, 2n^2) = S(2m'n \pm 1, 2n^2)
\]
where \(m' = m + an\). Therefore Theorem 1 is a generalization of Kanenobu-Murakami’s theorem [7].

(ii) \(S(0, 1)\) is a trivial link. So the condition (ii) of Theorem 1 also says that \(d_G(S(p, q), S(0, 1)) = 1\) if and only if
\[
S(p, q) = S(-2n^2, 2bn^2 + 2mn \pm 1) = S(-2n^2, 2m'n \pm 1)
\]
where \(m' = m + bn\). Therefore Theorem 1 is also a generalization of Kohn’s theorem [8].

3. Preliminaries

Let \(N(k)\) be a regular neighborhood of a knot \(k\) in a closed orientable 3-manifold \(M\), with \(\mu\) a meridian of \(N(k)\). Let \(E(k)\) be the exterior of \(k\) in \(M\), that is, \(E(k) = M - \text{int}N(k)\). Now, let \(k(\gamma)\) denote the closed manifold obtained by attaching a solid torus \(V\) to \(E(k)\) so that a curve of slope \(\gamma\) on \(\partial E(k)\) bounds a disk in \(V\). Here the slope indicates the isotopy class of a nontrivial simple closed curve in \(\partial E(k)\). We shall say that \(k(\gamma)\) is the result of \(\gamma\)-surgery on \(k\) in \(M\). For two slopes \(\gamma\) and \(\delta\) in \(\partial E(k)\), let \(\Delta(\gamma, \delta)\) be their minimal geometric intersection number.

For oriented manifolds \(M\) and \(N\), \(M \cong N\) means \(M\) and \(N\) are homeomorphic by an orientation preserving homeomorphism.

**Lemma 3.** If \(d_G(S(p, q), S(r, s)) = 1\), then \(L(p, q)\) is obtained by \(\gamma\)-surgery on some knot in \(L(r, s)\), where \(\Delta(\gamma, \mu) = 2\).
Proof. This is obtained by an argument similar to that of the proof of [9, Lemma 1] (cf. [7], [8]). In fact this follows from Montesinos’ technique [11] and the fact that the double branched covering of $S^3$ along $S(r, s)$ is $L(r, s)$.

Let $m_g^c((\alpha_1, \beta_1), \cdot \cdot \cdot , (\alpha_t, \beta_t))$ be a Seifert fibred space, where $g$ is the genus of the orbit surface $F$, $c$ is the number of boundary components, and $t$ is the number of surgery instructions used to obtain the Seifert fibred space from the genuine (orientable) $S^1$-bundle over $F$. Each pair $(\alpha_i, \beta_i)$ specifies a particular surgery (for example, see [10, Chapter Four]). Our convention shall be that when $g$ is nonnegative, $F$ is orientable, while $g$ negative implies that $F$ is nonorientable ($F = \biguplus_{i=1}^{|g|} RP^2$). For later use, we note that $m_g^c((\alpha_1, \beta_1), \cdot \cdot \cdot , (\alpha_t, \beta_t), (1, e)) \cong m_g^c((\alpha_1, \beta_1), \cdot \cdot \cdot , (\alpha_t, \beta_t + e\alpha_i), \cdot \cdot \cdot , (\alpha_t, \beta_t))$ for any integer $e$ and any number $i$ ($1 \leq i \leq t$).

**Lemma 4.** Let $k$ be a knot in $L(r, s)$. If $E(k)$ is a Seifert fibred space, then $k$ is a (possibly singular) fiber in some Seifert fibration of $L(r, s)$.

Proof. Let $V$ be a solid torus with meridian $\mu$. By hypothesis $E(k)$ is Seifert fibred and $L(r, s)$ is the union of $E(k)$ and $V$ along the boundary. The core of $V$ is isotopic to $k$ in $L(r, s)$. If $E(k) = m_g^c((\alpha_1, \beta_1), \cdot \cdot \cdot , (\alpha_t, \beta_t))$, then by using the fact that $H_1(k(\mu))$ is a cyclic group (cf. [8]), we see $g = 0$ or $-1$. There are also two cases to consider: $\mu$ is either identified with a fiber of $E(k)$ or not. If $\mu$ is not a fiber, then the Seifert fibration extends on the resulting manifold and $k$ is a fiber. Therefore hereafter we assume $\mu$ is a fiber. First we assume $g = 0$, then $t \leq 1$ because if otherwise $L(r, s)$ has a separating essential 2-sphere by a standard argument (see [6]), a contradiction. Therefore in this case $E(k)$ is a solid torus. Hence the statement follows immediately by re-fibering $E(k)$. Second we assume $g = -1$. Then $t \leq 0$ by a standard argument as above. Then $E(k)$ is a twisted $S^1$-bundle over a Möbius band and it is homeomorphic to $S^2 \times S^1$ minus a regular neighborhood of a “(2,1)-torus knot” (see [8, Lemma 4]), and this is homeomorphic to a twisted annulus bundle over a circle. And the fibers are parallel circles on the annuli. Therefore as in [8, Lemma 4], the statement of Lemma 4 follows by re-fibering $E(k)$.

