THE QUADRATIC FORM IN THE LÉVY-KHINCHIN FORMULA ON SEMIGROUPS

DRAGU ATANASIU

(Communicated by Palle E. T. Jorgensen)

Abstract. In this paper we obtain the quadratic form in the Lévy-Khinchin formula on a commutative involutive semigroup, with a neutral element, as a sum of two simpler quadratic forms.

The Lévy-Khinchin representation for a negative definite function, with the real part bounded below, defined on a commutative involutive semigroup with a neutral element, was proved in [1, p. 108, Theorem 3.19].

In Section 2 of this paper we give a proof for this Lévy-Khinchin representation in which we obtain the quadratic form $q$ from [1] as a sum of an additive function and a quadratic form associated with a bi-additive function.

Section 3 is concerned with negative definite functions which have real part bounded below, defined on the semigroup $(\mathbb{Z}^2, +)$ with the involution $(m, n)^* = (n, m)$. We obtain the Lévy-Khinchin representation of these functions using the result of Section 2.

1. Notation

Let $(S, +, *)$ be a commutative involutive semigroup with neutral element ([1, p. 86]). We say that a function $\varphi : S \to \mathbb{C}$ is positive definite if for each natural number $n \geq 1$, each family $c_1, \ldots, c_n$ of complex numbers and each family $x_1, \ldots, x_n$ of elements of $S$, we have

$$\sum_{j,k=1}^n c_j \bar{c}_k \varphi(x_j + x_k^*) \geq 0.$$

A function $\varphi : S \to \mathbb{C}$ is hermitian if $\varphi(x^*) = \overline{\varphi(x)}$ for each $x \in S$.

We say that an hermitian function $\varphi : S \to \mathbb{C}$ is negative definite if for each natural number $n \geq 2$, each family $c_1, \ldots, c_n$ of complex numbers such that $c_1 + \ldots + c_n = 0$, and each family $x_1, \ldots, x_n$ of elements of $S$ we have

$$\sum_{j,k=1}^n c_j \bar{c}_k \varphi(x_j + x_k^*) \leq 0.$$
We denote by $\Gamma$ the set
\[
\{ \rho : S \to \mathbb{C} | \rho(x + y) = \rho(x)\rho(y); \rho(x^*) = \overline{\rho(x)}; |\rho(x)| \leq 1; \rho(0) = 1 \}
\]
and by $\Omega$ the set $\{ \rho \in \Gamma | \rho \neq 1 \}$.

With the product topology $\Gamma$ is a compact space and $\Omega$ a locally compact space.

We denote by $\mathcal{M}(S)$ the set of positive Radon measure on $\Omega$ such that the functions $(\rho \to 1 - \Re \rho(x))_{x \in S}$ are $\mu$-integrable for each $\mu$ in $\mathcal{M}(S)$.

An element of $\mathcal{M}(S)$ is called a Lévy measure. We denote by $\mathcal{Q}(S)$ the set
\[
\{ q : S \to [0, \infty) | 2(q(x) + q(y)) = q(x + y) + q(x + y^*), x, y \in S \}.
\]
An element of $\mathcal{Q}(S)$ is called a quadratic form on $S$. This notion of quadratic form was introduced in [5, p. 211].

Let $\mathcal{A}(S)$ be the set
\[
\{ a : S \to [0, \infty) | a(x + y) = a(x) + a(y), a(x^*) = a(x), x, y \in S \}
\]
and $\mathcal{B}(S)$ the set
\[
\{ b : S \times S \to \mathbb{R} | b(x, y) = b(y, x), b(x^*, y) = -b(x, y),
\]
\[
b(x + y, z) = b(x, z) + b(y, z), b(x, x) \in [0, \infty], x, y, z \in S \}.
\]
We denote by $\mathcal{L}(S)$ the set
\[
\{ L : S \times \Omega \to \mathbb{R} | L(x + y, \rho) = L(x, \rho) + L(y, \rho), L(x^*, \rho) = -L(x, \rho), x, y \in S;
\]
\[
\rho \mapsto L(x, \rho) \text{ is continuous on } \Omega, x \in S;
\]
\[
\rho \mapsto L(x, \rho) - \Im \rho(x) \text{ is } \mu\text{-integrable, } x \in S, \mu \in \mathcal{M}(S) \}.
\]
An element of $\mathcal{L}(S)$ is called a Lévy function. In [4] is proved that the set $\mathcal{L}(S)$ is nonvoid.

