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A VERSION OF ZABRODSKY'S LEMMA

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Abstract. Zabrodsky's Lemma says: Suppose given a fibrant space Y and a homotopy fiber sequence $F \to E \to X$ with X connected. If the map $Y = \mathrm{map}(*, Y) \to \mathrm{map}(F, Y)$ which is induced by $F \to *$ is a weak equivalence, then $\mathrm{map}(X, Y) \to \mathrm{map}(E, Y)$ is a weak equivalence. This has been generalized by Bousfield. We improve on Bousfield's generalization and give some applications.

The lemma we refer to here is the following: Suppose given a fibrant space Y and a homotopy fiber sequence $F \to E \to X$ with X connected. If the map $Y = \mathrm{map}(*, Y) \to \mathrm{map}(F, Y)$ which is induced by $F \to *$ is a weak equivalence, then $\mathrm{map}(X, Y) \to \mathrm{map}(E, Y)$ is a weak equivalence. This result has been generalized by Bousfield in [B, Theorem 4.6]. Our statement is almost the same as [B, Theorem 4.6]. The only difference is that we assume in Theorem 1.1 that $Y = \mathrm{map}(*, Y) \to \mathrm{map}(F_k, Y)$ is a weak equivalence while Bousfield assume that $\mathrm{map}(\Omega k, Y)$ is a weak equivalence, where F_k is the homotopy fiber of the map k. Our assumption is indeed weaker. Our interest in Theorem 1.1 lies is the fact that it is a basic tool in comparing f-localization functors, and we will give some simple applications in this direction. The technique we use in the proof of Theorem 1.1 has been developed by Chachólski and Dror Farjoun.

0. Notations and preliminaries

Let Δ denote the simplicial category in which the objects are the ordered sets $[n] = \{0, 1, ..., n\}$ and morphisms are nondecreasing maps of sets. A simplicial set (or space) is a contravariant functor K from Δ to the category of sets. As usual we write K_n for $K([n])$, the n-simplices. Let *S.sets* denote the category of simplicial sets (or spaces). The standard n-simplex $\Delta[n]$ is the simplicial set with k-simplices $\mathrm{Hom}_\Delta([k], [n])$. There is a distinguished n-dimensional simplex $\tau \in \Delta[n]$ which comes from the identity map $[n] \to [n]$. And $\dot{\Delta}[n]$ denotes the 'boundary' simplex of $\Delta[n]$.

For each space K, there is a canonical way to associate a category to K, called the transport category of K or the Grothendieck construction of K. The objects of this category are pairs $([n], \sigma)$, where $[n]$ is an object of Δ^{op} and $\sigma \in K_n$. A morphism $([n], \sigma) \to ([m], \tau)$ of this category consists of a map $\phi\colon [n] \to [m]$ in Δ^{op} such that $K(\phi)(\sigma) = \tau$. We will use the same symbol K to denote both the space K

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and its associated category. A *diagram with shape* K is a functor $F\colon K \to S.sets$. The homotopy colimit is the following functor:

$$\oint_K : \{\text{diagrams over } K\} \to S.sets,$$

$$\oint_K F = \Big(\coprod_{\sigma\in K} \Delta[\dim\sigma] \times F(\sigma)\Big)/\sim,$$

where $\sim$ is an equivalence relation generated by $(\Delta[\phi](t), x) \sim (t, F(\phi)(x))$ for $\phi \in \mathrm{Hom}_\Delta([n],[m])$, $\tau \in K_m$, $x \in F(\tau)$ and $t \in \Delta[n]$.

A diagram F is called *bounded* if for every degeneracy morphism $s_i\colon \sigma \to s_i\sigma$, $F(\sigma) = F(s_i\sigma)$ and $F(s_i) = id_{F(\sigma)}$. By [C1, 3.12], we know that every map $f\colon X \to Y$ is weakly equivalent to a map $\oint_K F \to K$ for some bounded diagram $F\colon K \to S.sets$ such that $F(\sigma)$ is weakly equivalent to the homotopy fiber of f for each simplex $\sigma \in K$. Moreover, for a bounded diagram $F\colon K = L \cup_{\dot\Delta[n]} \Delta[n] \to S.sets$, by [C1, 3.13], $\oint_K F$ is the homotopy push-out of the diagram

$$\oint_L F \longleftarrow \dot\Delta[n] \times F(\tau) \longrightarrow \Delta[n] \times F(\tau),$$

where τ is the distinguished n-simplex in $\Delta[n]$.

For any map f, the homotopy fiber and homotopy cofiber of f are denoted by F_f and C_f respectively.

1. Zabrodsky's Lemma

We are going to prove the following version of Zabrodsky's Lemma (cf. [B, Theorem 4.6]). The technique we use here has been developed by Chachólski and Dror Farjoun.