**Lemma 5** (The classification of the Seifert fibration for a lens space). Suppose $m_g^0((\alpha_1, \beta_1), \cdot \cdot \cdot , (\alpha_t, \beta_t))$ ($\alpha_i \geq 2$) is a lens space. Then one of the following conditions holds.

1. $g = 0$ and $t \leq 2$.
2. $g = -1$ and $t \leq 1$.

Proof. This can be proved by a standard argument as in the proof of Lemma 4.

**Lemma 6.** If $d_G(S(p, q), S(r, s)) = 1$, then for some Seifert fibering of $L(r, s)$, $L(p, q)$ is obtained by $\gamma$-surgery along a fiber, where $\Delta(\gamma, \mu) = 2$.

Proof. From Lemma 3 we know that $L(p, q)$ is obtained by $\gamma$-surgery on some knot $k$ in $L(r, s)$, where $\Delta(\gamma, \mu) = 2$. So both $\gamma$ and $\mu$ are cyclic slopes; that is, $\gamma$ and $\mu$-surgery yield the manifold with cyclic fundamental groups. By the Cyclic Surgery Theorem [3], $E(k)$ is reducible or Seifert fibred. If $E(k)$ is reducible, then $k(\gamma)$ has $L(r, s)$ as a connected summand. Hence $L(p, q) = k(\gamma) = L(r, s)$, which contradicts our assumption. Therefore $E(k)$ is a Seifert fibred space. Lemma 4 implies that $k$ is a fiber in some Seifert fibration of $L(r, s)$.
Then, since $L$ is a meridian and a longitude of $(i)$ This is obtained by using the fact that an ordinary fiber on $V$ image of the meridian of $V$ and $\lambda$ be integers such that $rb - sa = 1$. Then we may assume $h(\lambda_1) = by_2 + a\lambda_2$ and $C_{m,n}$ is isotopic to $(sm + bn)\mu_2 + (rm + an)\lambda_2$ on $\partial V_2 = \partial V$. Let $c/d$-surgery along $L(r, s)$ such that $\Delta(\gamma, \mu) = 2$. By Lemma 7 (i), $L(p, q) \cong m_0^0((n, x), (rm + an, y), (c - 2mn, 2))$, where $L(r, s) \cong m_0^0((n, x), (rm + an, y))$. Then, since $|m|, |rm + an| \geq 2$, we have $c - 2mn = \epsilon$ where $\epsilon = \pm 1$ by Lemma 5. So by Lemma 7 (ii), $L(p, q) \cong L(2am^2 + r(2mn + 1), 2bm^2 + s(2mn + 1))$. Therefore $S(p, q)$ is equivalent to $S(2am^2 + r(2mn + 1), 2bm^2 + s(2mn + 1))$, where $(m, n)$ and $(a, b)$ satisfy the condition of the statement (ii). Secondly we consider the case where $|n| = 1$ or $|rm + an| = 1$. Here, without loss of generality, we can assume $|n| = 1$. Then if $c = 2m^2 + 1$ for some integer $m$, we can put $k = C_{m, 1}$. Therefore as in the case $n \geq 2$, by Lemma 7 (ii), the statement follows.
Next we consider the $RP^2$ case. Then $L(r, s) = E(k) \cup_{id} N(k)$ and $L(p, q) = E(k) \cup_h N(k)$, where $\Delta(h(\mu), \mu) = 2$. If $k$ is an ordinary fiber, then the core of $N(k) \subset E(k) \cup_h N(k)$ is a singular fiber, since $\Delta(h(\mu), \mu) = 2$. Therefore by Lemma 5, $E(k)$ is a twisted $S^1$-bundle over a Möbius band whether $k$ is a singular fiber or not. As in the proof of Lemma 4, $L(r, s)$ and $L(p, q)$ are obtained by surgery along a “(2,1)-torus knot” in $S^2 \times S^1$. Therefore, by re-fibering $E(k)$, we see $L(r, s) \cong m^0_0((2, 1), (2, 1), (1, 1))$ and $L(p, q) \cong m^0_0((2, 1), (2, 1), (1, 1))$ for some $d$ (cf. [8, p.1139]). Hence, by changing invariants $L(r, s) \cong m^0_0((2, 1), (2, 2d - 1))$ and $L(p, q) \cong m^0_0((2, 1), (2, 2d - 1), (1, 1))$. Then $L(p, q)$ is obtained by $c/2$-surgery along $C_{2, 1} \subset V_1 \cup V_2 = L(r, s)$ for some $z$, where $h(\mu_1) = r\lambda_2 + s\mu_2$, $h(\lambda_1) = a\lambda_2 + b\mu_2$, $rb - sa = 1$ and $c = 4z \pm 1$. Therefore as in the $S^2$ case, $(p, q)$ has the desired description.

