ON THE TOPOLOGICAL BOUNDARY
OF SEMI-FREDHOLM OPERATORS

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Abstract. We prove several distance formulas from a fixed operator in $B(H)$ to some classes of operators connected with the semi-Fredholm ones. Here $H$ is a separable Hilbert space. In particular, Fredholm and upper and lower semi-Fredholm operators have the same boundary in $B(H)$.

Let $H$ be a separable Hilbert space and let $B(H)$ be the algebra of all bounded linear operators on $H$. For an operator $T \in B(H)$, we will denote by $T^*$, $R(T)$, $N(T)$ and $\sigma(T)$ its adjoint, range, kernel and spectrum, respectively. Let $K(H)$ be the ideal of compact operators and $C(H) = B(H)/K(H)$ be the Calkin algebra. Denote by $\pi : B(H) \to C(H)$ the canonical projection. Endowed with the essential norm $|||T|||_e = |||\pi(T)|||$, $C(H)$ is a $C^*$-algebra.

The index of an operator $T \in B(H)$ will be denoted by $\text{ind}(T)$ and is defined by $\text{ind}(T) = \dim N(T) - \dim N(T^*)$, with the convention $\infty - \infty = 0$.

We introduce the following notation for several classes of operators:

- $F_+ = \{ T \in B(H) : R(T) \text{ is closed}, \dim N(T) < \infty \}$ is the set of all upper semi-Fredholm operators.
- $F_- = \{ T \in B(H) : R(T) \text{ is closed}, \dim N(T^*) < \infty \}$ is the set of all lower semi-Fredholm operators.
- $F_{\pm} = F_+ \cup F_-$ is the set of semi-Fredholm operators.
- $F = F_+ \cap F_-$ is the set of Fredholm operators.
- $I_n = \{ T \in B(H) : \text{ind}(T) = n \}$ with $n \in \mathbb{Z} = \mathbb{Z} \cup \{-\infty, +\infty\}$.
- $I^n = F_{\pm} \cap I_n$, with $n \in \mathbb{Z}$, the connected component of index $n$ in $F_{\pm}$.

For a set $X$ in $B(H)$, we will denote by $\text{int}X$, $\overline{X}$ and $\partial X$ the interior, closure and (topological) boundary, respectively.

For a linear operator $T \in B(H)$, we will denote by $\sigma_e(T) = \{ \lambda \in \mathbb{C} : T - \lambda I \notin F \}$ the essential spectrum of $T$. Let

$$m_e(T) = \inf\{\sigma_e(||T||)\}$$

(cf. [1]), where $||T|| = (T^*T)^{1/2}$, and

$$M_e(T) = \max\{m_e(T); m_e(T^*)\}.$$

Using Theorem 1.1 of [4] and Theorem 3.1 of [7], we easily obtain the following result:

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Theorem 1. Let $T \in B(H)$ and $n \in \mathbb{Z}$. Then
\[
\text{dist}(T, F^n_{\pm}) = \begin{cases} 
M_e(T) & \text{if } T \notin I_n, \\
0 & \text{if not.}
\end{cases}
\]

We recall the following theorem, which gives a characterization of the boundary of connected components of semi-Fredholm operators.

Theorem 2 ([5], Corollary 1.4). The boundary of $F^j_{\pm}$ does not depend on $j$, and, if $\Delta = \partial F^j_{\pm}$, then $\Delta = \partial F^j_{\pm}$ and $B(H) = \Delta \cup \left( \bigcup_{j \in \mathbb{Z}} F^j_{\pm} \right)$.

Remarks. 1) We have $\Delta = \{ T \in B(H) : M_e(T) = 0 \}$. Indeed, using [4], Theorem 2.1 and Corollary 2.2, we obtain $\Delta = \partial F^0_{\pm} = \partial G = \{ T \in B(H) : M_e(T) = 0 \}$, where $G$ is the group of invertible operators in $B(H)$.

2) The set $\Delta$ is stable with respect to compact perturbations: $\Delta + K(H) = \Delta$.

3) The set $\Delta$ is arcwise connected. Indeed, if $0 \in \Delta$ and $T \in \Delta$, then $tT \in \Delta$ for all $t \in [0, 1]$.

We prove now the following result.

