

THE QUADRATIC FORM IN THE LÉVY-KHINCHIN FORMULA ON SEMIGROUPS

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ABSTRACT. In this paper we obtain the quadratic form in the Lévy-Khinchin formula on a commutative involutive semigroup, with a neutral element, as a sum of two simpler quadratic forms.

The Lévy-Khinchin representation for a negative definite function, with the real part bounded below, defined on a commutative involutive semigroup with a neutral element, was proved in [1, p. 108, Theorem 3.19].

In Section 2 of this paper we give a proof for this Lévy-Khinchin representation in which we obtain the quadratic form q from [1] as a sum of an additive function and a quadratic form associated with a bi-additive function.

Section 3 is concerned with negative definite functions which have real part bounded below, defined on the semigroup $(\mathbb{Z}^2, +)$ with the involution $(m, n)^* = (n, m)$. We obtain the Lévy-Khinchin representation of these functions using the result of Section 2.

1. NOTATION

Let $(S, +, *)$ be a commutative involutive semigroup with neutral element ([1, p. 86]). We say that a function $\varphi : S \rightarrow \mathbb{C}$ is positive definite if for each natural number $n \geq 1$, each family c_1, \dots, c_n of complex numbers and each family x_1, \dots, x_n of elements of S , we have

$$\sum_{j,k=1}^n c_j \bar{c}_k \varphi(x_j + x_k^*) \geq 0.$$

A function $\varphi : S \rightarrow \mathbb{C}$ is hermitian if $\varphi(x^*) = \overline{\varphi(x)}$ for each $x \in S$.

We say that an hermitian function $\varphi : S \rightarrow \mathbb{C}$ is negative definite if for each natural number $n \geq 2$, each family c_1, \dots, c_n of complex numbers such that $c_1 + \dots + c_n = 0$, and each family x_1, \dots, x_n of elements of S we have

$$\sum_{j,k=1}^n c_j \bar{c}_k \varphi(x_j + x_k^*) \leq 0.$$

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We denote by Γ the set

$$\{\rho : S \rightarrow \mathbb{C} \mid \rho(x+y) = \rho(x)\rho(y); \rho(x^*) = \overline{\rho(x)}; |\rho(x)| \leq 1; \rho(0) = 1\}$$

and by Ω the set $\{\rho \in \Gamma \mid \rho \neq 1\}$.

With the product topology Γ is a compact space and Ω a locally compact space.

We denote by $\mathcal{M}(S)$ the set of positive Radon measure on Ω such that the functions $(\rho \mapsto 1 - \operatorname{Re} \rho(x))_{x \in S}$ are μ -integrable for each μ in $\mathcal{M}(S)$.

An element of $\mathcal{M}(S)$ is called a Lévy measure. We denote by $\mathcal{Q}(S)$ the set

$$\{q : S \rightarrow [0, \infty[\mid 2(q(x) + q(y)) = q(x+y) + q(x+y^*), x, y \in S\}.$$

An element of $\mathcal{Q}(S)$ is called a quadratic form on S . This notion of quadratic form was introduced in [5, p. 211].

Let $\mathcal{A}(S)$ be the set

$$\{a : S \rightarrow [0, \infty[\mid a(x+y) = a(x) + a(y), a(x^*) = a(x), x, y \in S\}$$

and $\mathcal{B}(S)$ the set

$$\begin{aligned} \{b : S \times S \rightarrow \mathbb{R} \mid b(x, y) = b(y, x), b(x^*, y) = -b(x, y), \\ b(x+y, z) = b(x, z) + b(y, z), b(x, x) \in [0, \infty[, x, y, z \in S\}. \end{aligned}$$

We denote by $\mathcal{L}(S)$ the set

$$\begin{aligned} \{L : S \times S \rightarrow \mathbb{R} \mid L(x+y, \rho) = L(x, \rho) + L(y, \rho), L(x^*, \rho) = -L(x, \rho), x, y \in S; \\ \rho \mapsto L(x, \rho) \text{ is continuous on } \Omega, x \in S; \\ \rho \mapsto L(x, \rho) - \operatorname{Im} \rho(x) \text{ is } \mu\text{-integrable}, x \in S, \mu \in \mathcal{M}(S)\}. \end{aligned}$$

An element of $\mathcal{L}(S)$ is called a Lévy function. In [4] is proved that the set $\mathcal{L}(S)$ is nonvoid.