Theorem 1.1. *Let*

$$\begin{tikzcd} F \arrow[r] \arrow[d, "g"'] & E \arrow[r] \arrow[d, "h"'] & X \arrow[d, "k"'] \\ F' \arrow[r] & E' \arrow[r] & X' \end{tikzcd}$$

be a map of homotopy fiber sequences with X and X' connected, and let F_k be the homotopy fiber of k. For any fibrant space Y, if the maps $\mathrm{map}(g, Y)$ *and* $Y = \mathrm{map}(*, Y) \to \mathrm{map}(F_k, Y)$ *which is induced by $F_k \to *$ are weak equivalences, then* $\mathrm{map}(h, Y)$ *is a weak equivalence. If Y is pointed and connected, we can use* $\mathrm{map}_*(-,-)$ *instead of* $\mathrm{map}(-,-)$. (*Of course, the assumption on F_k becomes* $\mathrm{map}_*(F_k, Y) \simeq *$.) *For a spectrum E, if g is an E^*-equivalence* (*resp., E_*-equivalence*) *and F_k is E^*-acyclic* (*resp., E_*-acyclic*), *then so is h.*

Proof. We will give the proof for the case $\mathrm{map}(-,-)$ only; other cases can be proved as in [B, Theorem 4.6].

Case 1: Assume $X \simeq X'$. Then the diagram can be replaced (up to weak equivalences) by

$$\begin{tikzcd} \oint_K F \arrow[r] \arrow[d, "h"'] & K \arrow[d, "1"'] \\ \oint_K F' \arrow[r] & K \end{tikzcd}$$

for some bounded diagrams F and F', where $K \simeq X' \simeq X$, $F(\sigma) \simeq F$ and $F'(\sigma) \simeq F'$ for each $\sigma \in K$. We proceed by induction on the dimension of K. If $\dim K = 0$, it is clear that Case 1 is true. Suppose $K = L \cup_{\dot{\Delta}[n]} \Delta[n]$, $\dim L < n$ and $\mathrm{map}(l, Y)\colon \mathrm{map}(\oint_L F', Y) \to \mathrm{map}(\oint_L F, Y)$ is a weak equivalence. Now by [C1, 3.13], $h\colon \oint_K F \to \oint_K F'$ is the homotopy push-out of the following diagram:

$$\begin{tikzcd}
\oint_L F \arrow[d, "l"'] & \dot{\Delta}[n] \times F(\tau) \arrow[l] \arrow[r] \arrow[d, "1\times g"'] & \Delta[n] \times F(\tau) \arrow[d, "1\times g"'] \\
\oint_L F' & \dot{\Delta}[n] \times F'(\tau) \arrow[l] \arrow[r] & \Delta[n] \times F'(\tau)
\end{tikzcd}$$

where τ is the distinguished n-simplex in $\Delta[n]$. Since $\mathrm{map}(1\times g, Y)$'s are weak equivalences, the homotopy push-out h also induces a weak equivalence in $\mathrm{map}(-, Y)$. This finishes the proof of Case 1.

Case 2: Assume the right hand square in Theorem 1.1 is a homotopy pull-back. Then $F \simeq F'$, and the right hand square can be replaced (up to weak equivalences) by

$$\begin{tikzcd}
\oint_K F \times G \arrow[r] \arrow[d, "h"'] & \oint_K G \arrow[d, "k"'] \\
\oint_K F \arrow[r] & K
\end{tikzcd}$$

for some bounded diagrams F and G, where $K \simeq X'$, $G(\sigma) = F_k$, $F(\sigma) \simeq F \simeq F'$ and $(F \times G)(\sigma) = G(\sigma) \times F(\sigma)$ for each $\sigma \in K$. Again we proceed by induction on the dimension of K. If $\dim K = 0$, it is clear that Case 2 is true. Suppose $K = L \cup_{\dot{\Delta}[n]} \Delta[n]$, $\dim L < n$ and $\mathrm{map}(F_k, Y) \simeq Y$. Now by [C1, 3.13], $h\colon \oint_K F \times G \to \oint_K F$ is the homotopy push-out of the diagram

$$\begin{tikzcd}
\oint_L F \times G \arrow[d, "l"'] & \dot{\Delta}[n] \times F(\tau) \times G(\tau) \arrow[l] \arrow[r] \arrow[d, "m"'] & \Delta[n] \times F(\tau) \times G(\tau) \arrow[d, "n"'] \\
\oint_L F & \dot{\Delta}[n] \times F(\tau) \arrow[l] \arrow[r] & \Delta[n] \times F(\tau)
\end{tikzcd}$$

where τ is the distinguished n-simplex in $\Delta[n]$. Using the adjunction of the mapping space $\mathrm{map}(-,-)$, it is easy to see that $\mathrm{map}(m, Y)$ and $\mathrm{map}(n, Y)$ are weak equivalences. By induction, $\mathrm{map}(l, Y)$ is also a weak equivalence. Hence, $\mathrm{map}(h, Y)$ is a weak equivalence. This finishes the proof of Case 2.

