

## A NOTE ON GREENBERG'S CONJECTURE AND THE ABC CONJECTURE

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ABSTRACT. For any totally real number field  $k$  and any prime number  $p$ , Greenberg's conjecture for  $(k, p)$  asserts that the Iwasawa invariants  $\lambda_p(k)$  and  $\mu_p(k)$  are both zero. For a fixed real abelian field  $k$ , we prove that the conjecture is "affirmative" for infinitely many  $p$  (which split in  $k$ ) if we assume the *abc* conjecture for  $k$ .

### 1. INTRODUCTION

For a number field  $k$  and a prime number  $p$ , let  $k_\infty/k$  be the cyclotomic  $\mathbb{Z}_p$ -extension over  $k$  with its  $n$ th layer  $k_n$  ( $k_0 = k$ ). Let  $A_n$  be the Sylow  $p$ -subgroup of the ideal class group of  $k_n$  and  $A_\infty = \varprojlim A_n$  the projective limit w.r.t. the relative norms. We denote by  $\lambda_p = \lambda_p(k)$  and  $\mu_p = \mu_p(k)$  the Iwasawa  $\lambda$ -invariant and the  $\mu$ -invariant associated to  $A_\infty$ , respectively. Greenberg's conjecture for  $k$  and  $p$  asserts that  $\lambda_p = \mu_p = 0$  for any totally real number field  $k$  and any  $p$  (cf. [Iw], p. 316, [Gr]). It is well known that the conjecture is valid if (1) there is only one prime ideal of  $k$  over  $p$  and it is totally ramified in  $k_\infty$  and further (2)  $A_0 = \{1\}$  (cf. [W], Proposition 13.22). In particular,  $\lambda_p(\mathbb{Q}) = \mu_p(\mathbb{Q}) = 0$  for all  $p$ . Further, it is known that  $\mu_p = 0$  when  $k$  is abelian over  $\mathbb{Q}$  (cf. [FW]). But, the conjecture for general  $k$  and  $p$  is far from being settled in spite of the efforts of several authors (see [IS] and its references).

In this note, we consider the following subproblem: "*For a fixed totally real number field  $k$  ( $\neq \mathbb{Q}$ ), do there exist infinitely many prime numbers  $p$  for which  $\lambda_p = \mu_p = 0$ ?*" In view of the proposition in [W] cited above, we should confine ourselves to those  $p$  which *split* in  $k$ . We prove that for a certain real abelian field  $k$ , the problem is "affirmative" if we assume the *abc* conjecture for  $k$ . Here, the *abc* conjecture is formulated as follows:

**Conjecture** (cf. [V], p. 84). Let  $K$  be a number field. For any  $\varepsilon$  ( $> 0$ ) and any finite set  $S$  of prime ideals of  $K$ , there exists a constant  $C$  ( $> 0$ ) depending only

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on  $K, \varepsilon$  and  $S$  such that

$$(1) \quad \prod_v \max(\|a\|_v, \|b\|_v, \|c\|_v) \leq C \left( \prod_{\mathfrak{p}|abc}' N\mathfrak{p} \right)^{1+\varepsilon}$$

for all integers  $a, b, c$  of  $K$  with  $a + b = c$ . Here,  $v$  runs over all absolute values of  $K$ ,  $\| * \|_v$  denotes the normalized valuation and  $\mathfrak{p}$  runs over all prime ideals of  $K$  with  $\mathfrak{p}|abc$  and  $\mathfrak{p} \notin S$ .