Let $\mathcal{T}(S)$ be the set
\[
\{ \ell : S \to \mathbb{R} | \ell(x + y) = \ell(x) + \ell(y), \ell(x^*) = -\ell(x), x, y \in S \}.
\]

2. The Lévy-Khinchin representation

**Theorem.** For a function $\varphi : S \to \mathbb{C}$ the following conditions are equivalent:

(i) the function $\varphi$ is negative definite and has real part bounded below;

(ii) for every $L \in \mathcal{L}(S)$ there are $C \in \mathbb{R}, a \in \mathcal{A}(S), b \in \mathcal{B}(S), \ell \in \mathcal{T}(S)$ and $\mu \in \mathcal{M}(S)$ which satisfy
\[
\varphi(x) = C + a(x) + b(x, x) + i\ell(x) + \int_{\Omega} (1 - \rho(x) + iL(x, \rho))d\mu(\rho), x \in S.
\]

$C, a, b$ and $\mu$ are uniquely determined by $\varphi; \ell$ is uniquely determined by $\varphi$ and $L$. We have the relations
\[
a(x) = \lim_{n \to \infty} \frac{\varphi(n(x + x^*))}{2n} \quad \text{and} \quad b(x, x) = \lim_{n \to \infty} \frac{\Re \varphi(nx)}{n^2}, x \in S.
\]

**Proof.** Let $U$ be the vector space
\[
\{ f : \Gamma \to \mathbb{R} | f(\rho) = \sum_{k=1}^{n} a_k \rho(x_k), \sum_{k=1}^{n} a_k = 0, n \in \mathbb{N}, n \geq 2, a_k \in \mathbb{C}, x_k \in S \}
\]
and $U_+ = \{ f \in U | f \geq 0 \}$.

For every $t \in [0, \infty]$ the function $\psi_t : S \to \mathbb{C}$ defined by $\psi_t(x) = e^{-t\varphi(x)}$ is positive definite (cf. [1, p. 74, Theorem 2.2]) and bounded.
It follows from [1, p. 93, Theorem 2.5] that for each \( t \in ]0, \infty[ \) there is a positive Radon measure \( \mu_t \) on \( \Gamma \) such that
\[
e^{-t\varphi(x)} = \int_{\Gamma} \rho(x) d\mu_t(\rho).
\]

Let \( n \geq 2 \) be a natural number, \( a_1, \ldots, a_n \) complex numbers such that \( a_1 + \ldots + a_n = 0 \), and \( x_1, \ldots, x_n \) elements of \( S \).

We have
\[
\sum_{k=1}^{n} a_k \left( \frac{e^{-t\varphi(x_k)} - 1}{t} \right) = \frac{1}{t} \int_{\Gamma} \sum_{k=1}^{n} a_k \rho(x_k) d\mu_t(\rho).
\]

Letting \( t \) tend to 0, we obtain that the function \( F : U \rightarrow \mathbb{R} \) defined by
\[
F(\rho \mapsto \sum_{k=1}^{n} a_k \rho(x_k)) = -\sum_{k=1}^{n} a_k \varphi(x_k)
\]
is well defined. If in (1) we take \( x_k + y \) instead of \( x_k \) and assume that the function \( \rho \mapsto \sum_{k=1}^{n} a_k \rho(x_k) \) is in \( U_+ \), we obtain, also letting \( t \) tend to 0, that the function
\[
y \mapsto -\sum_{k=1}^{n} a_n \varphi(x_n + y)
\]
is positive definite and bounded.

