CHARACTERIZATIONS OF W-TYPE SPACES

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Abstract. In this paper we obtain new characterizations of certain spaces of W-type.

1. Introduction

The spaces of W-type were studied by B.L. Gurevich [5] and I.M. Gelfand and G.E. Shilov [4]. They investigated the behaviour of the Fourier transformation on the W-spaces. Also W-spaces are applied to the theory of partial differential equations. These spaces are generalizations of spaces of S-type [3].


In this paper, motivated by the work of R.S. Pathak and S.K. Upadhyay [7], we give new characterizations of the spaces of W-type introduced in [2].

In our investigation the Hankel integral transformation defined by

$$ h_{\mu}(\phi)(x) = \int_0^{\infty} y^{2\mu+1}(xy)^{-\mu} J_{\mu}(xy)\phi(y)dy , \quad x \in (0, \infty) , $$

plays an important role, where as usual $J_{\mu}$ denotes the Bessel function of the first kind and order $\mu$. Throughout this paper $\mu$ will always represent a real number greater than $-1/2$.

It is known (Corollary 4.8, [1]) that $h_{\mu}$ is an automorphism of the space $Se$ constituted by all those complex valued even smooth functions $\phi = \phi(x), \ x \in \mathbb{R}$, such that

$$ \gamma_{m,n}(\phi) = \sup_{x \in \mathbb{R}} |x^m D^n \phi(x)| < \infty , \quad \text{for every } m, n \in \mathbb{N} . $$

Moreover $h_{\mu}^{-1}$, the inverse of $h_{\mu}$, coincides with $h_{\mu}$ on $Se$.

Throughout this paper we will denote by $K$ the following set of functions:

$$ K = \{ M \in C^2([0, \infty)) : M(0) = M'(0) = 0, M'(\infty) = \infty \text{ and } M''(x) > 0, x \in (0, \infty) \} . $$

For every $M \in K$ we will represent by $M^\times$ the Young dual function of $M$ ([4], p.19). Interesting and useful properties of the functions in $K$ can be found in [2] and [4].

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In [4] the W-spaces were defined as follows. Let \( M, \Omega \in K \) and \( a, b > 0 \).

The space \( W_{M,a}^{\Omega,b} \) consists of all those complex valued and smooth functions \( \phi \) on \( \mathbb{R} \) such that for every \( m \in \mathbb{N} - \{0\} \) and \( k \in \mathbb{N} \) there exists \( C_{m,k} > 0 \) for which

\[
|D^k \phi(x)| \leq C_{m,k} \exp \left( -M(a(1 - \frac{1}{m})|x|) \right), \; x \in \mathbb{R}.
\]

The space \( W_{\Omega}^{\Omega,b} \) consists of all entire functions \( \phi \) such that for every \( m \in \mathbb{N} - \{0\} \) and \( k \in \mathbb{N} \) there exists \( C_{m,k} > 0 \) for which

\[
|z^k \phi(z)| \leq C_{m,k} \exp \left( \Omega(b(1 + \frac{1}{m})|z|) \right), \; z \in \mathbb{C}.
\]

An entire function \( \phi \) is in \( W_{M,a}^{\Omega,b} \) if, and only if, for each \( m, k \in \mathbb{N} - \{0\} \) there exists \( C_{m,k} \) for which

\[
|\phi(z)| \leq C_{m,k} \exp \left( -M(a(1 - \frac{1}{m})|\Re z|) + \Omega(b(1 + \frac{1}{k})|\Im z|) \right), \; z \in \mathbb{C}.
\]

S.J.L. van Eijndhoven and M.J. Kerkhof [2] investigated the behaviour of the subspaces of the W-spaces defined as follows.

A function \( \phi \) is in \( W_{M,a} \) (respectively, \( W_{\Omega}^{\Omega,b} \) and \( W_{M,a}^{\Omega,b} \)) when \( \phi \) is even and \( \phi \) in \( W_{M,a} \) (respectively, \( W_{\Omega}^{\Omega,b} \) and \( W_{M,a}^{\Omega,b} \)).