General case: Let P be the homotopy pull-back of $E' \to X'$, and $k\colon X \to X'$. Then by Case 2, $i\colon P \to E'$ induces a weak equivalence on $\mathrm{map}(-, Y)$. Since the fiber of $P \to X$ is F', the induced map from F to F' is g itself. Hence, we have the following commutative diagram, and it satisfies Case 1:

$$\begin{tikzcd}
F \arrow[r] \arrow[d, "g"'] & E \arrow[d, "j"'] \\
F' \arrow[r] & P.
\end{tikzcd}$$

By Case 1, $\mathrm{map}(j, Y)$ is a weak equivalence. Since the required map h is the composite $i \circ j$ for which both i and j induce weak equivalences on $\mathrm{map}(-, Y)$, $\mathrm{map}(h, Y)$ is a weak equivalence. □

Remarks. 1. The difference between the assumptions of Theorem 1.1 and [B, Theorem 4.6] is that we assume $\operatorname{map}(F_k, Y) \simeq Y$ while Bousfield assumes that $\operatorname{map}(\Omega k, Y)$ is a weak equivalence. Our assumption is actually weaker. Indeed, consider the principle fibration $\Omega X \to \Omega X' \to F_k$. We know that F_k can be built as a homotopy colimit from the cofiber $C_{\Omega k}$ of the map Ωk (cf. [C2, Corollary 9.2]). So, if $\operatorname{map}(\Omega k, Y)$ is a weak equivalence, then $\operatorname{map}(C_{\Omega k}, Y) \simeq Y$. Hence, $\operatorname{map}(F_k, Y) \simeq Y$. Here is an example of a case when $\operatorname{map}(F_k, Y) \simeq Y$ but $\operatorname{map}(\Omega k, Y)$ is not a weak equivalence. Consider the trivial fiber sequence

$$S^2 \xrightarrow{r} S^2 \times S^1 \xrightarrow{k} S^1,$$

where r is the inclusion to the first factor of $S^2 \times S^1$ and k is the projection to the second factor of $S^2 \times S^1$. Take $Y = S^1$. It is clear that $\operatorname{map}_*(S^2, S^1) \simeq *$, and since S^1 is connected, $\operatorname{map}(S^2, S^1) \simeq S^1$. Now, if $\operatorname{map}(\Omega k, S^1)$ were a weak equivalence, $\operatorname{map}_*(\Omega k, S^1)$ would be a weak equivalence too. However, $\operatorname{map}_*(\Omega S^1, S^1)$ is connected but $\operatorname{map}_*(\Omega(S^2 \times S^1), S^1)$ is not. Thus, $\operatorname{map}(\Omega k, S^1)$ is not a weak equivalence.

2. We may ask whether the condition $\operatorname{map}(F_k, Y) \simeq Y$ (or F_k is E_*-acyclic) can be further weakened to the condition that $\operatorname{map}(k, Y)$ is a weak equivalence (or k is a E_*-equivalence). The following example shows that this cannot be done. Consider the following maps between homotopy fiber sequences:

$$\begin{tikzcd}
K(\mathbb{Z}/p^2,1) \arrow[r] \arrow[d, "id"'] & K(\mathbb{Z}/p,1) \arrow[r] \arrow[d] & K(\mathbb{Z}/p,2) \arrow[d, "k"'] \\
K(\mathbb{Z}/p^2,1) \arrow[r] & * \arrow[r] & K(\mathbb{Z}/p^2,2)
\end{tikzcd}$$

where k is induced by the inclusion $\mathbb{Z}/p \to \mathbb{Z}/p^2$. Since both $K(\mathbb{Z}/p,2)$ and $K(\mathbb{Z}/p^2,2)$ are complex K-theory acyclic, the map $k\colon K(\mathbb{Z}/p,2) \to K(\mathbb{Z}/p^2,2)$ is a complex K-theory equivalence. However, the space $K(\mathbb{Z}/p,1)$ is not complex K-theory acyclic.

3. Zabrodsky's Lemma in its original form is obtained if we take $* \to X \to X$ for the fiber sequence $F' \to E' \to X'$ in Theorem 1.1.