Let $\mathcal{T}(S)$ be the set

$$\{\ell : S \rightarrow \mathbb{R} \mid \ell(x+y) = \ell(x) + \ell(y), \ell(x^*) = -\ell(x), x, y \in S\}.$$

2. THE LÉVY-KHINCHIN REPRESENTATION

Theorem. For a function $\varphi : S \rightarrow \mathbb{C}$ the following conditions are equivalent:

- (i) the function φ is negative definite and has real part bounded below;
- (ii) for every $L \in \mathcal{L}(S)$ there are $C \in \mathbb{R}, a \in \mathcal{A}(S), b \in \mathcal{B}(S), \ell \in \mathcal{T}(S)$ and $\mu \in \mathcal{M}(S)$ which satisfy

$$\varphi(x) = C + a(x) + b(x, x) + i\ell(x) + \int_{\Omega} (1 - \rho(x) + iL(x, \rho)) d\mu(\rho), x \in S.$$

C, a, b and μ are uniquely determined by φ ; ℓ is uniquely determined by φ and L . We have the relations

$$a(x) = \lim_{n \rightarrow \infty} \frac{\varphi(n(x+x^*))}{2n} \quad \text{and} \quad b(x, x) = \lim_{n \rightarrow \infty} \frac{\operatorname{Re} \varphi(nx)}{n^2}, x \in S.$$

Proof. Let U be the vector space

$$\{f : \Gamma \rightarrow \mathbb{R} \mid f(\rho) = \sum_{k=1}^n a_k \rho(x_k), \sum_{k=1}^n a_k = 0, n \in \mathbb{N}, n \geq 2, a_k \in \mathbb{C}, x_k \in S\}$$

and $U_+ = \{f \in U \mid f \geq 0\}$.

For every $t \in]0, \infty[$ the function $\psi_t : S \rightarrow \mathbb{C}$ defined by $\psi_t(x) = e^{-t\varphi(x)}$ is positive definite (cf. [1, p. 74, Theorem 2.2]) and bounded.

It follows from [1, p. 93, Theorem 2.5] that for each $t \in]0, \infty[$ there is a positive Radon measure μ_t on Γ such that

$$e^{-t\varphi(x)} = \int_{\Gamma} \rho(x) d\mu_t(\rho).$$

Let $n \geq 2$ be a natural number, a_1, \dots, a_n complex numbers such that $a_1 + \dots + a_n = 0$, and x_1, \dots, x_n elements of S .

We have

$$(1) \quad \sum_{k=1}^n a_k \left(\frac{e^{-t\varphi(x_k)} - 1}{t} \right) = \frac{1}{t} \int_{\Gamma} \sum_{k=1}^n a_k \rho(x_k) d\mu_t(\rho).$$

Letting t tend to 0, we obtain that the function $F : U \rightarrow \mathbb{R}$ defined by

$$F(\rho \mapsto \sum_{k=1}^n a_k \rho(x_k)) = - \sum_{k=1}^n a_k \varphi(x_k)$$

is well defined. If in (1) we take $x_k + y$ instead of x_k and assume that the function $\rho \mapsto \sum_{k=1}^n a_k \rho(x_k)$ is in U_+ , we obtain, also letting t tend to 0, that the function

$$y \mapsto - \sum_{k=1}^n a_k \varphi(x_k + y)$$

is positive definite and bounded.