We now introduce new spaces of W-type.

Let \( M, \Omega \in K, a, b > 0 \) and \( 1 \leq p \leq \infty \). A complex valued and smooth function \( \phi = \phi(x), \; x \in I = (0, \infty) \), is in \( W_{p,M,a}^{\Omega,b} \) if, and only if, \( \phi \) belongs to \( Se \) and

\[
\left\| \exp \left( M[a(1 - \frac{1}{m})x] \Delta_k^p \phi(x) \right) \right\|_p < \infty \text{ for every } m \in \mathbb{N} - \{0\} \text{ and } k \in \mathbb{N}.
\]

Here and in the sequel \( \| \|_p \) denotes the norm in the Lebesgue space \( L_p(0, \infty) \).

By \( \Delta_k \) we denote the Bessel operator \( x^{-2\mu-1}Dx^{2\mu+1}D \).

The space \( W_{p,M,a}^{\Omega,b} \) consists of \( \phi \in Se \) that admit a holomorphic extension to the whole complex plane and that satisfy the following two conditions:

(i) there exists \( \epsilon > 0 \) such that for every \( k \in \mathbb{N} \) we can find \( C_k > 0 \) for which

\[
|z^k \phi(z)| \leq C_k \exp \left( \Omega(b|\Im z|) \right), \; z \in \mathbb{C},
\]

(ii) \( \sup_{y \in \mathbb{R}} \left\| \exp \left( -\Omega[b(1 + \frac{1}{n})|y|] \right)(x + iy)^m \phi(x + iy) \right\|_p < \infty \), for every \( n \in \mathbb{N} - \{0\} \) and \( m \in \mathbb{N} \).

A complex valued and smooth function \( \phi = \phi(x), x \in I \), is in \( W_{p,M,a}^{\Omega,b} \) if, and only if, \( \phi \) is in \( Se \) admitting a holomorphic extension to the whole complex plane and \( \phi \) satisfies (i) and

(iii) \( \sup_{y \in \mathbb{R}} \left\| \exp \left( M[a(1 - \frac{1}{m})x] - \Omega[b(1 + \frac{1}{n})|y|] \right) \phi(x + iy) \right\|_p < \infty \) for every \( m, n \in \mathbb{N} - \{0\} \).

In Section 2 we establish that \( W_{p,M,a}^{\Omega,b} = W_{M,a}^{\Omega,b} \), \( W_{p,M,a}^{\Omega,b} = W_{\Omega}^{\Omega,b} \) and \( W_{p,M,a}^{\Omega,b} = W_{M,a}^{\Omega,b} \), for every \( \mu > -1/2 \) and \( 1 \leq p \leq \infty \).
Throughout this paper for every $1 < p < \infty$ we denote by $p'$ the conjugate of $p$ (i.e., $p' = \frac{p}{p-1}$). Also by $C$ we always represent a suitable positive constant, not necessarily the same in each occurrence.

2. CHARACTERIZATIONS OF We-spaces

In this Section we prove, by using the Hankel transformation $h_\mu$, that $W_{\mu,M,a}^p = W_{\mu,M,a}^p$ and $W_{\mu,M,a}^\Omega = W_{\mu,M,a}^\Omega$, for every $\mu > -1/2$ and $1 \leq p \leq \infty$.

**Lemma 2.1.** Let $1 \leq p \leq \infty$ and $\mu > -1/2$. Then $W_{\mu,M,a}^p$ is contained in $W_{\mu,M,a}$.

**Proof.** Assume first that $1 < p < \infty$. Let $\phi$ be in $W_{\mu,M,a}^p$. Define

$$
\psi(y) = h_\mu(\phi)(y) = \int_0^\infty (xy)^{-\mu} J_\mu(xy) \phi(x) x^{2\mu+1} dx, \quad y \in \mathbb{C}.
$$

