

## ON THE GROWTH OF POLYNOMIALS

D. DRYANOV AND Q. I. RAHMAN

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*Dedicated to the memory of Professor Paul Erdős*

ABSTRACT. Let  $f$  be a polynomial of degree  $n$  having only real zeros and none in  $(-1, 1)$ . We look for a sharp upper bound for  $|f(z)|$  at an arbitrary point of the complex plane  $\mathbb{C}$  in terms of the supremum norm on  $[-1, 1]$ .

### 1. INTRODUCTION

Let  $\mathcal{P}_n$  be the class of all polynomials of degree at most  $n$ . As usual, we denote by  $T_n$  the  $n$ -th Chebyshev polynomial of the first kind. According to a classical result of P.L. Chebyshev (see [5] or [6]), if  $f \in \mathcal{P}_n$  and  $|f(x)| \leq 1$  for  $-1 \leq x \leq 1$ , then

$$(1) \quad |f(x)| \leq |T_n(x)| \quad \text{for } x \in \mathbb{R} \setminus [-1, 1].$$

In (1) equality holds at some point  $x_0 \in \mathbb{R} \setminus [-1, 1]$  if and only if  $f(z) \equiv e^{i\gamma} T_n(z)$  for some real  $\gamma$ . It was noted by S. Bernstein [2] that if  $f(z)$  is real for real  $z$ , then

$$(2) \quad |f(z)| \leq |T_n(z)| \quad \text{for } |z| \geq 1,$$

even if  $|f|$  is bounded by 1 only at the  $n + 1$  points  $\cos \frac{\nu\pi}{n}$ ,  $\nu = 0, 1, \dots, n$ , which are the extrema of  $T_n$  in  $[-1, 1]$ . Bernstein's paper went unnoticed, and his result was rediscovered by P. Erdős [4].

The polynomial  $T_n$  is extremal for several other problems. For example, it was proved by A. Markov (see [5] or [6]) that if  $f \in \mathcal{P}_n$  and  $|f(x)| \leq 1$  for  $-1 \leq x \leq 1$ , then

$$(3) \quad \max_{-1 \leq x \leq 1} |f'(x)| \leq n^2 = \max_{-1 \leq x \leq 1} |T'_n(x)|.$$

All the zeros of  $T_n$  are real and lie in the open interval  $(-1, 1)$ . This suggests that the above inequalities can be considerably improved if the zeros of  $f$  are all real but none of them lies in  $(-1, 1)$ . As regards (3), P. Erdős [3]; see in particular the second half of p. 311 proved the following result.

**Theorem A.** *Let  $f \in \mathcal{P}_n$  be such that  $|f(x)| \leq 1$  for  $-1 \leq x \leq 1$ . If the zeros of  $f$  are all real and lie on  $\mathbb{R} \setminus (-1, 1)$ , then*

$$(4) \quad \max_{-1 \leq x \leq 1} |f'(x)| \leq \frac{1}{2} \left(1 - \frac{1}{n}\right)^{-n+1} n.$$

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The inequality becomes an equality for polynomials of the form

$$\frac{e^{i\gamma}n^n(1+x)^{n-1}(1-x)}{2^n(n-1)^{n-1}}, \frac{e^{i\gamma}n^n(1-x)^{n-1}(1+x)}{2^n(n-1)^{n-1}}, \text{ where } \gamma \in \mathbb{R}.$$

Here we obtain a result which may be seen as an analogue of (2) for polynomials with zeros restricted as in Theorem A.

For each positive integer  $n$  and  $k = 0, 1, \dots, n$  let  $\eta_{n,k} := -1 + \frac{2k}{n}$ . Denote by  $\mathcal{P}_{n,\mathbb{R},1}$  the family of all polynomials  $f$  in  $\mathcal{P}_n$  which have only real zeros, none of which lies in  $(-1, 1)$ , and satisfy  $|f(x)| \leq 1$  for all  $x \in F_n := \{\eta_{n,k} : 0 \leq k \leq n\}$ . Furthermore, for  $k = 0, 1, \dots, n$ , let

$$q_{n,k}(z) := \frac{n^n}{2^n k^k (n-k)^{n-k}} (1+z)^k (1-z)^{n-k}.$$

Note that  $\max_{-1 \leq x \leq 1} |q_{n,k}(x)| = q_{n,k}(\eta_{n,k}) = 1$  for each  $k$ . We prove

**Theorem 1.** *Let  $\Omega$  be the union of the open disks*

$$\left\{ z \in \mathbb{C} : \left| z \pm \frac{i}{\sqrt{3}} \right| < \frac{2}{\sqrt{3}} \right\},$$

and  $\mathbb{S} := \mathbb{C} \setminus \Omega$ . If  $f \in \mathcal{P}_{n,\mathbb{R},1}$ , then for all  $z \in \mathbb{S}$

$$(5) \quad |f(z)| \leq \max_{0 \leq k \leq n} |q_{n,k}(z)|.$$

*Remark 1.* Our proof of the theorem will show that except when  $z = \pm i\sqrt{3}$  and  $n = 1$ , strict inequality holds in (5) unless  $f(z) \equiv e^{i\gamma} q_{n,k}(z)$  for some  $k$  and  $\gamma \in \mathbb{R}$ . If  $n = 1$ , then  $|f(\pm i\sqrt{3})|$  is also maximized by each constant of modulus 1.