Now, let  $k/\mathbb{Q}$  be a real abelian extension with  $k \neq \mathbb{Q}$ , and  $\Delta = \text{Gal}(k/\mathbb{Q})$ . For a prime number  $p$  with  $p \nmid [k:\mathbb{Q}]$  and a  $\mathbb{Q}_p$ -character  $\Psi$  of  $\Delta$ , let  $\lambda_p(\Psi)$  and  $\mu_p(\Psi)$  be the  $\lambda$ -invariant and the  $\mu$ -invariant associated to the  $\Psi$ -component  $e_\Psi A_\infty$ , respectively. Here, a  $\mathbb{Q}_p$ -character means a  $\mathbb{Q}_p$ -valued character of  $\Delta$  defined and irreducible over  $\mathbb{Q}_p$ , and  $e_\Psi$  is the idempotent of  $\mathbb{Q}_p[\Delta]$  corresponding to  $\Psi$ , which is an element of  $\mathbb{Z}_p[\Delta]$  as  $p \nmid [k:\mathbb{Q}]$ . By [FW],  $\mu_p(\Psi) = 0$ . We have  $\lambda_p = \sum_\Psi \lambda_p(\Psi)$ ,  $\Psi$  running over all  $\mathbb{Q}_p$ -characters of  $\Delta$ . Further, for the trivial character  $\Psi_0$  of  $\Delta$ , we have  $\lambda_p(\Psi_0) = 0$  since  $\lambda_p(\Psi_0) = \lambda_p(\mathbb{Q})$ .

**Theorem 1.** *Let  $k/\mathbb{Q}$  be a real cyclic extension with  $[k:\mathbb{Q}]$  an odd prime number. If the abc conjecture for  $k$  is valid, then there exist infinitely many pairs  $(p, \Psi)$  of a prime number  $p$  (with  $p \nmid [k:\mathbb{Q}]$ ) and a nontrivial  $\mathbb{Q}_p$ -character  $\Psi$  of  $\Delta$  satisfying (I)  $p$  splits in  $k$  and (II)  $\lambda_p(\Psi) = 0$ .*

**Theorem 2.** *Let  $k/\mathbb{Q}$  be a real quadratic extension for which the norm of a fundamental unit is  $-1$ . If the abc conjecture for  $k$  is valid, then there exist infinitely many prime numbers  $p$  satisfying (I)  $p$  splits in  $k$  and (II)  $\lambda_p = 0$ .*

When (i)  $k/\mathbb{Q}$  is noncyclic or (ii)  $k/\mathbb{Q}$  is cyclic and  $[k:\mathbb{Q}]$  is a composite, an assertion similar to the above theorems holds *without* assuming the abc conjecture (see §4).

## 2. SOME LEMMAS

First, we introduce some notation. Let  $k/\mathbb{Q}$  be a real abelian extension with  $k \neq \mathbb{Q}$ ,  $p$  an odd prime number with  $p \nmid [k:\mathbb{Q}]$  and  $\Psi$  a  $\mathbb{Q}_p$ -character of  $\Delta = \text{Gal}(k/\mathbb{Q})$ . We fix  $p$  and  $\Psi$  in this section. Let  $\psi$  be a fixed irreducible component of  $\Psi$  over an algebraic closure  $\overline{\mathbb{Q}_p}$  of  $\mathbb{Q}_p$ , and let  $O = O_\psi$  be the subring of  $\overline{\mathbb{Q}_p}$  generated by the values of  $\psi$  over  $\mathbb{Z}_p$ . We identify the subring  $e_\Psi \mathbb{Z}_p[\Delta]$  of  $\mathbb{Z}_p[\Delta]$  with  $O$  by  $e_\Psi \sigma \leftrightarrow \psi(\sigma)$  ( $\sigma \in \Delta$ ). Then, for a  $\mathbb{Z}_p[\Delta]$ -module  $X$  (e.g.  $A_n, A_\infty$ ), its  $\Psi$ -component  $X(\Psi) = e_\Psi X$  (or  $X^{e_\Psi}$ ) is considered as an  $O$ -module. Therefore,  $A_\infty(\Psi)$  is regarded as a module over the completed group ring  $\Lambda_{p,\Psi} = O[[\text{Gal}(k_\infty/k)]]$ . It is known to be torsion over  $\Lambda_{p,\Psi}$  by [Iw], Theorem 5. Let  $r$  be the degree of the quotient field of  $O$  over  $\mathbb{Q}_p$ . The invariant  $\lambda_p(\Psi)$  (resp.  $\mu_p(\Psi)$ ) mentioned in §1 is  $r$  times the  $\lambda$ -invariant (resp.  $\mu$ -invariant) of the torsion  $\Lambda_{p,\Psi}$ -module  $A_\infty(\Psi)$ .