According to Corollary 4.8 in [1], $\psi$ is in $S_\epsilon$. Moreover, the last integral is defined for every $y \in \mathbb{C}$. In fact, for every $y \in \mathbb{C}$ and $n \in \mathbb{N} - \{0\}$, by virtue of (5.3.b) of [2] and Hölder’s inequality we have

$$
\int_0^\infty (xy)^{-\mu} J_\mu(xy)|\phi(x)| x^{2\mu+1} dx \leq C \int_0^\infty \exp \left( x|\Im y| \right) |\phi(x)| x^{2\mu+1} dx
$$

$$
\leq C \int_0^\infty \exp \left[ x|\Im y| - M \left( a(1 - \frac{1}{n}) x \right) \right] \exp \left[ M \left( a(1 - \frac{1}{n}) x \right) \right] |\phi(x)| x^{2\mu+1} dx
$$

$$
\leq C \left( \int_0^\infty \exp \left[ x|\Im y| - M \left( a(1 - \frac{1}{n}) x \right) \right] x^{2\mu+1} dx \right)^{1/p'} \left( \int_0^\infty \exp \left[ M \left( a(1 - \frac{1}{n}) x \right) \right] \phi(x) dx \right)^{1/p}
$$

$$
\leq C \left( \int_0^\infty \exp \left[ x|\Im y| - M \left( a(1 - \frac{1}{n}) x \right) \right] x^{2\mu+1} dx \right)^{1/p'} \left( \int_0^\infty x|\Im y| \left( a(1 - \frac{1}{n}) \right) x \right)^{1/p'}
$$

Moreover, denoting as usual by $M^\xi$ the Young dual of $M$, according to well-known properties of $M^\xi$ ([4]) we obtain for every $x \in I$, $y \in \mathbb{C}$, $n \in \mathbb{N} - \{0\}$, where $1 < m < n$,

$$
x|\Im y| - M \left( a(1 - \frac{1}{n}) x \right) = \frac{x|\Im y|}{a(1 - 1/m)} a(1 - \frac{1}{m}) - M \left( a(1 - \frac{1}{n}) x \right)
$$

$$
\leq M \left( a(1 - \frac{1}{m}) x \right) - M \left( a(1 - \frac{1}{n}) x \right) + M^\xi \left( \frac{|\Im y|}{a(1 - 1/m)} \right)
$$

$$
\leq - M \left( a \left( \frac{1}{m} - \frac{1}{n} \right) \right) + M^\xi \left( \frac{|\Im y|}{a(1 - 1/m)} \right).
$$
Hence for every \( m, n \in \mathbb{N} - \{0\} \) with \( 1 < m < n \) we can write
\[
\int_0^\infty |(xy)^-\mu J_{\mu}(xy)||\phi(x)|x^{2\mu+1}dx \\
\leq C \left( \int_0^\infty \left( \exp\left[-M\left(a\left(\frac{1}{m} - \frac{1}{n}\right)x\right)\right]\right)^{1/p'}dx \right)^{1/p'} \exp\left[M^x\left(\frac{|3y|}{a(1-1/m)}\right)\right] \\
\leq C \exp\left[M^x\left(\frac{|3y|}{a(1-1/m)}\right)\right], \quad y \in \mathbb{C},
\]
because \( \lim_{x \to \infty} M'(x) = \infty. \)

If \( p = 1 \) or \( p = \infty \) we can argue in a similar way.

Thus we conclude that the integral in the right hand side of (1) is a continuous extension of \( \psi \) to the whole complex plane. Moreover, by proceeding in a similar way we can see that it also is entire. Such an extension will be denoted again by \( \psi \). Note that \( \psi \) is an even function.