## 2. PROOF OF THEOREM 1

There is nothing to prove for  $z = \pm 1$ . Besides, for reasons of symmetry, it is enough to prove (5) for all

$$z \in \mathbb{E} := \mathbb{S} \cap \{x + iy : x \geq 0\} \setminus \{1\}.$$

The result is obtained in several steps.

**Step 1.** Given any point  $\zeta \in \mathbb{E}$ , let  $\sigma_\zeta := \sup_{f \in \mathcal{P}_{n,\mathbb{R},1}} |f(\zeta)|$ . There exists a polynomial  $g \in \mathcal{P}_{n,\mathbb{R},1}$  such that  $|g(\zeta)| = \sigma_\zeta$ . For this we observe that if

$$\psi(z) := \prod_{k=0}^n (z - \eta_{n,k}),$$

then for all  $z \in \mathbb{C}$  we have

$$f(z) = \sum_{k=0}^n f(\eta_{n,k}) \frac{\psi(z)}{\psi'(\eta_{n,k})(z - \eta_{n,k})}.$$

From this it follows that the polynomials in  $\mathcal{P}_{n,\mathbb{R},1}$  are uniformly bounded on every compact subset of  $\mathbb{C}$ , and so they form a normal family [1, p.216]. Consequently, there exists a polynomial  $g \in \mathcal{P}_n$  such that  $|g(\eta_{n,k})| \leq 1$  for  $k = 0, 1, \dots, n$  and  $|g(\zeta)| = \sigma_\zeta$ . Obviously,  $g$  cannot be identically zero. So by a well-known theorem of Hurwitz [1, p.176] it must belong to  $\mathcal{P}_{n,\mathbb{R},1}$ . We call such a polynomial **extremal**.

**Step 2.** Let

$$\mathcal{E}_\zeta := \{g \in \mathcal{P}_{n,\mathbb{R},1} : |g(\zeta)| = \sigma_\zeta\}.$$

It is clear that if  $g \in \mathcal{E}_\zeta$ , then  $\max_{1 \leq k \leq n} |g(\eta_{n,k})|$  must be equal to 1. Besides, it must be of degree  $n$  unless  $n = 1$  and  $\zeta = i\sqrt{3}$ . Indeed, if a polynomial  $g \in \mathcal{P}_{n,\mathbb{R},1}$  is of degree  $n - j < n$ , then

$$g_+(z) := \left(\frac{1+z}{2}\right)^j g(z)$$

also belongs to  $\mathcal{P}_{n,\mathbb{R},1}$ , and  $|g_+(\zeta)| > |g(\zeta)|$  if  $|\zeta + 1| > 2$ . In particular,  $|g_+(\zeta)| > |g(\zeta)|$  for all  $\zeta \in \mathbb{E}$ ,  $\zeta \neq i\sqrt{3}$ , and so  $g$  cannot be extremal for any  $\zeta \in \mathbb{E}$  except possibly for  $\zeta = i\sqrt{3}$ . Now let  $n \geq 2$  and suppose that a polynomial  $g \in \mathcal{P}_{n,\mathbb{R},1}$  of degree  $n - j < n$  is extremal for  $\zeta = i\sqrt{3}$ . Without loss of generality we may assume  $g$  to be positive on  $(-1, 1)$ . Since  $f(z) := 1 - z^2$  belongs to  $\mathcal{P}_{n,\mathbb{R},1}$  for  $n \geq 2$  and  $|f(i\sqrt{3})| = 4 > 1$ , it follows that  $g(z)$  cannot be identically equal to 1, i.e.  $g$  cannot be of degree 0. Next we note that  $g$  cannot attain its maximum on the set  $\{\eta_{n,k} : k = 0, 1, \dots, n\}$  at both  $-1$  and  $+1$ . If it did, it would have at least three critical points between the largest zero in  $(-\infty, -1]$  and the smallest zero in  $[1, \infty)$ . But that is not possible since the zeros of  $g$  are all real. Hence if  $g_-(z) := ((1-z)/2)^j g(z)$ , then either  $\max_{0 \leq k \leq n} |g_+(\eta_{n,k})| < 1$  or  $\max_{0 \leq k \leq n} |g_-(\eta_{n,k})| < 1$ . Since  $|g_+(i\sqrt{3})| = |g_-(i\sqrt{3})| = |g(i\sqrt{3})|$ , we see that  $g \notin \mathcal{E}_{i\sqrt{3}}$ .

**Step 3.** In order to prove the theorem we need to show that if  $g \in \mathcal{E}_\zeta$  and  $g(\alpha) = 0$ , then  $\alpha$  is either  $+1$  or  $-1$ . First of all we show that  $\alpha$  cannot lie in  $(-\infty, -1)$ .