(ii) $\Rightarrow$ (iii). As above $L(p, q)$ can be obtained by $c/2$-surgerying along $C_{m, n} \subset V_1 \cup V_2 = L(r, s)$, where $V_1 \cup V_2$ is as in Section 3. Then $L(p, q) \cong m^0_0((n, x), (rn + an, y), (1, 1))$, where $L(r, s) \cong m^0_0((n, x), (rn + an, y))$ similarly. Hence $L(p, q)$ and $L(r, s)$ are the double covers of the Montesinos knots or links (see [2, Chapter 12]) as in the statement (iii) of Theorem 1, where $r_1 = x/n$ and $r_2 = y/(rn + an)$. Therefore $S(p, q)$ and $S(r, s)$ have the desired description.

(iii) $\Rightarrow$ (i). Since $\left(\begin{array}{c}2n \\ m\end{array}\right) = \gcd(2n, m) or \left(\begin{array}{c}2n \\ m\end{array}\right)$, we can easily see $d_G(S(p, q), S(r, s)) \leq 1$.

It remains to prove $d_G(S(p, q), S(r, s)) \neq 0$. Suppose $S(p, q)$ is not equivalent to $S(r, s)$. Then applying the above arguments to a crossing change in $\left(\begin{array}{c}2n \\ m\end{array}\right)$ by (ii), there exist pairs of relatively prime integers $(m, n)$ and $(a, b)$ such that $n \neq 0$, $rn + an \neq 0$, $rb - sa = 1$ and $(2an^2 + r(2mn \pm 1), 2bn^2 + s(2mn \pm 1))$ is equivalent to $S(r, s)$. Therefore it follows that $r = 2an^2 + r(2mn \pm 1)$ and (I) $s = 2bn^2 + s(2mn \pm 1)$ (mod $r$) or (II) $s = 2bn^2 + s(2mn \pm 1)$ (mod $r$). But under the above conditions, elementary number theory for the congruences easily proves these cases never occur. This makes a contradiction.

This completes the proof of Theorem 1.

5. Addendum

In this section we measure the Gordian distance between a Montesinos knot (or link) and a two-bridge knot (or link). Though the proof of Theorem 1 depends on the Cyclic Surgery Theorem, that of the following theorem will depend on the recent Boyer-Zhang’s theorem [1].

**Theorem 8.** Let $K = M((\alpha_1, \beta_1), \cdots, (\alpha_t, \beta_t))$ be a Montesinos knot or link with $\alpha_i \geq 2$, $t \geq 4$. Then $d_G(K, S(p, q)) \geq 2$, for any two-bridge knot or link $S(p, q)$.

**Remark 9.** In [14] Motegi independently proved Theorem 8 for the knot case.

**Proof of Theorem 8** (cf. [13, 17]). We assume that $d_G(K, S(p, q)) = 1$. Then by analogy of Lemma 3, the double cover of $M_K$ of $K$ is obtained by $\gamma$-surgery on some knot $k$ in $L(p, q)$, where $\Delta(\gamma, \mu) = 2$. Here $M_K = m^0_0((\alpha_1, \beta_1), \cdots, (\alpha_t, \beta_t))$. We remark that $E(k)$ is irreducible. Because if otherwise, then $k(\gamma)$ must be a reducible Seifert fibred space. This contradicts our assumption. Hence by Boyer-Zhang’s result [1, Theorem 1] $E(k)$ is a Seifert fibred space or a cable on a Seifert fibred space.
If $E(k)$ is the former, then, by Lemma 4, $k$ is a fiber for some Seifert fibration of $L(p, q)$. Then, by an argument as in Section 3, we can prove $k(\gamma)$ can never be such a Seifert fibred space as the number of singular fibers $\geq 4$. If $E(k)$ is the latter, then there exists a cable space $C$ such that $\partial C = T_1 \amalg T_2$, where $\partial E(k) = T_1$ and

$$E(k) = m^k_g((\alpha_1, \beta_1), \ldots, (\alpha_u, \beta_u)) \cup T_2 C$$

for some integers $g$ and $u$. For any slope $\delta$ on $T_1 = \partial(k)$, let $C(\delta)$ denote the manifold obtained by $\gamma$-surgery on $k$ in $C$. Since $k(\mu) = L(p, q)$, $C(\mu)$ must be a solid torus. Let $k'$ be a core of $C(\mu)$. We can regard $k'$ as a knot in $L(p, q)$. Then

$$E(k') = m^k_g((\alpha_1, \beta_1), \ldots, (\alpha_u, \beta_u)).$$

Again by Lemma 4, $k'$ is a fiber for some Seifert fibration of $L(p, q)$. Hence we can also prove that $k(\gamma) = E(k') \cup C(\mu)$ cannot be such a Seifert fibred space.

This makes a contradiction.

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