We prove that \( \psi \in W_{eM^x,1/a}. \)

It is not difficult to deduce from Lemma 5.4-1 of [9] that for every \( k \in \mathbb{N} \)
\[
y^{2k}\psi(y) = (-1)^k \int_0^\infty (xy)^-\mu J_{\mu}(xy)\Delta_{\mu}^k[\phi(x)]x^{2\mu+1}dx, \quad y \in \mathbb{C}.
\]

Then, proceeding as above, we get for every \( k, m \in \mathbb{N}, m > 1, \)
\[
|y^{2k}\psi(y)| \leq \int_0^\infty |(xy)^-\mu J_{\mu}(xy)||\Delta_{\mu}^k[\phi(x)]|x^{2\mu+1}dx \\
\leq C \int_0^\infty \exp\left|x|3y|\right|x^{2\mu+1}|\Delta_{\mu}^k[\phi(x)]|dx \\
\leq C \exp\left[M^x\left(\frac{|3y|}{a(1-1/m)}\right)\right], \quad y \in \mathbb{C}.
\]

Hence \( \psi \) is in \( W_{eM^x,1/a}. \)

Since \( h_\mu = h_{-\mu}^{-1} \) on \( S_\epsilon \), according to Lemma 7.4 of [2], we conclude that \( W_{e\mu,M,a}^p \) is contained in \( W_{eM,a} \).

**Lemma 2.2.** Let \( 1 \leq p \leq \infty \) and \( \mu > -1/2 \). Then \( W_{eM,a} \) is contained in \( W_{e\mu,M,a}^p \).

**Proof.** By virtue of Lemma 7.3 of [2], \( h_{\mu}(W_{eM,a}) \subset W_{eM^x,1/a} \). Let \( \phi \in W_{eM^x,1/a} \).

Since \( h_{\mu} = h_{-\mu}^{-1} \) on \( S_\epsilon \), our result will be established when we see that \( h_{\mu}(\phi) \) is in \( W_{e\mu,M,a} \).

Note first that according to Corollary 4.8 of [1], \( h_{\mu} \phi \) is in \( S_\epsilon \).

Let \( k \in \mathbb{N} \). By invoking Lemma 5.4-1 of [9] we can obtain that
\[
\Delta_{\mu}^k h_{\mu}(\phi)(x) = (-1)^k h_{\mu}(z^{2k}\phi(z))(x), \quad x \in I.
\]

A procedure similar to the one developed in the proof of Lemma 6.1 of [2] allows us to write, for every \( x > 1 \) and \( \tau > 0, \)
\[
\Delta_{\mu}^k h_{\mu}(\phi)(x) = \frac{1}{2} \int_{-\infty}^{\infty} (x(\sigma + i\tau))^{-\mu}H_{\mu}^{(1)}(x(\sigma + i\tau))\phi(\sigma + i\tau)(\sigma + i\tau)^{2\mu+2k+1}d\sigma,
\]
where \( H^{(1)}_\mu \) denotes the Hankel function ([8], p. 73).

Now for every \( x > 1 \) and \( \tau > 0 \) we divide the last integral as follows:

\[
\int_{-\infty}^{\infty} (x(\sigma + i\tau))^{-\mu} H^{(1)}_\mu(x(\sigma + i\tau)) \phi(\sigma + i\tau)(\sigma + i\tau)^{2\mu + 2k + 1} d\sigma
\]

\[
= \left( \int_{|x(\sigma + i\tau)| \leq 1} + \int_{|x(\sigma + i\tau)| > 1} \right) (x(\sigma + i\tau))^{-\mu} H^{(1)}_\mu \cdot (x(\sigma + i\tau)) \phi(\sigma + i\tau)(\sigma + i\tau)^{2\mu + 2k + 1} d\sigma.
\]

We will analyze each of the integrals separately.