Let us agree to denote by  $A, B, P$  and  $X$  the points of the complex plane which represent the numbers  $-1, +1, \zeta$  and  $\alpha$ , respectively. Assume that  $\alpha \in (-\infty, -1)$ . It is easily seen that in this situation

$$\left| \frac{1 - \alpha}{2} \frac{z + 1}{z - \alpha} \right| \leq 1 \quad \text{for } z \in [-1, 1].$$

Furthermore,

$$\left| \frac{1 - \alpha}{2} \frac{\zeta + 1}{\zeta - \alpha} \right| = \left| \frac{1 - \alpha}{\zeta - \alpha} \right| \left| \frac{\zeta + 1}{2} \right| = \frac{\sin \widehat{XPB} \sin \widehat{ABP}}{\sin \widehat{XBP} \sin \widehat{APB}} = \frac{\sin \widehat{XPB}}{\sin \widehat{APB}} > 1.$$

Hence, the polynomial

$$g_1(z) := \frac{1 - \alpha}{2} \frac{z + 1}{z - \alpha} g(z)$$

belongs to  $\mathcal{P}_{n,\mathbb{R},1}$ , but  $|g_1(\zeta)| > |g(\zeta)|$ , contradicting the assumption that  $g \in \mathcal{E}_\zeta$ .

*It remains to show that if  $g \in \mathcal{E}_\zeta$  and  $g(\alpha) = 0$ , then  $\alpha$  cannot lie in  $(1, \infty)$ .*

**Step 4.** For each  $\varphi \in (0, \frac{\pi}{2})$ , we denote by  $\mathcal{L}_\varphi$  the line

$$z = -1 + te^{i\varphi} \quad (-\infty < t < \infty)$$

and by  $\mathbb{H}_\varphi$  that half-plane bounded by  $\mathcal{L}_\varphi$  which contains the infinite interval  $[-1, \infty)$ . Let

$$t_1 = t_1(\varphi) := \inf \{t : -1 + te^{i\varphi} \in \mathbb{E}\} \quad (0 < \varphi < \frac{\pi}{2})$$

and consider the half-lines

$$\mathcal{L}_\varphi^+ : -1 + te^{i\varphi} \quad (t_1 \leq t < \infty).$$

Take any point  $P$  on  $\mathcal{L}_\varphi^+$ , i.e.  $P$  represents  $\zeta = \zeta(t) = -1 + te^{i\varphi}$ , where  $t \geq t_1$ . Take another point  $Q$  on the same half-line such that  $P$  is an interior point of the line segment  $AQ$ .

Before we go on, we need to take a point  $C$  on the positive real axis and draw three half-lines  $\Lambda_1, \Lambda_2, \Lambda_3$  contained in  $\mathbb{H}_\varphi$  with  $P$  as initial point. The half-lines are drawn so that  $\Lambda_1$  makes with  $\vec{PB}$  an angle equal to  $\widehat{APB}$ ,  $\Lambda_2$  makes with  $\vec{PB}$  an angle equal to  $\widehat{PBC}$ , and  $\Lambda_3$  makes with  $\vec{PQ}$  an angle equal to  $\widehat{APB}$ .

At this stage the reader will find it useful to note that the chord  $AB$  subtends an angle  $\frac{\pi}{3}$  at each point of the circle  $|z - \frac{i}{\sqrt{3}}| = \frac{2}{\sqrt{3}}$  which lies in the upper half-plane. It should be clear that except when  $\zeta = i\sqrt{3}$ , the half-line  $\Lambda_1$  intersects the positive real axis. Denote the point of intersection by  $L(\zeta)$ , or by  $L$  for brevity. We may say that  $L(i\sqrt{3}) = \infty$ . Let  $a = a(\zeta)$  be the distance of  $L$  from the origin. Thus  $a(\zeta) < +\infty$  unless  $\zeta = i\sqrt{3}$ .

The half-line  $\Lambda_2$  intersects the positive real axis if and only if  $\widehat{APB} + \widehat{PBC} < \pi - \widehat{PAB}$ , which is equivalent to

$$\widehat{APB} + \widehat{PAB} < \frac{\pi}{2}$$

since  $\widehat{PBC} = \widehat{PAB} + \widehat{APB}$ . Thus  $\Lambda_2$  intersects the positive real axis if and only if  $\zeta \in \mathbb{E} \cap \{x + iy : x > 1, y > 0\}$ . Denote the point of intersection by  $M(\zeta)$ , or by  $M$  if there is no ambiguity. We say that  $M$  is the point at infinity for all  $\zeta \in \mathbb{E} \setminus \{x + iy : x > 1, y > 0\}$ . Let  $b = b(\zeta)$  be the distance of  $M$  from the origin. Note that  $b(-1 + te^{i\varphi}) < +\infty$  for each  $t \geq t_1$  when  $0 < \varphi < \frac{\pi}{6}$ , and only for  $t > 2 \sec \varphi$  when  $\varphi \in [\frac{\pi}{6}, \frac{\pi}{2})$ .