For a prime ideal  $\mathfrak{p}$  of  $k$  over  $p$ , let  $k_\mathfrak{p}$  be the completion of  $k$  at  $\mathfrak{p}$  and  $\mathcal{U}_\mathfrak{p}$  the group of principal units of  $k_\mathfrak{p}$ . We denote by  $\mathcal{U}$  the group of semi-local units of  $k$  at  $p$ , namely,  $\mathcal{U} := \prod_{\mathfrak{p}|p} \mathcal{U}_\mathfrak{p}$ ,  $\mathfrak{p}$  running over all prime ideals of  $k$  with  $\mathfrak{p}|p$ . The group  $E$  of global units of  $k$  is considered as a subgroup of  $\prod_{\mathfrak{p}|p} k_\mathfrak{p}^\times$ . Denote by  $\mathcal{E}$  the closure of  $E \cap \mathcal{U}$  in  $\mathcal{U}$ . The groups  $\mathcal{U}$  and  $\mathcal{E}$  can be regarded as  $\mathbb{Z}_p[\Delta]$ -modules in a natural way, and hence  $\mathcal{U}(\Psi)$  and  $\mathcal{E}(\Psi)$  are  $O$ -modules.

We regard  $\psi$  as a primitive Dirichlet character, and we denote its “dual” character by  $\psi^*$ . Namely,  $\psi^*$  is the primitive Dirichlet character associated to  $\omega\psi^{-1}$ , where  $\omega$  is the Teichmüller character  $\mathbb{Z}/p\mathbb{Z} \rightarrow \mathbb{Z}_p$ .

**Lemma 1** (cf. [IS], Remark 4). *If  $\psi(p) \neq 1$  and  $A_0(\Psi) = \{1\}$ , then we have  $\lambda_p(\Psi) = \mu_p(\Psi) = 0$ .*

**Lemma 2.** *Assume that  $\Psi$  is nontrivial. If  $A_0(\Psi) = \{1\}$  and  $\mathcal{U}(\Psi) = \mathcal{E}(\Psi)$ , then we have  $\lambda_p(\Psi) = \mu_p(\Psi) = 0$ .*

Lemma 1 is a refinement of the proposition in [W] cited in §1. Lemma 2 is already known when  $k$  is a real quadratic field by [FK]. The assertion for the general case and its proof were communicated to the author by Hiroki Sumida.

*Proof of Lemma 2.* Let  $M/k_\infty$  be the maximal pro- $p$  abelian extension unramified outside  $p$  and  $L/k_\infty$  the maximal unramified pro- $p$  abelian extension. Further, let  $M_0$  be the maximal abelian extension of  $k$  contained in  $M$  and  $K_0$  the Hilbert  $p$ -class field of  $k$ . The Galois groups  $\text{Gal}(M/k_\infty)$ ,  $\text{Gal}(L/k_\infty)$ , etc. are regarded as modules over  $\mathbb{Z}_p[\Delta]$  in a natural way. By class field theory,  $\text{Gal}(L/k_\infty)$  is canonically isomorphic to  $A_\infty$ . Therefore, as  $M \supset L$ , it suffices to show that  $\text{Gal}(M/k_\infty)(\Psi) = \{1\}$ . We have a canonical isomorphism  $\text{Gal}(M_0/K_0) \simeq \mathcal{U}/\mathcal{E}$  by class field theory (cf. [C], Theorem 1). From this, we see that  $\text{Gal}(M_0/K_0k_\infty)(\Psi)$  is isomorphic to  $\mathcal{U}(\Psi)/\mathcal{E}(\Psi)$  since  $\text{Gal}(M_0/K_0k_\infty)(\Psi) = \text{Gal}(M_0/K_0)(\Psi)$  as  $\Psi \neq \Psi_0$ . On the other hand,  $\text{Gal}(K_0k_\infty/k_\infty)(\Psi)$  is naturally isomorphic to  $A_0(\Psi)$ . Therefore, under the assumptions of Lemma 2, we obtain  $\text{Gal}(M_0/k_\infty)(\Psi) = \{1\}$  and hence  $\text{Gal}(M/k_\infty)(\Psi) = \{1\}$  by Nakayama’s lemma.  $\square$