Consequently Theorem 2.5 from [1], p. 93, implies that for every $g \in U_+$ there is a positive Radon measure μ_g on Γ such that if $g(\rho) = \sum_{k=1}^n a_k \rho(x_k)$, $\sum_{k=1}^n a_k = 0$, we have

$$- \sum_{k=1}^n a_k \varphi(x_k + y) = \int_{\Gamma} \rho(y) d\mu_g(\rho), \quad y \in S.$$

Let $f \in U, g \in U_+$. We suppose that

$$f(\rho) = \sum_{k=1}^n a_k \rho(x_k), \sum_{k=1}^n a_k = 0 \quad \text{and} \quad g(\rho) = \sum_{\ell=1}^m b_{\ell} \rho(y_{\ell}), \sum_{\ell=1}^m b_{\ell} = 0.$$

We have

$$(2) \quad F(fg) = - \sum_{k=1}^n \sum_{\ell=1}^m a_k b_{\ell} \varphi(x_k + y_{\ell}) = \int_{\Gamma} f(\rho) d\mu_g(\rho).$$

This implies that if $h, g \in U_+$ we have

$$F(\rho \mapsto \rho(x)h(\rho)g(\rho)) = \int_{\Gamma} \rho(x)h(\rho)d\mu_g(\rho) = \int_{\Gamma} \rho(x)g(\rho)d\mu_h(\rho), \quad x \in S,$$

and consequently that

$$(3) \quad h\mu_g = g\mu_h.$$

It follows from (3) that we can define a positive Radon measure μ on Ω such that

$$\mu|_{O_g} = \frac{1}{g|_{O_g}} \mu_g|_{O_g}$$

where $O_g = \{\rho \in \Gamma | g(\rho) > 0\}$.

It is easy to verify that

$$(4) \quad \mu_g|_{\Omega} = g|_{\Omega} \mu.$$

The measure μ is the Lévy measure for φ (cf. [1, p. 103, 3.12]).

Using (4), for g in U_+ we obtain

$$(5) \quad \int_{\Omega} g(\rho) d\mu(\rho) = \int_{\Omega} d\mu_g \leq \int_{\Gamma} d\mu_g = F(g).$$

We denote by E the set

$$\{f \in U \mid F(f) = \int_{\Omega} f(\rho) d\mu(\rho)\}.$$

The relations (2) and (4) yield

$$(6) \quad fg \in E \quad \text{for } f \in U \quad \text{and } g \in U_+$$

because $f(\theta) = 0$, where $\theta : S \rightarrow \mathbb{R}$ is identically 1.

We denote by $B : S \times S \rightarrow \mathbb{R}$ the function defined by

$$B(x, y) = F(\rho \mapsto \operatorname{Im} \rho(x) \operatorname{Im} \rho(y)) - \int_{\Omega} \operatorname{Im} \rho(x) \operatorname{Im} \rho(y) d\mu(\rho).$$

It is immediate that

$$B(x^*, y) = -B(x, y) \quad \text{and} \quad B(x, x) \geq 0.$$

It follows from (6) that the functions

$$\rho \mapsto (1 - \operatorname{Re} \rho(x)) \operatorname{Im} \rho(y) \operatorname{Im} \rho(z)$$

and

$$\rho \mapsto (1 - \operatorname{Re} \rho(y)) \operatorname{Im} \rho(x) \operatorname{Im} \rho(z)$$

are in E .

The equality

$$\begin{aligned} & F(\rho \mapsto (1 - \operatorname{Re} \rho(x)) \operatorname{Im} \rho(y) \operatorname{Im} \rho(z) + (1 - \operatorname{Re} \rho(y)) \operatorname{Im} \rho(x) \operatorname{Im} \rho(z)) \\ &= \int_{\Omega} ((1 - \operatorname{Re} \rho(x)) \operatorname{Im} \rho(y) \operatorname{Im} \rho(z) + (1 - \operatorname{Re} \rho(y)) \operatorname{Im} \rho(x) \operatorname{Im} \rho(z)) d\mu(\rho) \end{aligned}$$

is equivalent to

$$B(x + y, z) = B(x, z) + B(y, z) \quad x, y, z \in S.$$

We have proved that $B \in \mathcal{B}(S)$.