Theorem 3. Let $T \in B(H)$ and $J \subseteq \mathbb{Z}$. Then
\[
\text{(a) dist} \left( T, \bigcup_{j \in J} F^j_{\pm} \right) = \begin{cases} 
M_e(T) & \text{if ind}(T) \notin J, \\
0 & \text{if not,}
\end{cases}
\]

and
\[
\text{(b) dist} \left( T, \bigcup_{j \in J} I_j \right) = \begin{cases} 
M_e(T) & \text{if ind}(T) \notin J, \\
0 & \text{if not.}
\end{cases}
\]

Proof. (a) Let $n = \text{ind}(T) \in \mathbb{Z}$. If $n \in J$, then $F^n_{\pm} \subseteq \bigcup_{j \in J} F^j_{\pm}$. Therefore
\[
0 \leq \text{dist}(T, \bigcup_{j \in J} F^j_{\pm}) \leq \text{dist}(T, F^n_{\pm}).
\]

But, using Theorem 1, we obtain $\text{dist}(T, F^n_{\pm}) = 0$. Hence, $\text{dist}(T, \bigcup_{j \in J} F^j_{\pm}) = 0$.

Suppose now that $n \notin J$ and let $j_0 \in J$. Then, using once again Theorem 1 and the relation $n \neq j_0$, we get
\[
\text{dist} \left( T, \bigcup_{j \in J} F^j_{\pm} \right) \leq \text{dist}(T, F^n_{\pm}) = M_e(T).
\]
We now show the converse inequality. For every $S \in \bigcup_{j \in J} F^j_{\pm}$, we have $\text{ind}(S) \neq n = \text{ind}(T)$. Then Theorem 1.1 of [6] implies $\|T - S\| \geq M_e(T)$. Therefore,

$$\text{dist}\left(T, \bigcup_{j \in J} F^j_{\pm}\right) \geq M_e(T),$$

and (a) is proved.

In order to prove (b), it suffices to remark that $\bigcup_{j \in J} F^j_{\pm} \subseteq \bigcup_{j \in J} I_j$ and to use (a). Indeed, if $\text{ind}(T) = n \in J$, then, using (a), we obtain

$$0 \leq \text{dist}\left(T, \bigcup_{j \in J} I_j\right) \leq \text{dist}\left(T, \bigcup_{j \in J} F^j_{\pm}\right) = 0.$$

On the other hand, if $n \notin J$, then for all $S \in \bigcup_{j \in J} I_j$, we have $\|T - S\| \geq M_e(T)$ (cf. [6], Theorem 1.1). This yields

$$\text{dist}\left(T, \bigcup_{j \in J} I_j\right) \geq M_e(T).$$

Using (a) again, we also find that

$$\text{dist}\left(T, \bigcup_{j \in J} I_j\right) \leq \text{dist}\left(T, \bigcup_{j \in J} F^j_{\pm}\right) = M_e(T),$$

which completes the proof.

As consequences, the results below easily follow. Assertions (b) and (c) of the following corollary are also in [2, Theorems 12 and 13].

**Corollary 4.** Let $T \in B(H)$. Then:

(a) $\text{dist}(T, F) = \begin{cases} M_e(T) & \text{if } \text{ind}(T) = \pm\infty, \\ 0 & \text{if not.} \end{cases}$

(b) $\text{dist}(T, F_+) = \begin{cases} m_e(T^*) & \text{if } \text{ind}(T) = +\infty, \\ 0 & \text{if } \text{ind}(T) \neq +\infty. \end{cases}$

(c) $\text{dist}(T, F_-) = \begin{cases} m_e(T) & \text{if } \text{ind}(T) = -\infty, \\ 0 & \text{if } \text{ind}(T) \neq -\infty. \end{cases}$

**Proof.** (a) follows from Theorem 3 above with $J = \mathbb{Z}$, while (b) and (c) follow from [2, Lemma 7] and Theorem 3 above with $J = \mathbb{Z} \cup \{-\infty\}$ and $J = \mathbb{Z} \cup \{+\infty\}$, respectively. 


Now we can prove the following equalities.

**Theorem 5.** Let $J \subseteq \mathbb{Z}$. Then

(a)  
$$\bigcup_{j \in J} F_j^\pm = \Delta \cup \left( \bigcup_{j \in J} F_j^1 \right) = \bigcup_{j \in J} F_j^1;$$

(b)  
$$\partial \left( \bigcup_{j \in J} F_j^\pm \right) = \Delta.$$

**Proof.**  
(a) We prove the first equality. If $J = \mathbb{Z}$, then the result follows from Theorem 2 since $\bigcup_{j \in J} F_j^\pm = B(H)$.