**Lemma 3.** *Assume  $\psi^*(p) \neq 1$ . Let  $X$  be a closed Galois submodule of  $\mathcal{U}(\Psi)$  such that  $u_{\mathfrak{q}} \not\equiv 1 \pmod{\mathfrak{q}^2}$  for some element  $u = (u_{\mathfrak{p}})_{\mathfrak{p}|p}$  in  $X$  and some prime ideal  $\mathfrak{q}$  with  $\mathfrak{q}|p$ . Then, we have  $X = \mathcal{U}(\Psi)$ .*

*Proof.* We have  $\mathcal{U}(\Psi) \simeq O$  because of  $\psi^*(p) \neq 1$  (cf. [Gi], §2). Therefore,  $X = \mathcal{U}(\Psi)^A$  for some ideal  $A$  of  $O$  since  $X$  is an  $O$ -submodule of  $\mathcal{U}(\Psi)$ . We have  $A = p^a O$  for some integer  $a$  ( $\geq 0$ ) since the quotient field of  $O$  is unramified over  $\mathbb{Q}_p$  as  $p \nmid [k: \mathbb{Q}]$ . If  $a \geq 1$ , then we must have  $u_{\mathfrak{p}} \equiv 1 \pmod{\mathfrak{p}^2}$  for all  $u = (u_{\mathfrak{p}})$  in  $X$  and all  $\mathfrak{p}|p$ . Therefore, we obtain  $A = O$  and  $X = \mathcal{U}(\Psi)$ .  $\square$

The following lemma is easily proved and we do not give its proof.

**Lemma 4.** *Let  $K$  be a number field,  $\mathfrak{p}$  a prime ideal of  $K$  and  $\alpha$  an element of  $K$  relatively prime to  $\mathfrak{p}$ . If  $\alpha^n \equiv 1 \pmod{\mathfrak{p}}$  but  $\alpha^n \not\equiv 1 \pmod{\mathfrak{p}^2}$  for some integer  $n$ , then we have  $\alpha^{N\mathfrak{p}-1} \not\equiv 1 \pmod{\mathfrak{p}^2}$ .*

### 3. PROOF OF THE THEOREMS

Let  $k/\mathbb{Q}$  be (A) a real cyclic extension with  $[k: \mathbb{Q}]$  an odd prime number or (B) a real quadratic extension for which the norm of a fundamental unit is  $-1$ . In the case (A), take a totally negative unit  $\varepsilon$  of  $k$  with  $\varepsilon \neq -1$ . Then,  $N\varepsilon = -1$  as  $[k: \mathbb{Q}]$  is odd. Here,  $N$  denotes the norm map from  $k$  to  $\mathbb{Q}$ . In the case (B), let  $\varepsilon$  be a fundamental unit of  $k$ , for which we have  $N\varepsilon = -1$  by assumption. Let  $\|\ast\|_i$  ( $1 \leq i \leq [k: \mathbb{Q}]$ ) be the real absolute values of  $k$ . Replacing  $\varepsilon$  by  $\varepsilon^x$  for some large odd integer  $x$  if necessary, we may well assume that  $\|\varepsilon\|_i$  is so large (resp. so small) for all  $i$  with  $\|\varepsilon\|_i > 1$  (resp.  $\|\varepsilon\|_i < 1$ ) that

$$(2) \quad |N(1 - \varepsilon^m)| > |N(1 - \varepsilon^n)| \quad \text{when } m > n \geq 1.$$

**Claim 1.** Let  $\mathfrak{p}$  be a prime ideal of  $k$  with  $\mathfrak{p} \nmid 2$ . If  $\varepsilon^n \equiv 1 \pmod{\mathfrak{p}}$  for some odd integer  $n$ , then  $p = \mathfrak{p} \cap \mathbb{Q}$  splits completely in  $k$ .

Actually: Assume that  $p$  does not split completely in  $k$ . Then,  $\mathfrak{p}$  is the unique prime ideal of  $k$  over  $p$  since  $[k: \mathbb{Q}]$  is a prime number. So,  $(\varepsilon^\sigma)^n \equiv 1 \pmod{\mathfrak{p}}$  for all  $\sigma \in \Delta$ . Therefore, as  $n$  is odd,  $-1 = (N\varepsilon)^n \equiv 1 \pmod{\mathfrak{p}}$ . This contradicts  $\mathfrak{p} \nmid 2$ .