The half-line  $\Lambda_3$  intersects the positive real axis if and only if  $\widehat{APB} > \widehat{PAB}$ . This cannot be the case if  $P$  lies on  $\mathcal{L}_\varphi^+$  for any  $\varphi \in [\frac{\pi}{3}, \frac{\pi}{2})$ . If  $\varphi \in (0, \frac{\pi}{3})$  and  $\zeta = -1 + te^{i\varphi}$ , then  $\widehat{APB} > \widehat{PAB}$  if and only if  $t_1(\varphi) = \frac{2}{\sqrt{3}} \sin \varphi + 2 \cos \varphi \leq t < 4 \cos \varphi$ . If  $\Lambda_3$  intersects the positive real axis, we denote the point of intersection by  $N(\zeta)$  or simply by  $N$ . Let  $c = c(\zeta)$  be the distance of  $N$  from the origin.

**Step 5.** We need to compare the quantities  $a(\zeta)$ ,  $b(\zeta)$  and  $c(\zeta)$ . Note that  $c(\zeta) = +\infty$  if  $\zeta \in \mathcal{L}_\varphi^+$ , where  $\frac{\pi}{3} \leq \varphi < \frac{\pi}{2}$ , and also when  $\zeta = \zeta(t) = -1 + te^{i\varphi}$ , where  $\varphi \in (0, \frac{\pi}{3})$  but  $t \geq 4 \cos \varphi$ . For each  $\varphi \in (0, \frac{\pi}{3})$  the quantity  $c(-1 + te^{i\varphi})$  increases as  $t$  increases from  $t_1(\varphi)$  to  $4 \cos \varphi$ .

It follows that

$$b(\zeta) \leq c(\zeta)$$

except possibly when  $\zeta$  is of the form  $-1 + te^{i\varphi}$ , where  $0 < \varphi < \frac{\pi}{3}$  and  $t_1 \leq t < 4 \cos \varphi$ .

Since  $b(\zeta) = +\infty$  for  $\zeta \in \mathbb{E} \setminus \{x + iy : x > 1, y > 0\}$ , we trivially have

$$c(\zeta) < b(\zeta)$$

if  $\zeta = -1 + te^{i\varphi}$ , where  $\frac{\pi}{4} \leq \varphi < \frac{\pi}{3}$  and  $t_1(\varphi) \leq t < 4 \cos \varphi$ . At least one of  $c(\zeta)$  and  $b(\zeta)$  will be  $+\infty$  for  $\varphi \in [\frac{\pi}{4}, \frac{\pi}{3})$ , because  $2 \sec \varphi \geq 4 \cos \varphi$  for such values of  $\varphi$ .

For each  $\varphi \in (0, \frac{\pi}{4})$  there exists a number

$$t_2(\varphi) \in (\max\{t_1(\varphi), 2 \sec \varphi\}, 4 \cos \varphi)$$

such that

$$c(-1 + t_2(\varphi)e^{i\varphi}) = b(-1 + t_2(\varphi)e^{i\varphi}),$$

whereas  $c(-1 + te^{i\varphi}) < b(-1 + te^{i\varphi})$  or  $c(-1 + te^{i\varphi}) > b(-1 + te^{i\varphi})$  according as  $t_1(\varphi) \leq t < t_2(\varphi)$  or  $t > t_2(\varphi)$ , respectively. Indeed, for  $\zeta = -1 + te^{i\varphi} \in \mathcal{L}_\varphi^+$  the angles  $\widehat{APB}, \widehat{PBC}$  decrease as  $t$  increases; so there is one and only one  $\zeta \in \mathcal{L}_\varphi^+$  corresponding to which  $\widehat{APB} + \widehat{PBC} = \pi - \widehat{APB}$ , and this occurs for  $\frac{\pi}{3} < \widehat{PBC} < \frac{\pi}{2}$ . For such  $\zeta$  the point  $M(\zeta)$  coincides with  $N(\zeta)$ ;  $N(\zeta)$  lies to the left of  $M(\zeta)$  or to the right of  $M(\zeta)$  according as  $\widehat{APB} + \widehat{PBC} > \pi - \widehat{APB}$  or  $\widehat{APB} + \widehat{PBC} < \pi - \widehat{APB}$ , respectively.

Since  $\widehat{PBC} = \widehat{APB} + \widehat{PAB} > \widehat{APB}$ , it follows that  $a(\zeta) < b(\zeta)$  for all  $\zeta \in \mathbb{E} \setminus \{i\sqrt{3}\}$ , whereas  $a(i\sqrt{3}) = b(i\sqrt{3}) = +\infty$ . If  $\zeta = -1 + te^{i\varphi}$ , where  $0 < \varphi < \frac{\pi}{2}$  and  $t > t_1(\varphi)$ , then  $\widehat{APB} < \frac{\pi}{3}$ ; so  $\widehat{APL} < \frac{2\pi}{3}$ , whereas  $\widehat{APN} > \frac{2\pi}{3}$ , which implies that  $a(\zeta) < c(\zeta)$ . The same can be said if  $t = t_1(\varphi)$  and  $\varphi \in (\frac{\pi}{3}, \frac{\pi}{2})$ . However, if  $t = t_1(\varphi)$  and  $\varphi \in (0, \frac{\pi}{3})$ , then  $a(\zeta) = c(\zeta) < +\infty$ , and for  $\varphi = \frac{\pi}{3}$  we have  $a(i\sqrt{3}) = c(i\sqrt{3}) = +\infty$ .