Assume first that \( \mu \geq 1/2 \). On the one hand, by using (5.3.c) of [2], we get for every \( n \in \mathbb{N} - \{0\} \)

\[
\int_{|x(\sigma + i\tau)| \leq 1} |(x(\sigma + i\tau))^{-\mu} H^{(1)}_\mu(x(\sigma + i\tau)) \phi(\sigma + i\tau)(\sigma + i\tau)^{2\mu + 2k + 1}| d\sigma
\]

\[
\leq C \exp(-x\tau) \int_{-\infty}^{\infty} |\phi(\sigma + i\tau)| d\sigma
\]

\[
\leq C \exp(-x\tau + M^x \left[ \frac{1}{a} (1 + \frac{1}{n}) \tau \right]), \quad \text{for } x > 1 \text{ and } \tau > 0;
\]

on the other hand, by using again (5.3.c) of [2], for every \( n \in \mathbb{N} - \{0\} \)

\[
\int_{|x(\sigma + i\tau)| > 1} |(x(\sigma + i\tau))^{-\mu} H^{(1)}_\mu(x(\sigma + i\tau)) \phi(\sigma + i\tau)(\sigma + i\tau)^{2\mu + 2k + 1}| d\sigma
\]

\[
\leq C \exp(-x\tau) \int_{-\infty}^{\infty} |\phi(\sigma + i\tau)(\sigma + i\tau)^{2\mu + 2k + 1}| d\sigma
\]

\[
\leq C \exp(-x\tau + M^x \left[ \frac{1}{a} (1 + \frac{1}{n}) \tau \right]), \quad \text{for } x > 1 \text{ and } \tau > 0.
\]

For fixed \( n \in \mathbb{N} - \{0\} \) we choose \( \tau > 0 \) such that

\[
M^x \left[ \frac{1}{a} (1 + \frac{1}{n}) \tau \right] = \frac{ax}{(1 + 1/n)}.
\]

Then from Lemma 2.4 of [2] we have

\[
-x\tau + M^x \left[ \frac{1}{a} (1 + \frac{1}{n}) \tau \right] = -M \left( \frac{ax}{(1 + 1/n)} \right).
\]

Hence by combining (4), (5) and (6) it follows that

\[
|\Delta^k h_\mu(\phi)(x)| \leq C \exp\left(-M \left[ x(1 - \frac{1}{n + 1}) \right] \right), \quad \text{for } x > 1 \text{ and } n \in \mathbb{N}.
\]

Note also that, if \(-1/2 < \mu < 1/2\), by invoking (5.3.d) of [2] one has

\[
|\Delta^k h_\mu(\phi)(x)| \leq C \exp(-x\tau) \int_{-\infty}^{\infty} |\phi(\sigma + i\tau)(\sigma + i\tau)^{\mu + 2k + 1/2}| d\sigma, \tau > 0 \text{ and } x > 1.
\]

Proceeding as above, we conclude that

\[
|\Delta^k h_\mu(\phi)(x)| \leq C \exp\left(-M \left[ ax(1 - \frac{1}{m}) \right] \right), \quad \text{for } x > 1 \text{ and } m \in \mathbb{N} - \{0\}.
\]
Now let \( x \in (0,1) \) and \( m \in \mathbb{N} - \{0\} \). According to (5.3.b) of [2] we have

\[
\left| \exp\left( M \left[ ax\left(1 - \frac{1}{m}\right) \right] \Delta^k h_{\mu}(\phi)(x) \right) \right| = \left| \exp\left( M \left[ ax\left(1 - \frac{1}{m}\right) \right] h_{\mu}(z^{2k}\phi(z))(x) \right) \right|
\]

\[
\leq C \int_0^\infty \sigma^{2\mu+2k+1} |\phi(\sigma)| \, d\sigma
\]

because \( M \) is an increasing function on \((0,\infty)\).

Hence, for every \( k \in \mathbb{N} \) and \( m \in \mathbb{N} - \{0\} \),

\[
\left| \exp\left( M \left[ ax\left(1 - \frac{1}{m}\right) \right] \Delta^k h_{\mu}(\phi)(x) \right) \right| \leq C , \; x > 0 ,
\]

and, if \( m \in \mathbb{N} - \{0\} \), \( k \in \mathbb{N} \) and \( 1 \leq p < \infty \), then

\[
\left\{ \int_0^\infty \left| \exp\left( M \left[ ax\left(1 - \frac{1}{m}\right) \right] \Delta^k h_{\mu}(\phi)(x) \right)^p \right| \, dx \right\}^{1/p} \leq C
\]

because \( \int_0^\infty \exp(-pM \left[ ax\left(1 - \frac{1}{m}\right) \right]) \, dx < \infty \).