Now, we assume that the *abc* conjecture holds for  $k$ . Then, applying the inequality (1) for  $\varepsilon^n + (1 - \varepsilon^n) = 1$ , we see that for some constant  $C_1$ ,

$$(3) \quad |N(1 - \varepsilon^n)| \leq C_1 \left( \prod_{\mathfrak{p}|(1-\varepsilon^n)}' N\mathfrak{p} \right)^{3/2}$$

for all integers  $n$ . Here,  $\mathfrak{p}$  runs over all prime ideals of  $k$  with  $\mathfrak{p}|(1 - \varepsilon^n)$  and  $\mathfrak{p} \nmid 2(1 - \varepsilon)$ . Using this inequality, we show

**Claim 2.** Under the *abc* conjecture for  $k$ , for all sufficiently large  $n$  satisfying

$$(4) \quad (n, 2(1 - \varepsilon)) = 1,$$

there exists a prime ideal  $\mathfrak{p}$  of  $k$  such that

$$(5)_n \quad \mathfrak{p} \nmid 2(1 - \varepsilon), \quad \varepsilon^n \equiv 1 \pmod{\mathfrak{p}} \quad \text{and} \quad \varepsilon^n \not\equiv 1 \pmod{\mathfrak{p}^2}.$$

Actually: For an integer  $n$  with (4) and a prime ideal  $\mathfrak{p}$  of  $k$  satisfying  $\mathfrak{p}|(1 - \varepsilon^n)$  and  $\mathfrak{p} \nmid 2(1 - \varepsilon)$ , we see that  $\text{ord}_{\mathfrak{p}}(1 - \varepsilon^n) \leq C_2$  for some constant  $C_2$  independent of  $n$  and  $\mathfrak{p}$ , where  $\text{ord}_{\mathfrak{p}}(*)$  is the normalized additive valuation at  $\mathfrak{p}$ . This follows from  $(1 - \varepsilon^n)/(1 - \varepsilon) \equiv n \pmod{1 - \varepsilon}$  and  $(n, 1 - \varepsilon) = 1$  for  $\mathfrak{p}$  with  $\mathfrak{p}|(1 - \varepsilon)$  and from  $2 \nmid n$  for  $\mathfrak{p}$  with  $\mathfrak{p} \nmid 2$ . Therefore, by (2), for all sufficiently large  $n$  with (4), there exists a prime ideal  $\mathfrak{p}$  such that  $\varepsilon^n \equiv 1 \pmod{\mathfrak{p}}$  and  $\mathfrak{p} \nmid 2(1 - \varepsilon)$ . Assume that there are infinitely many  $n$  with (4) such that  $\varepsilon^n \equiv 1 \pmod{\mathfrak{p}^2}$  for all  $\mathfrak{p}$  satisfying  $\varepsilon^n \equiv 1 \pmod{\mathfrak{p}}$  and  $\mathfrak{p} \nmid 2(1 - \varepsilon)$ . For these  $n$ , we have

$$\prod_{\mathfrak{p}|(1-\varepsilon^n)}' N\mathfrak{p} \leq |N(1 - \varepsilon^n)|^{1/2}.$$

Combining this inequality with (3), we obtain

$$|N(1 - \varepsilon^n)| \leq C_1 |N(1 - \varepsilon^n)|^{3/4}.$$

This is a contradiction since the last inequality holds only for a finite number of  $n$  because of (2), and hence, Claim 2 is proved.

Let  $n_1$  and  $n_2$  be (sufficiently large) integers satisfying (4) and  $(n_1, n_2) = 1$ , and let  $\mathfrak{p}_i$  be a prime ideal of  $k$  satisfying  $(5)_{n_i}$  with  $n = n_i$  ( $i = 1, 2$ ). Assume  $\mathfrak{p}_1 = \mathfrak{p}_2$  ( $:= \mathfrak{p}$ ). Then, from  $\varepsilon^{n_i} \equiv 1 \pmod{\mathfrak{p}}$  and  $(n_1, n_2) = 1$ , we have  $\varepsilon \equiv 1 \pmod{\mathfrak{p}}$ , contradicting  $(5)_n$ . Thus, we must have  $\mathfrak{p}_1 \neq \mathfrak{p}_2$ . Therefore, by Claims 1, 2 and Lemma 4, we see that there exist *infinitely many* prime ideals  $\mathfrak{p}$  of  $k$  for which  $p = \mathfrak{p} \cap \mathbb{Q}$  splits completely in  $k$  and

$$(6) \quad \varepsilon^{N\mathfrak{p}-1} = \varepsilon^{p-1} \not\equiv 1 \pmod{\mathfrak{p}^2}.$$