The above observations about  $a(\zeta), b(\zeta)$  and  $c(\zeta)$  allow us to conclude that the interval  $(1, +\infty)$  can be the same as  $(1, c(\zeta))$ . But in the case  $c(\zeta) < +\infty$ , we can express  $(1, +\infty)$  as  $(1, c(\zeta)) \cup [a(\zeta), b(\zeta)) \cup [b(\zeta), +\infty)$  if  $b(\zeta) < +\infty$  and as  $(1, c(\zeta)) \cup [a(\zeta), b(\zeta))$  if  $b(\zeta) = +\infty$ .

**Step 6.** We are finally ready to prove that if  $g \in \mathcal{E}_\zeta$  and  $g(\alpha) = 0$ , then  $\alpha$  cannot lie in  $(1, +\infty)$ .

**I.** Let  $\alpha \in (1, c]$  if  $c < +\infty$ ; otherwise let  $\alpha \in (1, +\infty)$ . It is clear that

$$\left| \frac{z-1}{z-\alpha} \frac{\alpha+1}{2} \right| \leq 1 \quad \text{for } z \in [-1, 1].$$

Since

$$\left| \frac{\zeta-1}{2} \frac{\alpha+1}{\zeta-\alpha} \right| = \frac{\sin \widehat{PAB} \sin \widehat{APX}}{\sin \widehat{APB} \sin \widehat{PAX}} = \frac{\sin \widehat{APX}}{\sin \widehat{APB}},$$

and

$$\widehat{APB} < \widehat{APX} \leq \widehat{APN} = \pi - \widehat{APB},$$

it follows that  $\sin \widehat{APB} \leq \sin \widehat{APX}$ , i.e.  $\left| \frac{\zeta-1}{\zeta-\alpha} \frac{\alpha+1}{2} \right| \geq 1$ , where the inequalities are strict unless  $X$  coincides with  $N(\zeta)$ . Hence the polynomial

$$g_2(z) := \frac{z-1}{z-\alpha} \frac{\alpha+1}{2} g(z)$$

belongs to  $\mathcal{P}_{n, \mathbb{R}, 1}$  and  $|g_2(\zeta)| \geq |g(\zeta)|$ , where the inequality is strict unless  $c(\zeta) < +\infty$  and  $\alpha = c(\zeta)$ . So the assumption  $g \in \mathcal{E}_\zeta$  is contradicted except in such a situation. Since  $c(i\sqrt{3}) = +\infty$ , we may hereafter assume  $\zeta \neq i\sqrt{3}$ .

**II.** Now let  $\alpha \in [a, b)$ . There exists a point  $X_1$  in  $(-\infty, -1)$  such that  $\widehat{X_1PB} = \widehat{BPX}$ . Denote by  $-\alpha_1$  the corresponding real number. The function

$$g_3(z) := \frac{z+\alpha_1}{z-\alpha} \frac{\alpha-1}{\alpha_1+1} g(z)$$

is easily seen to belong to  $\mathcal{P}_{n,\mathbb{R},1}$ . Besides,

$$\left| \frac{\zeta + \alpha_1}{1 + \alpha_1} \right| \left| \frac{\alpha - 1}{\zeta - \alpha} \right| = \frac{\sin \widehat{X_1BP} \sin \widehat{BPX}}{\sin \widehat{X_1PB} \sin \widehat{PBX}},$$

i.e.  $|g_3(\zeta)| = |g(\zeta)|$ . Thus the function

$$h_3(z) := \frac{z + 1}{2} \frac{1 + \alpha_1}{z + \alpha_1} g_3(z)$$

belongs to  $\mathcal{P}_{n,\mathbb{R},1}$  and

$$|h_3(z)| = \left| \frac{\zeta + 1}{2} \right| \left| \frac{1 + \alpha_1}{\zeta + \alpha_1} \right| |g(\zeta)| = \frac{\sin \widehat{ABP} \sin \widehat{X_1PB}}{\sin \widehat{APB} \sin \widehat{X_1BP}} |g(\zeta)| \geq |g(\zeta)|,$$

where the inequality is strict unless  $\alpha = a(\zeta)$ . So, we get a contradiction except when  $\alpha = a(\zeta)$ .

**III.** Next, let  $b < +\infty$  and  $\alpha \geq b$ . The polynomial

$$g_4(z) := \frac{\alpha - 1}{z - \alpha} g(z)$$

belongs to  $\mathcal{P}_{n-1,\mathbb{R},1}$ , and

$$|g_4(\zeta)| = \frac{\sin \widehat{BPX}}{\sin \widehat{PBX}} |g(\zeta)| \geq |g(\zeta)|,$$

where the inequality is strict unless  $\alpha = b(\zeta)$ . But there is really no extremal polynomial of degree less than  $n$ , since  $\zeta \neq i\sqrt{3}$ . This is a contradiction.

