AN ESTIMATE ON THE DISTORTION OF THE LOGARITHMIC CAPACITY

MARÍA J. GONZÁLEZ

(Communicated by Linda Keen)

Abstract. Let \( \Gamma \) be a Fuchsian group. We show that the existence of a set on \( \partial \mathbb{D} \) with no \( \Gamma \)-equivalent points and positive logarithmic capacity does not imply that \( \Gamma \) is of convergence type.

Let \( \Gamma \) be a Fuchsian group, that is, a discontinuous group of Möbius transformation of \( \mathbb{D} = \{ z \in \mathbb{C} : |z| < 1 \} \) onto itself.

In [P] Pommerenke asks the following question: If there exists a Borel set on \( \partial \mathbb{D} \) of positive capacity that contains no \( \Gamma \)-equivalent points, does it follow that \( \Gamma \) is of convergence type?

The following theorem shows that the answer is NO.

Theorem 1. There exist a Denjoy domain \( \Omega \) and a normal fundamental domain \( F \) associated to \( \Omega \) such that

(a) \( \text{cap}(\partial F \cap \partial \mathbb{D}) > 0 \), and

(b) \( \text{cap} \partial \Omega = 0 \) (\( \text{cap} \) denotes the logarithmic capacity).

The rest of this paper will be devoted to proving the theorem, but first I would like to thank C. Bishop and P. Jones for many helpful conversations.

Proof. We will construct a Denjoy domain satisfying (a) and (b), but before doing that we need an estimate on harmonic measure.

Suppose \( \Omega \subset \mathbb{R}^+ = \{ y > 0 \} \) is a domain bounded by two orthogonal circles of radius 1 at distance \( \delta \). Denote by \( I \) this interval on \( \mathbb{R} \) of length \( \delta \), which can be assumed to be centered at 0, i.e., \( I = (-\delta, \delta) \). Let \( z_0 = 2i \) (Figure 1). Then

\[
\omega(z_0, I, \Omega) \leq c_1 e^{-c_2 \sqrt{\delta}},
\]

where \( c_1, c_2 \) are universal constants.

We prove this statement by using an extremal length argument.

Let \( F \) be the family of curves in \( \Omega \) separating \( z_0 \) from \( I \), and let \( \rho \) be admissible for \( F(\rho \in \mathcal{A}(F)) \). For each \( t \), \( 0 \leq t \leq 1 \), consider the line \( \gamma_t = it \) joining both orthogonal circles. Then

\[
\int_{\gamma_t} \rho \, ds \geq 1.
\]
By Hölder’s inequality

\[ 1 \leq \left( \int_{\gamma_t} \rho^2 \, ds \right) l(\gamma_t), \]

where \( l(\gamma_t) \) denotes the length of \( \gamma_t \). Integrating on \( t \) we get

\[ \int \int \rho^2 \, dx \, dy \geq \int_0^1 \left( \int_{\gamma_t} \rho^2 \, ds \right) \, dt \geq \int_0^1 \frac{dt}{l(\gamma_t)}. \]

Since \( l(\gamma_t) \simeq \delta + 2t^2 \), we obtain

\[ \int_0^1 \frac{dt}{l(\gamma_t)} \simeq \int_0^1 \frac{dt}{\delta + 2t^2} \simeq \int_0^{\sqrt{\delta}/2} \frac{dt}{\delta} + \int_{\sqrt{\delta}/2}^1 \frac{dt}{2t^2} \simeq \frac{1}{\sqrt{\delta}}. \]

Hence,

\[ M(F) = \inf_{\rho \in A(F)} \int \rho^2 \, dx \, dy \geq \frac{c}{\sqrt{\delta}}, \]

Therefore, by Beurling’s Theorem

\[ \omega(z_0, I, \Omega) \leq c_1 e^{-c_2 M(F)} \leq c_1 e^{-c_2 \frac{1}{\sqrt{\delta}}}, \]

which ends the proof of (1).

Next we consider a Cantor set \( E = \bigcap E_n \subset \mathbb{R} \), where its \( n \)-th approximation \( E_n \) consists of \( 2^n \) intervals \( \{I_n\} \) of length \( l_n = e^{-n^2} \) (Figure 2). Then \( E \) has positive capacity if and only if \( \sum 2^{-n} \log \frac{1}{l_n} < \infty \). (See [C, pg. 29].) In this case

\[ \sum 2^{-n} \log e^{2n} = \sum n^2/2^n < \infty, \]

and therefore \( \text{cap}(E) > 0 \).

We are now ready to construct “half” of the fundamental domain, \( \widetilde{F} \). To do so we draw orthogonal circles supported on the intervals \([-1, 1] \setminus E \) (Figure 2).

We obtain a normal fundamental domain \( F \) by reflecting \( \widetilde{F} \) across the orthogonal circle that contains the points 1 and -1. Note that \( \text{cap}(\partial F \cap \mathbb{R}) > 0 \).

To get the correspondent Denjoy domain \( \Omega \) we send \( \widetilde{F} \) conformally onto \( \mathbb{R}_+^2 \). Denote by \( \Phi \) the conformal map so that \( \Phi(i) = \infty \). Then \( \partial \Omega = \Phi(E) \). Besides, \( \partial \Omega \) looks like a “Cantor set” where the length of the intervals at the \( n \)-th stage are less than \( c_1 e^{-c_2 e^n} \) (this is just a consequence of (1)). Therefore \( \text{cap}(\partial \Omega) = 0 \). \( \square \)
AN ESTIMATE ON THE DISTORTION OF THE LOGARITHMIC CAPACITY

REFERENCES


DEPARTAMENT DE MATEMÀTIQUES, UNIVERSITAT AUTÒNOMA DE BARCELONA, 08193 Bella-
terra, Barcelona, Spain

E-mail address: gonzalez@mat.uab.es

Current address: Departamento de Matemáticas, CASEM, Universidad de Cádiz, 11519 Puerto Real, Cádiz, Spain

E-mail address: majose.gonzalez@uca.es