Let  $\mathfrak{p}$  be a prime ideal of  $k$  satisfying the above two conditions. We may well assume that  $p = \mathfrak{p} \cap \mathbb{Q}$  is so large that

$$p \nmid [k: \mathbb{Q}] \cdot d_k \cdot h_k,$$

where  $d_k$  (resp.  $h_k$ ) is the discriminant (resp. the class number) of  $k$ . By (6) (and  $p \nmid [k: \mathbb{Q}]$ ), there exists a nontrivial  $\mathbb{Q}_p$ -character  $\Psi$  of  $\Delta$  such that  $(\varepsilon^{p-1})^{\varepsilon_{\Psi}} \neq 1$

mod  $\mathfrak{p}^2$ . Let  $\psi$  be, as before, an irreducible component of  $\Psi$  over  $\overline{\mathbb{Q}}_p$ . Then, by  $p \nmid d_k$ , the conductor of the dual character  $\psi^*$  of  $\psi$  is divisible by  $p$ , and hence  $\psi^*(p) \neq 1$ . Therefore, we have  $\mathcal{U}(\Psi) = \mathcal{E}(\Psi)$  by Lemma 3. Now, we obtain  $\lambda_p(\Psi) = \mu_p(\Psi) = 0$  from Lemma 2 and  $p \nmid h_k$ . Further, in the case (B) (= the real quadratic case), we have  $\lambda_p = \lambda_p(\Psi) + \lambda_p(\Psi_0) = 0$ . Thus, we have proved Theorems 1 and 2.  $\square$

*Remark 1.* Lang [L], p. 41, presents an argument which derives the existence of infinitely many primes  $p$  with  $2^{p-1} \not\equiv 1 \pmod{p^2}$  from the *abc* conjecture for  $\mathbb{Q}$ . In the above proof of Theorems 1 and 2, we have used this classical argument.

*Remark 2.* In the above proof of Theorems 1 and 2, the existence of a unit  $\varepsilon$  with  $N\varepsilon = -1$  is quite essential. The author could not handle a real quadratic field whose fundamental unit has norm 1 by the method in this note.

#### 4. REMARK

Let  $k/\mathbb{Q}$  be a real abelian extension with  $k \neq \mathbb{Q}$  and  $\psi$  a fixed nontrivial homomorphism from  $\Delta = \text{Gal}(k/\mathbb{Q})$  to  $\overline{\mathbb{Q}}^\times$ , where  $\overline{\mathbb{Q}}$  is an algebraic closure of  $\mathbb{Q}$ . Fixing an embedding of  $\overline{\mathbb{Q}}$  into  $\overline{\mathbb{Q}}_p$  for each prime  $p$ , we denote by  $\Psi_p$  the  $\mathbb{Q}_p$ -character of  $\Delta$  for which  $\psi$  is an irreducible component over  $\overline{\mathbb{Q}}_p$ . We also denote by  $k_\psi$  the subfield of  $k$  corresponding to  $\ker \psi$  by Galois theory.

Assume that (C)  $k/\mathbb{Q}$  is non-cyclic or (D)  $k/\mathbb{Q}$  is cyclic with  $[k:\mathbb{Q}]$  a composite. In the case (D), we further assume that  $k_\psi = k$ . Then, there exist infinitely many primes  $p$  satisfying (I)  $p$  splits in  $k$  and (II)  $\lambda_p(\Psi_p) = 0$ .

Actually: As is easily seen, there exist infinitely many  $p$  which remain prime in  $k_\psi$  but split in  $k$  (resp. which split but not completely in  $k$ ) in the case (C) (resp. (D)). For these  $p$ , we have  $\psi(p) \neq 1$ , and hence  $\lambda_p(\Psi_p) = 0$  if  $p \nmid [k:\mathbb{Q}]$  and  $p \nmid h_k$  by Lemma 1.

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