Summarizing the above conclusions, we see that we have got a contradiction except when  $c(\zeta) < +\infty$  and  $\alpha = c(\zeta)$  or when  $\alpha = a(\zeta)$ . Comparing parts **I**, **II** and **III**, we see that  $\alpha = c(\zeta)$  and  $\alpha = a(\zeta)$  are covered if  $a(\zeta) < c(\zeta)$ . All that remains is the case  $\alpha = a(\zeta) = c(\zeta) < +\infty$ . But this occurs only when  $\zeta = \zeta_1 := -1 + t_1 e^{i\varphi}$ , where  $0 < \varphi < \frac{\pi}{3}$ , i.e.  $\zeta$  lies on the circle  $\left| z - \frac{i}{\sqrt{3}} \right| = \frac{2}{\sqrt{3}}$ . In that case we introduce the polynomial

$$g_5(z) := \frac{a(\zeta) + 1}{z - a(\zeta)} \frac{z - 1}{2} g(z).$$

First we note that  $g$  must vanish at  $-1$ . If not, the polynomial  $g(-z)$  would have all its zeros in  $(-\infty, -1]$ , and  $|g(-\zeta)|$  would be larger than  $|g(\zeta)|$ , contrary to the assumption that  $g \in \mathcal{E}_\zeta$ . Next we note that  $\frac{a(\zeta)+1}{x-a(\zeta)} \frac{x-1}{2}$  decreases from 1 to 0 as  $x$  increases from  $-1$  to  $+1$ . This means that  $\max_{x \in F_n} |g(x)|$  and  $\max_{x \in F_n} \left| \frac{a(\zeta)+1}{x-a(\zeta)} \frac{x-1}{2} \right|$  are not attained at the same point of  $F_n$ . Hence

$$\max_{x \in F_n} |g_5(x)| = \mu \max_{x \in F_n} |g(x)|,$$

where  $\mu \in (0, 1)$ . We get a contradiction with the fact that if  $g_5 \in \mathcal{E}_\zeta$  then  $\max \{|g_5(\eta_{n,k})| : k = 0, 1, \dots, n\}$  must be equal to 1.

It remains to consider points  $\zeta \in (1, +\infty)$ . Clearly  $g(\zeta)$  cannot be zero. So we have to consider two possibilities, namely,  $\alpha \in (\zeta, +\infty)$  and  $\alpha \in (1, \zeta)$ . In the first case, we may consider  $g_4(z) := \frac{\alpha-1}{\alpha-z} g(z)$  to see that  $g$  cannot belong to  $\mathcal{E}_\zeta$  if  $g(\alpha) = 0$ . In the second case,  $g_2(z) = \frac{1-z}{\alpha-z} \frac{\alpha+1}{2} g(z)$  shows the same.

3. DETERMINATION OF  $\max_{0 \leq k \leq n} |q_{n,k}(z)|$  FOR A GIVEN  $z$

First let  $Re(z) \geq 0$ , i.e.  $\left| \frac{1+z}{1-z} \right| \geq 1$ . Then, for  $\left[ \frac{n+1}{2} \right] \leq k \leq n$ , we have  $|q_{n,k}(z)/q_{n,n-k}(z)| \geq 1$ . Hence

$$\Theta(z) := \max_{0 \leq k \leq n} |q_{n,k}(z)| = \max_{\left[ \frac{n+1}{2} \right] \leq k \leq n} |q_{n,k}(z)|.$$

For  $0 \leq t \leq n$ , let

$$\rho(t) := \frac{(t+1)^{t+1}(n-t-1)^{n-t-1}}{t^t(n-t)^{n-t}},$$

where, as usual,  $0^0 = 1$ . Examining its logarithmic derivative, we see that  $\rho(t)$  is strictly increasing on  $[0, n-1]$ . Hence

$$(7) \quad \rho\left(\left[\frac{n-1}{2}\right]\right) < \rho\left(\left[\frac{n+1}{2}\right]\right) < \dots < \rho(n-1).$$

Furthermore, it is easily checked that

$$(8) \quad \rho\left(\left[\frac{n-1}{2}\right]\right) < 1 < \rho\left(\left[\frac{n+1}{2}\right]\right) \quad \text{if } n \text{ is even,}$$

whereas

$$(9) \quad \rho\left(\left[\frac{n-1}{2}\right]\right) = 1 < \rho\left(\left[\frac{n+1}{2}\right]\right) \quad \text{if } n \text{ is odd.}$$

It is clear that if  $z \neq 1$  and  $w(z) := \frac{1+z}{1-z}$ , then

$$(10) \quad |q_{n,k+1}(z)/q_{n,k}(z)| \geq 1 \quad \text{for } 0 \leq k \leq n-1$$

if and only if

$$(11) \quad |w(z)| = \left| \frac{1+z}{1-z} \right| \geq \rho(k),$$

where equality holds in (10) if and only if it does in (11). Hence for  $\left[ \frac{n-1}{2} \right] \leq k \leq n-1$ ,

$$|q_{n,k+1}(z)| \geq |q_{n,k}(z)| \quad \text{if and only if } \rho(k) \leq |w(z)| < +\infty.$$