Define $a : S \rightarrow \mathbb{R}$ by

$$\begin{aligned} a(x) &= -\varphi(0) + \operatorname{Re} \varphi(x) - \frac{1}{2} B(x, x) - \int_{\Omega} (1 - \operatorname{Re} \rho(x)) d\mu(\rho) \\ &= F(\rho \mapsto 1 - \operatorname{Re} \rho(x) - \frac{1}{2} (\operatorname{Im} \rho(x))^2) - \int_{\Omega} (1 - \operatorname{Re} \rho(x) - \frac{1}{2} (\operatorname{Im} \rho(x))^2) d\mu(\rho). \end{aligned}$$

First we have $a(x^*) = a(x)$.

Using the relations

$$\begin{aligned} & 1 - \operatorname{Re} \rho(x) - \frac{1}{2} (\operatorname{Im} \rho(x))^2 \\ &= \frac{1}{2} (1 - \operatorname{Re} \rho(x))^2 + \frac{1}{2} (1 - (\operatorname{Re} \rho(x))^2 - (\operatorname{Im} \rho(x))^2) \geq 0, \end{aligned}$$

we see from (5) that $a(x) \geq 0$ for every $x \in S$.

The fact that the function

$$\rho \mapsto (1 - \operatorname{Re} \rho(x))(1 - \operatorname{Re} \rho(y))$$

is in E yields

$$\begin{aligned}
 (7) \quad B(x, y) &= F(\rho \mapsto \operatorname{Im} \rho(x)\operatorname{Im} \rho(y) - (1 - \operatorname{Re} \rho(x))(1 - \operatorname{Re} \rho(y))) \\
 &\quad - \int_{\Omega} (\operatorname{Im} \rho(x)\operatorname{Im} \rho(y) - (1 - \operatorname{Re} \rho(x))(1 - \operatorname{Re} \rho(y))) d\mu(\rho) \\
 &= F(\operatorname{Re} \rho(x) + \operatorname{Re} \rho(y) - 1 - \operatorname{Re} \rho(x + y)) \\
 &\quad - \int_{\Omega} (\operatorname{Re} \rho(y) + \operatorname{Re} \rho(x) - 1 - \operatorname{Re} \rho(x + y)) d\mu(\rho).
 \end{aligned}$$

Using (7) and the equality

$$B(x, y) = \frac{1}{2}(B(x + y, x + y) - B(x, x) - B(y, y)),$$

we obtain

$$a(x + y) = a(x) + a(y), \quad x, y \in S.$$

We have proved that $a \in \mathcal{A}(S)$.

The function $\ell : S \rightarrow \mathbb{R}$ defined by

$$\ell(x) = \operatorname{Im} \varphi(x) - \int_{\Omega} ((-\operatorname{Im} \rho(x)) + L(x, \rho)) d\mu(\rho)$$

satisfies

$$\ell(x^*) = -\ell(x).$$

We have

$$\begin{aligned}
 &\operatorname{Im} \varphi(x) + \operatorname{Im} \varphi(y) - \operatorname{Im} \varphi(x + y) \\
 &= \int_{\Omega} (\operatorname{Im} \rho(x + y) - \operatorname{Im} \rho(x) - \operatorname{Im} \rho(y)) d\mu(\rho)
 \end{aligned}$$

because the function

$$\begin{aligned}
 \rho &\mapsto \operatorname{Im} \rho(x) + \operatorname{Im} \rho(y) - \operatorname{Im} \rho(x + y) \\
 &= (1 - \operatorname{Re} \rho(x))\operatorname{Im} \rho(y) + (1 - \operatorname{Re} \rho(y))\operatorname{Im} \rho(x)
 \end{aligned}$$

is in E . Consequently the function ℓ also satisfies

$$\ell(x + y) = \ell(x) + \ell(y).$$

Taking $b(x, y) = \frac{1}{2}B(x, y)$, we finish the proof of the implication (i) \Rightarrow (ii).

(ii) \Rightarrow (i). Let $b \in \mathcal{B}(S)$. Using the relation

$$b(x + y^*, x + y^*) = b(x, x) + b(y, y) - 2b(x, y)$$

we see, as in [1, p. 103], that the function $x \mapsto b(x, x)$ is negative definite. Now the implication (ii) \Rightarrow (i) is clear.