Suppose that $J \neq \mathbb{Z}$. Then, using Theorem 2 again, we have

$$B(H) = \Delta \cup \left( \bigcup_{j \in J} F_j^1 \right) \cup \left( \bigcup_{j \in \mathbb{Z} \setminus J} F_j^1 \right).$$

It follows that $\Delta \cup \left( \bigcup_{j \in J} F_j^1 \right)$ is closed, being the complement of the open set $\left( \bigcup_{j \in \mathbb{Z} \setminus J} F_j^1 \right)$ in $B(H)$. Therefore

$$\bigcup_{j \in J} F_j^1 \subseteq \Delta \cup \left( \bigcup_{j \in J} F_j^1 \right).$$

In order to show the other inclusion, it suffices to see that $\Delta \subseteq \bigcup_{j \in J} F_j^1$ (see Theorem 3).

The second equality follows from

$$\bigcup_{j \in J} F_j^\pm = \bigcup_{j \in J} \left( \Delta \cup F_j^1 \right).$$

(b) Since $\bigcup_{j \in J} F_j^1$ is open in $B(H)$, we have

$$\partial \left( \bigcup_{j \in J} F_j^\pm \right) = \left( \bigcup_{j \in J} F_j^\pm \right) \setminus \left( \bigcup_{j \in J} F_j^\pm \right) = \left( \Delta \cup \left( \bigcup_{j \in J} F_j^1 \right) \right) \setminus \left( \bigcup_{j \in J} F_j^1 \right) = \Delta.$$

The proof is complete.

This easily implies the following consequence:

**Corollary 6.** We have

(a)  
$$\overline{F} = F \cup \Delta; \quad F_+ = F_+ \cup \Delta \quad \text{and} \quad F_- = F_- \cup \Delta;$$

(b)  
$$\partial F = \partial F_+ = \partial F_- = \Delta.$$

We also have

**Corollary 7.** Let $J \subseteq \mathbb{Z}$. Then

$$\bigcup_{j \in J} I_j = \Delta \cup \left( \bigcup_{j \in J} F_j^1 \right) = \Delta \cup \left( \bigcup_{j \in J} I_j \right).$$
Proof. Using Theorems 3 and 5, we obtain
\[ \Delta \cup \left( \bigcup_{j \in J} I_j \right) \subseteq \bigcup_{j \in J} F_j^3 = \Delta \cup \left( \bigcup_{j \in J} F_j^3 \right) \subseteq \Delta \cup \left( \bigcup_{j \in J} I_j \right). \]

For the interior and for the boundary of the closure of sets considered in Theorem 5, we have

**Theorem 8.** Let \( J \subseteq \mathbb{Z}, J \neq \mathbb{Z} \). Then

(a) \( \text{int} \left( \bigcup_{j \in J} F_j^3 \right) = \bigcup_{j \in J} F_j^3 = \text{int} \left( \bigcup_{j \in J} F_j^3 \right) \);

(b) \( \partial \left( \bigcup_{j \in J} F_j^3 \right) = \partial \left( \bigcup_{j \in J} F_j^3 \right) = \Delta. \)

Proof. (a) We show the first equality. The inclusion
\[ \bigcup_{j \in J} F_j^3 \subseteq \text{int} \left( \bigcup_{j \in J} F_j^3 \right) \]

is clear. In order to show the other one, let \( T \in \text{int} \left( \bigcup_{j \in J} F_j^3 \right) \). If \( T \) is semi-Fredholm, then \( T \in \bigcup_{j \in J} F_j^3 \). Indeed, using Theorem 5, we have
\[ T \in \text{int} \left( \bigcup_{j \in J} F_j^3 \right) = \text{int} \left( \Delta \cup \bigcup_{j \in J} F_j^3 \right) \subseteq \Delta \cup \left( \bigcup_{j \in J} F_j^3 \right) \]

and, because \( T \notin \Delta \), we get \( T \in \bigcup_{j \in J} F_j^3 \).

Suppose now that \( T \) is not semi-Fredholm. Then \( T \in \Delta \). Consider \( n \notin J \) (this is always possible since \( J \neq \mathbb{Z} \)). Using Theorem 2, we obtain \( T \in \Delta = \partial F_n^3 \). Therefore
\[ T \in \text{int} \left( \bigcup_{j \in J} F_j^3 \right) \cap \partial F_n^3, \]

which implies the non-voidness of \( \left( \bigcup_{j \in J} F_j^3 \right) \cap F_n^3 \). The continuity of the index yields a contradiction.