Besides,  $|q_{n,k+1}(z)| = |q_{n,k}(z)|$  only if  $|w(z)| = \rho(k)$ . Because of (7), it follows that for each integer  $j$  such that  $n \geq j \geq \left[ \frac{n-1}{2} \right]$ , we have

$$|q_{n,j}(z)| \geq |q_{n,k}(z)| \quad \text{for all } k \leq j-1 \text{ if } \rho(j-1) \leq |w(z)|,$$

where  $|q_{n,j}(z)| = |q_{n,k}(z)|$  only if  $k = j-1$  and  $|w(z)| = \rho(j-1)$ . In addition, for  $n > j \geq \left[ \frac{n-1}{2} \right]$ ,

$$|q_{n,j}(z)| \geq |q_{n,k}(z)| \quad \text{for } j < k \leq n \text{ if } |w(z)| \leq \rho(j),$$

where  $|q_{n,j}(z)| = |q_{n,k}(z)|$  only if  $k = j+1$  and  $|w(z)| = \rho(j)$ . Thus, setting  $\rho(n) = +\infty$ , we see that for each integer  $j$  such that  $n \geq j \geq \left[ \frac{n+1}{2} \right]$ ,

$$\max_{0 \leq k \leq n, k \neq j} |q_{n,k}(z)| < |q_{n,j}(z)| \quad \text{if } \rho(j-1) < |w(z)| < \rho(j).$$

It may be added that if  $|w(z)| = \rho(j-1)$ , then  $|q_{n,j}(z)| = |q_{n,j-1}(z)|$ . Thus, for any given  $z$  belonging to the closed right half-plane,  $\Theta(z)$  is attained by  $|q_{n,j}(z)|$  alone if  $\rho(j-1) < |w(z)| < \rho(j)$ ; it is also attained by  $|q_{n,j-1}(z)|$  if  $|w(z)| = \rho(j-1)$ . Each point  $z$  belonging to the closed right half-plane  $\mathbb{H}^+$  is covered, since for each such  $z$  there exists, in view of (8) and (9), an integer  $j$  in  $\left[ \left[ \frac{n+1}{2} \right], n \right]$  such that

$\rho(j - 1) \leq |w(z)| = \left| \frac{1+z}{1-z} \right| < \rho(j)$ . To determine  $\Theta(z)$  for points belonging to the left half-plane, it suffices to observe that  $\Theta(z) = \Theta(-z)$  for all  $z$ .

For  $\left\lceil \frac{n+1}{2} \right\rceil \leq k \leq n - 1$  let

$$c_k := \frac{(\rho(k))^2 + 1}{(\rho(k))^2 - 1}, \quad r_k := \frac{2\rho(k)}{(\rho(k))^2 - 1}.$$

It is easily checked that  $c_k - r_k$  increases with  $k$ , whereas  $c_k + r_k$  decreases. So if  $\mathbb{D}_k := \{z \in \mathbb{C} : |z - c_k| < r_k\}$ , then

$$\mathbb{D}_l \supset \bar{\mathbb{D}}_m \quad \text{if} \quad \left\lceil \frac{n+1}{2} \right\rceil \leq l < m \leq n - 1.$$

Now note that  $|w(z)| \geq \rho(k)$  for some integer  $k$  such that  $\left\lceil \frac{n+1}{2} \right\rceil \leq k \leq n - 1$  if and only if  $z \in \bar{\mathbb{D}}_k$ . We therefore have the following:

**Theorem 2.** *Let  $\Theta(z)$ ,  $\mathbb{H}^+$  and  $\Omega_k$  be as above. If  $z \in \bar{\mathbb{D}}_{n-1}$ , then  $\Theta(z) = |q_{n,n}(z)|$ ; if  $z \in \bar{\mathbb{D}}_{n-2} \setminus \mathbb{D}_{n-1}$ , then  $\Theta(z) = |q_{n,n-1}(z)|$ . More generally, if  $z \in \bar{\mathbb{D}}_k \setminus \mathbb{D}_{k+1}$  for some  $k$  such that  $\left\lceil \frac{n+1}{2} \right\rceil \leq k \leq n - 2$ , then  $\Theta(z) = |q_{n,k+1}(z)|$ . If  $z \in \mathbb{H}^+ \setminus \mathbb{D}_{\left\lceil \frac{n+1}{2} \right\rceil}$ , then  $\Theta(z) = \left| q_{n,\left\lceil \frac{n+1}{2} \right\rceil}(z) \right|$ .*

*Remark 2.* It may be noted that if  $|z - c_k| = r_k$  for some  $k$  such that  $\left\lceil \frac{n+1}{2} \right\rceil \leq k \leq n - 1$ , then  $\Theta(z) = |q_{n,k+1}(z)| = |q_{n,k}(z)|$ . Furthermore, if  $n$  is odd and  $Re(z) = 0$ , then  $\Theta(z) = \left| q_{n,\left\lceil \frac{n+1}{2} \right\rceil}(z) \right| = \left| q_{n,\left\lfloor \frac{n+1}{2} \right\rfloor}(z) \right|$ .

#### 4. SOME ADDITIONAL REMARKS

*Remark 3.* Even if we assume  $|f(x)|$  to be bounded by 1 for all  $x \in [-1, 1]$ , it is clearly not possible to improve upon (5). But for (5) to remain true for all  $z \in \mathbb{S}$ , do we have to assume that  $|f(\eta_{n,k})| \leq 1$  for  $0 \leq k \leq n$ ? The answer is yes. In fact, if we require  $|f(x)|$  to be bounded above by 1 on any closed subset  $F$  of  $[-1, 1]$  which does not contain one of the above  $n + 1$  points, say  $\eta_{n,j}$ , then (5) will fail at least for all  $z \in \mathbb{C}$ , where  $\Theta(z) = |q_{n,j}(z)|$ . This is because there exists  $\delta > 0$  such that  $|(1 + \delta)q_{n,j}(x)| \leq 1$  for all  $x \in F$ .