Thus we establish that \( h_{\mu}\phi \in W^p_{\mu,M,a} \), \( 1 \leq p \leq \infty \), and the proof is finished.

\[\square\]

From Lemmas 2.1 and 2.2 we deduce

**Theorem 2.1.** For every \( 1 \leq p \leq \infty \) and \( \mu > -1/2 \), \( W^p_{\mu,M,a} = W_{\mu,M,a} \).

**Lemma 2.3.** Let \( 1 \leq p \leq \infty \). Then \( W^p_{\mu,\Omega,b} \) is contained in \( W_{\mu,\Omega,b} \).

**Proof.** Let \( \phi \) be in \( W^p_{\mu,\Omega,b} \). Assume that \( \mu > -1/2 \). Proceeding as in the proof of Lemma 2.2, we can establish that for every \( k \in \mathbb{N} \) there exists \( l = l(k) \) such that

\[
|\Delta^k h_{\mu}(\phi)(x)| \leq C \exp(-x\tau) \int_{-\infty}^\infty |\phi(\sigma + i\tau)| \left( |\sigma + i\tau|^l + 1 \right) \, d\sigma , \; x \in (0, \infty) .
\]

Hence, according to Hölder’s inequality and (6), we obtain for each \( k \in \mathbb{N} \), \( m \in \mathbb{N} - \{0\} \) and suitable \( \tau > 0 \)

\[
\exp\left( \Omega^x \left[ \frac{1}{b} \left(1 - \frac{1}{m}\right) x \right] \right) |\Delta^k h_{\mu}(\phi)(x)|
\]

\[
\leq C \exp\left( \Omega^x \left[ \frac{1}{b} \left(1 - \frac{1}{m}\right) x \right] - \Omega^x \left[ \frac{1}{b} \left(1 - \frac{1}{m+1}\right) x \right] \right) \left\{ \int_{-\infty}^{\infty} \frac{d\sigma}{\left( 1 + \sigma^2 \right)^{p'}} \right\}^{1/p'}
\]

\[
\cdot \left\{ \int_{-\infty}^{\infty} \left( \exp\left[ -\Omega b(1 + \frac{1}{m})\tau \right] \left( |\sigma + i\tau| + 1 \right) \left( |\sigma + i\tau|^l + 1 \right) \right)^p \, d\sigma \right\}^{1/p}
\]

\[
\leq C , \; x \in (0, \infty) ,
\]

provided that \( 1 < p < \infty \). When \( p = 1 \) or \( p = \infty \) we can proceed in a similar way. Thus we prove that \( h_{\mu}(\phi) \in W^{\infty}_{\mu,\Omega^x,1/b} \). Therefore Theorem 2.1 shows that \( h_{\mu}(\phi) \in W_{\mu,\Omega^x,1/b} \).

Since \( h_{\mu} = h_{\mu}^{-1} \) on \( \mathbb{S} \), it is sufficient to take into account Lemma 7.3 of [2] to see that \( \phi \in W_{\mu,\Omega^x,b} \), and the proof of this lemma is complete.

\[\square\]
The next result is not hard to see.

**Lemma 2.4.** Let $1 \leq p \leq \infty$. Then $W^\Omega_{1,0}$ is contained in $W^p_{1,0}$.

As an immediate consequence of Lemmas 2.3 and 2.4 we obtain the following

**Theorem 2.2.** Let $1 \leq p \leq \infty$. Then $W^p_{1,0} = W^\Omega_{1,0}$.

**Lemma 2.5.** Let $1 \leq p \leq \infty$. Then $W^p_{M,a}$ is contained in $W^\Omega_{M,a}$.

**Proof.** Let $\phi$ be in $W^p_{M,a}$. Choose $\mu \geq 1/2$. Since $h_\mu = h^{-1}_\mu$ on $Se$, by virtue of Lemma 7.5 of [2], to prove this lemma it is sufficient to see that $h_\mu \phi$ is in $W^1_{M,1/a}$. The Hankel transformation $h_\mu \phi$ of $\phi$ is in $Se$ (Corollary 4.8 [1]). Moreover, proceeding as in the proof of Lemma 2.1, we can see that $h_\mu \phi$ can be holomorphically extended to the whole complex plane.