The second equality follows from Theorem 5.
(b) Using (a) and Theorem 5, we obtain
\[\partial \left( \bigcup_{j \in J} F_{\pm}^j \right) = \partial \left( \bigcup_{j \in J} F_{\pm}^j \right) \]
\[= \left( \bigcup_{j \in J} F_{\pm}^j \right) \cap \text{int} \left( \bigcup_{j \in J} F_{\pm}^j \right) \]
\[= \left( \Delta \cup \left[ \bigcup_{j \in J} F_{\pm}^j \right] \right) \cap \left( \bigcup_{j \in J} F_{\pm}^j \right) = \Delta.\]

The proof is complete. ■

The following consequence can be easily obtained.

Corollary 9. We have
(a) \(\text{int}(F) = F\); \(\text{int}(F_+) = F_+\) and \(\text{int}(F_-) = F_-\);
and
(b) \(\partial F = \partial F_- = \partial F_+ = \partial F_- = \partial F_+ = \Delta\).

We also have

Corollary 10. Let \(J \subseteq \mathbb{Z}, J \neq \mathbb{Z}\). Then
(a) \(\text{int} \left( \bigcup_{j \in J} I_j \right) = \bigcup_{j \in J} F_{\pm}^j = \text{int} \left( \bigcup_{j \in J} I_j \right) \);
(b) \(\partial \left( \bigcup_{j \in J} I_j \right) = \partial \left( \bigcup_{j \in J} I_j \right) = \Delta.\)

Proof. (a) The first equality follows from Corollary 7 and Theorems 5 and 8.
For the second equality, it is sufficient to note that
\[\bigcup_{j \in J} F_{\pm}^j \subseteq \text{int} \left( \bigcup_{j \in J} I_j \right) \subseteq \text{int} \left( \bigcup_{j \in J} F_{\pm}^j \right) = \bigcup_{j \in J} F_{\pm}^j.\]
Indeed, the first two inclusions are obvious and the last equality is (a).
(b) is a direct consequence of (a) and of Corollary 7. ■

We introduce the following notation:
• \(G_+ = \{ T \in B(H) : T \text{ is left invertible} \}\).
• \(G_- = \{ T \in B(H) : T \text{ is right invertible} \}\).
• \(G_{\pm} = G_+ \cup G_- \): the set of one-sided invertible operators.
• \(G = G_+ \cap G_- \): the set of invertible operators.

For \(n \in \mathbb{Z}\), let \(G^n_{\pm} = G_{\pm} \cap I_n\). For \(n \in \mathbb{Z}_- = \mathbb{Z}_- \cup \{ -\infty \}\), we denote \(G^n_+ = G_+ \cap I_n\),
while for \(n \in \mathbb{Z}_+ = \mathbb{Z}_+ \cup \{ +\infty \}\), we set \(G^n_- = G_- \cap I_n\).

Theorem 11. Let \(T \in B(H)\) and \(J \subseteq \mathbb{Z}\). Then
\[\text{dist} \left( T, \bigcup_{j \in J} G_{\pm}^j \right) = \begin{cases} M_c(T) & \text{if ind}(T) \notin J, \\ 0 & \text{if not.} \end{cases}\]
Proof. Let \( n = \text{ind}(T) \). If \( n \in J \), then \( G^+_n \subseteq \bigcup_{j \in J} G^j_\pm \), so, using Theorem 3.1 of [7], we have

\[
0 \leq \text{dist}\left( T, \bigcup_{j \in J} G^j_\pm \right) \leq \text{dist}(T, G^+_n) = 0.
\]

Suppose now that \( n \notin J \). Let \( j_0 \in J \). Using Theorem 3.1 of [7], we get

\[
0 \leq \text{dist}\left( T, \bigcup_{j \in J} G^j_\pm \right) \leq \text{dist}(T, G^{j_0}_\pm) = M_e(T).
\]

On the other hand, for all \( L \in \bigcup_{j \in J} G^j_\pm \), \( \text{ind}(L) \neq n = \text{ind}(T) \). Therefore, using Theorem 1.1 of [6], we have

\[
\|T - L\| \geq M_e(T).
\]

Thus

\[
\text{dist}\left( T, \bigcup_{j \in J} G^j_\pm \right) \geq M_e(T).
\]

Now (1) and (2) imply the desired equality. \( \square \)

We obtain the following consequence.