*Remark 4.* What can we say about  $|f(z)|$  when  $z \notin \mathbb{S}$ , i.e.  $z \in \Omega$ ? First of all we wish to point out that  $\mathbb{S}$  is independent of  $n$ . The answer to the question depends on  $n$ . Note that inequality (5) does not hold for any  $z$  belonging to the intersection  $\Delta$  of the two disks  $|1 + z| < 2$  and  $|1 - z| < 2$  in case  $n = 1$ . This follows from the definition of  $\Delta$ . Now consider the polynomial  $f(z) := \frac{1}{2}(1 + z)(2 - z)$ , which satisfies the conditions of Theorem 1 in case  $n = 2$ . Comparing  $|f(1 + iy)|$  with  $|q_{2,0}(1 + iy)|$ ,  $|q_{2,1}(1 + iy)|$  and  $|q_{2,2}(1 + iy)|$ , we see that

$$|f(z)| > \max_{0 \leq k \leq 2} |q_{2,k}(z)|$$

if  $z = 1 + iy$ ,  $0 < |y| < \frac{1}{\sqrt{3}}$ . Note that these points lie in  $\Omega \setminus \Delta$ . On the other hand, not only for  $n = 2$  but for all even  $n$ , we have  $|f(iy)| = |f(0)| \prod_{\nu=1}^n |1 - iyx_\nu|$ , where  $-1 \leq x_\nu \leq 1$  for  $\nu = 1, \dots, n$ . Hence,

$$|f(iy)| \leq |f(0)| (1 + y^2)^{\frac{n}{2}} = \left| q_{n,\frac{n}{2}}(iy) \right|,$$

i.e. (5) holds at all points of the imaginary axis in case  $n$  is even.

*Remark 5.* Although a polynomial  $f$  of degree  $n$  having only real zeros none of which lies in  $(-1, 1)$  is not completely determined by the value it takes at any point  $\alpha$  in  $(-1, 1)$ , its modulus at any point  $z \in \mathbb{C}$  can be estimated in terms of  $n$  and  $|f(\alpha)|$ . To see this note that  $f(z) = f(0) \prod_{\nu=1}^n (1 - zx_{\nu})$ , where  $-1 \leq x_{\nu} \leq 1$  for  $\nu = 1, \dots, n$ . For all  $z$ ,

$$|1 - zx_{\nu}| \leq \max\{|1 + z|, |1 - z|\}.$$

Hence, if  $|f(0)| \leq 1$ , then

$$(12) \quad |f(z)| \leq |f(0)| \max\{|1 + z|^n, |1 - z|^n\} = \begin{cases} |1 + z|^n & \text{if } \operatorname{Re}(z) \geq 0, \\ |1 - z|^n & \text{if } \operatorname{Re}(z) \leq 0. \end{cases}$$

If  $|f(\alpha)| \leq 1$ , where  $\alpha \in (-1, 0) \cup (0, 1)$ , then

$$F(w) := (\alpha w + 1)^n f\left(\frac{w + \alpha}{\alpha w + 1}\right)$$

is a polynomial of degree at most  $n$  having only real zeros none of which lies in  $(-1, 1)$ . Furthermore,  $|F(0)| = |f(\alpha)| \leq 1$ , and so by (12)

$$|\alpha w + 1|^n f\left(\frac{w + \alpha}{\alpha w + 1}\right) = |F(w)| \leq \max\{|1 + w|^n, |1 - w|^n\}.$$

Replacing  $\frac{w + \alpha}{\alpha w + 1}$  by  $z$ , we easily conclude that if  $|f(\alpha)| \leq 1$ , where  $\alpha \in (-1, 0) \cup (0, 1)$ , then for all  $z \in \mathbb{C}$ ,

$$f(z) \leq \begin{cases} \left|\frac{1+z}{1+\alpha}\right|^n & \text{if } \left|z - \frac{1+\alpha^2}{2\alpha}\right| \leq \frac{1-\alpha^2}{2\alpha}, \\ \left|\frac{1-z}{1-\alpha}\right|^n & \text{if } \left|z - \frac{1+\alpha^2}{2\alpha}\right| \geq \frac{1-\alpha^2}{2\alpha}. \end{cases}$$

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DEPARTMENT OF MATHEMATICS, UNIVERSITY OF SOFIA, JAMES BOUCHER 5, 1126 SOFIA, BULGARIA

*Current address:* Département de Mathématiques et de Statistique, Université de Montréal, Montréal, Canada H3C 3J7

*E-mail address:* dryanovd@ere.UMontreal.CA

DÉPARTEMENT DE MATHÉMATIQUES ET DE STATISTIQUE, UNIVERSITÉ DE MONTRÉAL, MONTRÉAL, CANADA H3C 3J7

*E-mail address:* rahmanqi@ere.UMontreal.CA