Let $\tau > 0$. An argument similar to the one developed in Lemma 6.1 of [2] allows us to write

$$(h_\mu \phi)(x) = \frac{1}{2} \int^\infty_{-\infty} \left( x(\sigma + i\tau) - |\sigma| \right)^{-\mu} H^{(1)}_{\mu}(x(\sigma + i\tau)) \phi(\sigma + i\tau)(\sigma + i\tau)^{2\mu + 1} d\sigma, \quad |x| > 1.$$

As in the proof of Lemma 2.2,

$$(h_\mu \phi)(x) = \frac{1}{2} \left( \int_{|x(\sigma + i\tau)| \leq 1} + \int_{|x(\sigma + i\tau)| > 1} \right) \left( x(\sigma + i\tau) - |\sigma| \right)^{-\mu} H^{(1)}_{\mu}(x(\sigma + i\tau)) \phi(\sigma + i\tau)(\sigma + i\tau)^{2\mu + 1} d\sigma, \quad |x| > 1.$$

We must analyze each of the two integrals.

According to (5.3.c) of [2] we have, for every $n, m \in \mathbb{N} \setminus \{0\},$

$$\int_{|x(\sigma + i\tau)| > 1} \left| \left( x(\sigma + i\tau) - |\sigma| \right)^{-\mu} H^{(1)}_{\mu}(x(\sigma + i\tau)) \phi(\sigma + i\tau)(\sigma + i\tau)^{2\mu + 1} \right| d\sigma$$

$$\leq C|x|^{-\mu - 1/2} \int_{-\infty}^{\infty} \exp \left( -|\Re x| \tau - |\Im x| \sigma \right) \left| \phi(\sigma + i\tau)(\sigma + i\tau)^{\mu + 1/2} \right| d\sigma$$

$$\leq C|x|^{-\mu - 1/2} \cdot \left\{ \int_{-\infty}^{\infty} \left( \exp \left[ -|\Re x| \tau + |\Im x| |\sigma| - M \left( a(1 - \frac{1}{n}) \right) + \Omega \left( b(1 + \frac{1}{m}) \right) \right] \right) \right. \left| \sigma + i\tau \right|^{\mu + 1/2} \right\}^{1/p'} d\sigma,$$

where $|x| > 1$, provided that $1 < p < \infty$. By Lemma 2.4 of [2],

$$|\Im x| |\sigma| \leq M^x \left( \frac{|\Im x|}{a(1 - 1/l)} \right) + M \left( a(1 - \frac{1}{l}) |\sigma| \right), \quad \sigma \in \mathbb{R}, x \in \mathbb{C} \text{ and } l \in \mathbb{N}, l > 1.$$

Then

$$|\Im x| |\sigma| - M \left( a(1 - \frac{1}{n}) |\sigma| \right) \leq M^x \left( \frac{|\Im x|}{a(1 - 1/l)} \right) - M \left( a(1 - \frac{1}{n}) |\sigma| \right),$$