If we have a representation as in (ii), we obtain

$$\begin{aligned}
 (8) \quad \varphi(n(x + x^*)) &= \varphi(0) + 2na(x) + n^2b(x + x^*, x + x^*) \\
 &\quad + \int_{\Omega} (1 - \rho(n(x + x^*))) d\mu(\rho), \quad x \in S, n \in \mathbb{N}^*.
 \end{aligned}$$

The dominated convergence theorem implies

$$(9) \quad \lim_{n \rightarrow \infty} \frac{1}{2n} \int_{\Omega} (1 - |\rho(x)|^{2n}) d\mu(\rho) = 0,$$

because $\frac{1 - |\rho(x)|^{2n}}{2n} \leq 1 - |\rho(x)|^2$.

The relations $b(x + x^*, x + x^*) = 0$, (8) and (9) yield

$$(10) \quad a(x) = \lim_{n \rightarrow \infty} \frac{1}{2n} \varphi(n(x + x^*)).$$

From the equality (7) we obtain

$$(11) \quad \begin{aligned} 2b(nx, nx^*) &= \varphi(0) - \varphi(nx) - \varphi(nx^*) + \varphi(n(x + x^*)) \\ &+ \int_{\Omega} 1 - ((\rho(x))^n)(1 - (\rho(x^*))^n) d\mu(\rho) \end{aligned}$$

We have, using again the dominated convergence theorem,

$$(12) \quad \lim_{n \rightarrow \infty} \frac{1}{n^2} \int_{\Omega} (1 - (\rho(x))^n)(1 - (\rho(x^*))^n) d\mu(\rho) = 0$$

because $\frac{1}{n^2} |(1 - (\rho(x))^n)(1 - (\rho(x^*))^n)| \leq |1 - \rho(x)|^2$.

The relations $b(x, x^*) = -b(x, x)$, (10), (11) and (12) now give

$$b(x, x) = \lim_{n \rightarrow \infty} \frac{1}{n^2} \operatorname{Re} \varphi(nx).$$

The representation from (ii) implies that

$$\begin{aligned} &\varphi(x + y) + \varphi(x + y^*) - 2\varphi(x) \\ &= 2(a(y) + b(y, y) + \int_{\Omega} \rho(x)(1 - \operatorname{Re} \rho(y)) d\mu(\rho)), \end{aligned}$$

which proves the unicity of the measure μ . This completes the proof of the theorem. □

Corollary. *A function $q : S \rightarrow [0, \infty[$ is an element of $\mathcal{Q}(S)$ if and only if there are $a \in \mathcal{A}(S)$ and $b \in \mathcal{B}(S)$ such that*

$$q(x) = a(x) + b(x, x).$$

The functions a and b are uniquely determined by q according to the relations

$$a(x) = \frac{1}{2}q(x + x^*) \quad \text{and} \quad b(x, x) = \lim_{n \rightarrow \infty} \frac{q(nx)}{n^2}, \quad x \in S.$$

The corollary is an immediate consequence of the unicity results in the theorem of this section and [1, p. 108, Theorem 3.19]. It also results from [1], p. 102, because if we denote by $a : S \rightarrow \mathbb{R}$ the function defined by

$$a(x) = q(x) - b(x, x),$$

where $b(x, y) = \frac{1}{2}(-q(x) - q(y) + q(x + y))$, we have

$$a(x + y) = a(x) + a(y), \quad a(x^*) = a(x),$$

and the definition of $\mathcal{Q}(S)$ gives

$$a(x) = q(x) - \frac{1}{2}(-2q(x) + q(2x)) = \frac{1}{2}q(x + x^*).$$

Remark 1. Using the method of Bloom and Ressel from [2], we deduce from the representation given in this paper that the quadratic form on some commutative hypergroups may be written as a sum of two quadratic forms (see [2, p. 248, Theorem 2.4, and p. 250, Theorem 2.6]).