2. Applications

We are going to list some applications in f-localization theory. Let us recall the definition of an f-localization functor L_f. Fix a map $f\colon W \to V$ between spaces. A space T is called *f-local* if it is fibrant and the map

$$\operatorname{map}(f,T)\colon \operatorname{map}(V,T) \to \operatorname{map}(W,T)$$

is a weak equivalence. A map $g\colon C \to D$ is called an *L_f-equivalence* if the map $\operatorname{map}(g,T)\colon \operatorname{map}(D,T) \to \operatorname{map}(C,T)$ is a weak equivalence for every f-local space T. For a space Y, an *f-localization* of Y is a map $\eta\colon Y \to L_fY$ which is an L_f-equivalence and for which L_fY is f-local. If the map f is $W \to *$, we call f-local spaces *W-null* and L_f-equivalence maps *P_W-equivalences.* We write P_W for L_f and call P_WX a *W-nullification* of X. Since L_fY (resp., P_WY) is clearly unique up to homotopy, we call it *the* f-localization (resp., W-nullification) of Y.

Corollary 2.1. *For any connected spaces W and X, let $\eta\colon X \to P_WX$ be the W-nullification map of X. If $\pi_1(\eta)$ is an isomorphism, then $P_W(X\langle 1\rangle) \simeq (P_WX)\langle 1\rangle$, where $X\langle 1\rangle$ is the simply connected cover of X.*

Proof. Consider the following diagram:

$$\begin{tikzcd}
X\langle 1\rangle \arrow[r] \arrow[d, "\mu"'] & X \arrow[r] \arrow[d, "\eta"'] & K(\pi_1 X, 1) \arrow[d, "\simeq"'] \\
(P_W X)\langle 1\rangle \arrow[r] & P_W X \arrow[r] & K(\pi_1 P_W X, 1).
\end{tikzcd}$$

The fiber of η is P_W-acyclic (i.e. $P_W F_\eta \simeq *$) by [D, Theorem 1.H.2]. Hence, by Theorem 1.1, the induced map $\mu\colon X\langle 1\rangle \to (P_W X)\langle 1\rangle$ is a P_W-equivalence. From the principle fibration $K(\pi_1 P_W X, 0) \to (P_W X)\langle 1\rangle \to P_W X$, since $K(\pi_1 P_W X, 0)$ is always W-null, $(P_W X)\langle 1\rangle$ is also W-null. Hence, μ is the W-nullification map of $X\langle 1\rangle$, so $P_W(X\langle 1\rangle) \simeq (P_W X)\langle 1\rangle$. □

Remarks. 1. It is clear from the proof of Corollary 2.1 that if $\pi_j(\eta)$ is an isomorphism for $j \leq n$, then $P_W(X\langle j\rangle) \simeq (P_W X)\langle j\rangle$ for $j \leq n$.

2. The interest in this corollary lies in the fact that the localization of a simply connected space is usually easier to compute.

Corollary 2.2. *Consider a homotopy cofiber sequence $A \xrightarrow{g} B \xrightarrow{h} C_g$ with A and B connected. For any map f, if $L_f F_h \simeq *$ and the natural map $F_g \to \Omega C_g$ is an L_f-equivalence, then $L_f A \simeq *$.*

Proof. Apply Theorem 1.1 to the following diagram between homotopy fiber sequences:

$$\begin{tikzcd}
F_g \arrow[r] \arrow[d] & A \arrow[r, "g"] \arrow[d] & B \arrow[d, "h"'] \\
\Omega C_g \arrow[r] & * \arrow[r] & C_g.
\end{tikzcd}$$

□

The following corollary improves the result [D, Theorem 3.D.2] of Dror Farjoun, which has several important special cases (cf. [D, Corollary 3.D.3]).

Corollary 2.3. *Suppose given a homotopy fiber sequence $F \to E \to X$ with E and X connected. Let F_η be the homotopy fiber of the f-localization map η of E. If $L_{\Sigma f} X \simeq L_f X$ and $L_f F_\eta \simeq *$, then $L_f F \to L_f E \to L_f X$ is a homotopy fiber sequence.*

Proof. Consider the following commutative diagram:

$$\begin{tikzcd}
\Omega X \arrow[r] \arrow[d, "u"'] & F \arrow[r] \arrow[d] & E \arrow[r, "g"] \arrow[d] & X \arrow[d] \\
\Omega L_f X \arrow[r] & F' \arrow[r] & L_f E \arrow[r, "L_f(g)"] & L_f X
\end{tikzcd}$$

where F' is the homotopy fiber of $L_f(g)$. Since $L_{\Sigma f} X \simeq L_f X$, it follows that $\Omega L_f X \simeq \Omega L_{\Sigma f} X \simeq L_f \Omega X$ by [B, Theorem 3.1] or [D, Theorem 3.A.1]. Thus, u is an L_f-equivalence. By Theorem 1.1, the map $F \to F'$ is also an L_f-equivalence, since $L_f F_\eta \simeq *$. Clearly, F' is f-local, since F' is the homotopy fiber of two f-local spaces. Consequently, $F \to F'$ is the f-localization of F, and we have a homotopy fiber sequence $L_f F \to L_f E \to L_f X$. □

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