where $\sigma \in \mathbb{R}$, $x \in \mathbb{C}$ and $l, n \in \mathbb{N}$, $n > l > 1.$
We assume now that $\Re x > 0$, and we choose $\tau > 0$ such that
\[
\Omega'\left(b(1 + \frac{1}{m})\tau\right) = \frac{\Re x}{b(1 + 1/m)}.
\]
Then, again by Lemma 2.4 of [2],
\[
\tau\Re x = \Omega\left(b(1 + \frac{1}{m})\tau\right) + \Omega^x\left(\frac{\Re x}{b(1 + 1/m)}\right).
\]
Hence, since $-\mu - \frac{1}{2} \leq 0$ and $1 < p < \infty$, we obtain for every $|x| \geq 1$ and $\Re x > 0$
\begin{equation}
\int_{|x(\sigma + i\tau)| > 1} |(x(\sigma + i\tau))^{-\mu}H^{(1)}_{\mu}(x(\sigma + i\tau))\phi(\sigma + i\tau)(\sigma + i\tau)^{2\mu + 1}|d\sigma
\leq C \exp\left[M^x\left(\frac{|3x|}{a(1 - 1/l)}\right) - \Omega^x\left(\frac{\Re x}{b(1 + 1/m)}\right)\right]
\cdot \left(\int_{-\infty}^{\infty} \exp\left[-M\left(a\left(\frac{1}{1 - \frac{1}{n}}\right)\right)|\sigma + i\tau|^{\mu + 1/2}\right)^{p'} d\sigma\right)^{1/p'}
\leq C \exp\left[M^x\left(\frac{|3x|}{a(1 - 1/l)}\right) - \Omega^x\left(\frac{\Re x}{b(1 + 1/m)}\right)\right] n, m, l \in \mathbb{N} - \{0\}, 1 < l < n,,
\end{equation}
because $\int_{-\infty}^{\infty} \exp\left[-M\left(a\left(\frac{1}{1 - \frac{1}{n}}\right)\right)|\sigma + i\tau|^{\mu + 1/2}\right)^{p'} d\sigma < \infty$.

If $p = 1$ or $p = \infty$, we can proceed in a similar way.

On the other hand, by (5.3.c) of [2]
\begin{equation}
\int_{|x(\sigma + i\tau)| \leq 1} |(x(\sigma + i\tau))^{-\mu}H^{(1)}_{\mu}(x(\sigma + i\tau))\phi(\sigma + i\tau)(\sigma + i\tau)^{2\mu + 1}|d\sigma
\leq C |x|^{-2\mu} \int_{-\infty}^{\infty} \exp\left[-(\Re x)\tau + |3x||\sigma|\right]|\phi(\sigma + i\tau)(\sigma + i\tau)|d\sigma
\leq C \exp\left[M^x\left(\frac{|3x|}{a(1 - 1/l)}\right) - \Omega^x\left(\frac{\Re x}{b(1 + 1/m)}\right)\right], |x| \geq 1 and \Re x > 0,,
\end{equation}
for $m, l \in \mathbb{N} - \{0\}, 1 < l$.

Hence from (7) and (8) we conclude that
\begin{equation}
|h_\mu \phi(x)| \leq C \exp\left[M^x\left(\frac{1}{a} + \frac{1}{l-1}||3x||\right) - \Omega^x\left(\frac{1}{b}\left|\frac{1}{m} + 1\right|\Re x\right)\right]
\end{equation}
for every $|x| \geq 1$ and $\Re x > 0$, $m, l \in \mathbb{N}$, where $1 < l$.

Since $h_\mu \phi$ is even, the corresponding inequality (9) also holds when $\Re x < 0$.

Now let $|x| < 1$. By using (5.3.b) of [2] we deduce that
\[
|h_\mu \phi(x)| \leq C \int_{0}^{\infty} \exp(t||3x||)|\phi(t)||2^{\mu + 1}dt.
\]

Proceeding as in the above case, we conclude that $h_\mu \phi \in W_{e^{M^x}}^{\Omega^x,1/a}$.

The following result can be proved without difficulty.

Lemma 2.6. Let $1 \leq p \leq \infty$. Then $W_{e^{M,a}}^{\Omega,b}$ is contained in $W_{e^{M,a}}^{p,\Omega,b}$. \hfill \Box
From Lemmas 2.5 and 2.6 we obtain

**Theorem 2.3.** Let $1 \leq p \leq \infty$. Then $\mathcal{W}^p_{M,a} = \mathcal{W}^p_{M,a}$. 

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**References**


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