Corollary 12. Let \( T \in B(H) \), and \( J \subseteq \mathbb{Z}_+ \). Then

\[
\text{dist}\left( T, \bigcup_{j \in J} G^j_- \right) = \begin{cases} M_e(T) & \text{if } \text{ind}(T) \notin J, \\ 0 & \text{if } \text{ind}(T) \in J. \end{cases}
\]

A similar result can be stated for \( \text{dist}(T, \bigcup_{j \in J} G^j_+) \) if \( J \subseteq \mathbb{Z}_- \). For the distance of \( T \) to the set \( G_- \setminus G \) we obtain the following formula:

Corollary 13. Let \( T \in B(H) \). Then

\[
\text{dist}(T, G_- \setminus G) = \begin{cases} M_e(T) & \text{if } \text{ind}(T) \leq 0, \\ 0 & \text{if not}. \end{cases}
\]

The following result gives a description of the closure, the interior and the boundary of \( \bigcup_{j \in J} G^j_\pm \) for \( J \subseteq \mathbb{Z} \).

Theorem 14. Let \( J \subseteq \mathbb{Z} \). Then:

(a) \( \bigcup_{j \in J} G^j_\pm = \Delta \cup \left( \bigcup_{j \in J} F^j_\pm \right) = \bigcup_{j \in J} G^j_\pm \).

If in addition \( J \neq \mathbb{Z} \), then:

(b) \( \text{int}\left( \bigcup_{j \in J} G^j_\pm \right) = \bigcup_{j \in J} F^j_\pm = \text{int}\left( \bigcup_{j \in J} G^j_\pm \right) \),

(c) \( \partial\left( \bigcup_{j \in J} G^j_\pm \right) = \partial\left( \bigcup_{j \in J} G^j_\pm \right) = \Delta. \)
Proof. (a) We prove the first equality. If $J = \mathbb{Z}$, then, using [3, Problem 109], $G_{\pm}$ is dense in $B(H)$ and the result follows from Theorem 2.

Suppose now that $J \neq \mathbb{Z}$. Using Theorem 5, we obtain that $\Delta \cup \left( \bigcup_{j \in J} F^j_{\pm} \right)$ is closed. But $\bigcup_{j \in J} G^j_{\pm} \subseteq \Delta \cup \left( \bigcup_{j \in J} F^j_{\pm} \right)$. Thus

(1) $\bigcup_{j \in J} G^j_{\pm} \subseteq \Delta \cup \left( \bigcup_{j \in J} F^j_{\pm} \right)$.

On the other hand, using Theorem 11, we have:

(2) $\Delta \cup \left( \bigcup_{j \in J} F^j_{\pm} \right) \subseteq \bigcup_{j \in J} G^j_{\pm}$.

The first equality follows from (1) and (2).

The second equality follows from $G^j_{\pm} = F^j_{\pm}$, for all $j \in \mathbb{Z}$. This follows easily from Theorem 1 and Theorem 11.

(b) is a direct consequence of (a), Theorem 5 and Theorem 8.

(c) Since the first equality follows from (a) and (b), it sufficient to show the second one. Using (a) and (b), we have

$$\partial \left( \bigcup_{j \in J} G^j_{\pm} \right) = \left( \Delta \cup \left( \bigcup_{j \in J} F^j_{\pm} \right) \right) \setminus \left( \bigcup_{j \in J} F^j_{\pm} \right) = \Delta.$$

The proof is complete.

We obtain as consequences the following formulas:

**Corollary 15.** 1) a) $G_+ = \Delta \cup \left( \bigcup_{j \leq 0} F^j_{\pm} \right)$,

b) $G_- = \Delta \cup \left( \bigcup_{j \geq 0} F^j_{\pm} \right)$;

2) a) $\text{int}(G_+) = \bigcup_{j \geq 0} F^j_{\pm}$,

b) $\text{int}(G_-) = \bigcup_{j \leq 0} F^j_{\pm}$;

3) $\partial G_+ = \partial G_- = \Delta$.

**Corollary 16.** 1) a) $G_+ \setminus G = \Delta \cup \left( \bigcup_{j < 0} F^j_{\pm} \right)$,

b) $G_- \setminus G = \Delta \cup \left( \bigcup_{j > 0} F^j_{\pm} \right)$;

2) a) $\text{int}(G_+ \setminus G) = \bigcup_{j < 0} F^j_{\pm}$,

b) $\text{int}(G_- \setminus G) = \bigcup_{j > 0} F^j_{\pm}$;

3) $\partial (G_+ \setminus G) = \partial (G_- \setminus G) = \Delta$.

Similar formulas can be given for $\bigcup_{j \in J} G^j_{\pm}, J \subseteq \mathbb{Z}_-$, and for $\bigcup_{j \in J} G^j_{\pm}, J \subseteq \mathbb{Z}_+$. 

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