Remark 2. The relation $q(x) = a(x) + b(x, x)$ of the corollary is also a consequence of [3, p. 636, Theorem 8].

3. AN APPLICATION

Proposition. Consider on the semigroup $(\mathbb{Z}^2, +)$ the involution $(m, n)^* = (n, m)$. For a function $\varphi : \mathbb{Z}^2 \rightarrow \mathbb{C}$ the following conditions are equivalent:

- (i) the function φ is negative definite and has real part bounded below;
- (ii) there are real numbers C, α, β, γ , such that $\alpha, \beta \geq 0$, and a positive Radon measure μ on $\{z \in \mathbb{C} \mid |z| = 1, z \neq 1\}$ such that the function $z \mapsto 1 - \operatorname{Re} z$ is μ -integrable, which satisfy

$$\begin{aligned} \varphi(m, n) &= C + (m + n)\alpha + (m - n)^2\beta + i(m - n)\gamma \\ &\quad + \int_{T \setminus \{1\}} (1 - z^m \bar{z}^n + (m - n)\operatorname{Im} z) d\mu(z), \end{aligned}$$

where $T = \{z \in \mathbb{C} \mid |z| = 1\}$.

C, α, β, γ and μ are uniquely determined by φ .

Proof. We note that the function

$$z \mapsto ((m, n) \mapsto z^m \bar{z}^n)$$

is a homeomorphism of T onto the space of bounded characters of \mathbb{Z}^2 .

It is easy to see that $a \in \mathcal{A}(\mathbb{Z}^2)$ if and only if there is $\alpha \in [0, \infty[$ such that $a(m, n) = (m + n)\alpha$, that $b \in \mathcal{B}(\mathbb{Z}^2)$ if and only if there is $\beta \in [0, \infty[$ such that $b((m, n), (p, q)) = (mp + nq - mq - np)\beta$, and that $\ell \in \mathcal{T}(\mathbb{Z}^2)$ if and only if there is a real number γ such that $\ell(m, n) = (m - n)\gamma$.

Let μ be a positive Radon measure on $T \setminus \{1\}$ such that the function $z \mapsto 1 - \operatorname{Re} z$ is μ -integrable. We note that the function $z \mapsto (1 - z)^2$ is also μ -integrable.

We show that the functions $z \mapsto 1 - z^m \bar{z}^n + m(z - 1) + n(\bar{z} - 1)$ and $z \mapsto 1 - \operatorname{Re} z^m \bar{z}^n$ are μ -integrable for every $(m, n) \in \mathbb{Z}^2$. Take, for example, $m < 0$ and $n \geq 0$. Using the binomial theorem, we obtain that the function

$$\left(\frac{1}{z}\right)^{-m} \bar{z}^n + m\left(\frac{1}{z} - 1\right) - n(\bar{z} - 1) - 1$$

is μ -integrable.

We have

$$\begin{aligned} & z^m \bar{z}^n - m(z - 1) - n(\bar{z} - 1) - 1 \\ &= \left(\frac{1}{z}\right)^{-m} \bar{z}^n + m\left(\frac{1}{z} - 1\right) - n(\bar{z} - 1) - 1 - m\frac{(1 - z)^2}{z}, \end{aligned}$$

which means that the functions

$$z \mapsto 1 - z^m \bar{z}^n + m(z - 1) + n(\bar{z} - 1)$$

and $z \mapsto 1 - \operatorname{Re} z^m \bar{z}^n$ are μ -integrable.

The other cases are proved in a similar way. It follows that we can choose the function $L : \mathbb{Z}^2 \times (T \setminus \{1\}) \rightarrow \mathbb{C}$ defined by $L((m, n), z) = (m - n)\operatorname{Im} z$ as a Lévy function for \mathbb{Z}^2 , and that $\mathcal{M}(\mathbb{Z}^2) = \{\mu \text{ positive Radon measure on } T \setminus \{1\} \mid \text{the function } z \mapsto 1 - \operatorname{Re} z \text{ is } \mu\text{-integrable}\}$.

Now the proposition is a particular case of the theorem given in Section 2. \square

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