Consequently Theorem 2.5 from [1], p. 93, implies that for every \( g \in U_+ \) there is a positive Radon measure \( \mu_g \) on \( \Gamma \) such that if \( g(\rho) = \sum_{k=1}^{n} a_k \rho(x_k), \sum_{k=1}^{n} a_k = 0 \), we have
\[
-\sum_{k=1}^{n} a_k \varphi(x_k + y) = \int_{\Gamma} \rho(y) d\mu_g(\rho), \ y \in S.
\]

Let \( f, g \in U_+ \). We suppose that
\[
f(\rho) = \sum_{k=1}^{n} a_k \rho(x_k), \sum_{k=1}^{n} a_k = 0 \quad \text{and} \quad g(\rho) = \sum_{\ell=1}^{m} b_\ell \rho(y_\ell), \sum_{\ell=1}^{m} b_\ell = 0.
\]

We have
\[
F(fg) = -\sum_{k=1}^{n} \sum_{\ell=1}^{m} a_k b_\ell \varphi(x_k + y_\ell) = \int_{\Gamma} f(\rho) d\mu_g(\rho).
\]

This implies that if \( h, g \in U_+ \) we have
\[
F(\rho \mapsto \rho(x)h(\rho)g(\rho)) = \int_{\Gamma} \rho(x)h(\rho) d\mu_g(\rho) = \int_{\Gamma} \rho(x)g(\rho) d\mu_h(\rho), \ x \in S,
\]
and consequently that
\[
h \mu_g = g \mu_h.
\]

It follows from (3) that we can define a positive Radon measure \( \mu \) on \( \Omega \) such that
\[
\mu|_{O_g} = \frac{1}{g|_{O_g}} \mu_g|_{O_g}
\]
where \( O_g = \{ \rho \in \Gamma | g(\rho) > 0 \} \).

It is easy to verify that
\[
\mu_g|_{\Omega} = g|_{\Omega} \mu.
\]
The measure $\mu$ is the Lévy measure for $\varphi$ (cf. [1, p. 103, 3.12]).

Using (4), for $g$ in $U_+$ we obtain

$$\int_{\Omega} g(\rho)d\mu(\rho) = \int_{\Omega} d\mu_g \leq \int_{\Gamma} d\mu_g = F(g).$$

We denote by $E$ the set

$$\{ f \in U | F(f) = \int_{\Omega} f(\rho)d\mu(\rho) \}.$$

The relations (2) and (4) yield

$$fg \in E \quad \text{for} \quad f \in U \quad \text{and} \quad g \in U_+$$

because $f(\theta) = 0$, where $\theta : S \to \mathbb{R}$ is identically 1.

We denote by $B : S \times S \to \mathbb{R}$ the function defined by

$$B(x, y) = F(\rho \mapsto \text{Im} \rho(x) \text{Im} \rho(y)) - \int_{\Omega} \text{Im} \rho(x) \text{Im} \rho(y)d\mu(\rho).$$

It is immediate that

$$B(x^*, y) = -B(x, y) \quad \text{and} \quad B(x, x) \geq 0.$$ 

It follows from (6) that the functions

$$\rho \mapsto (1 - \text{Re} \rho(x))\text{Im} \rho(y)\text{Im} \rho(z)$$

and

$$\rho \mapsto (1 - \text{Re} \rho(y))\text{Im} \rho(x)\text{Im} \rho(z)$$

are in $E$.

The equality

$$F(\rho \mapsto (1 - \text{Re} \rho(x))\text{Im} \rho(y)\text{Im} \rho(z) + (1 - \text{Re} \rho(y))\text{Im} \rho(x)\text{Im} \rho(z))$$

$$= \int_{\Omega} ((1 - \text{Re} \rho(x))\text{Im} \rho(y)\text{Im} \rho(z) + (1 - \text{Re} \rho(y))\text{Im} \rho(x)\text{Im} \rho(z))d\mu(\rho)$$

is equivalent to

$$B(x + y, z) = B(x, z) + B(y, z) \quad x, y, z \in S.$$ 

We have proved that $B \in B(S)$.

Define $a : S \to \mathbb{R}$ by

$$a(x) = -\varphi(0) + \text{Re} \varphi(x) - \frac{1}{2}B(x, x) - \int_{\Omega} (1 - \text{Re} \rho(x))d\mu(\rho)$$

$$= F(\rho \mapsto 1 - \text{Re} \rho(x) - \frac{1}{2}(\text{Im} \rho(x))^2) - \int_{\Omega} (1 - \text{Re} \rho(x) - \frac{1}{2}(\text{Im} \rho(x))^2)d\mu(\rho).$$

First we have $a(x^*) = a(x)$.

Using the relations

$$1 - \text{Re} \rho(x) - \frac{1}{2}(\text{Im} \rho(x))^2$$

$$= \frac{1}{2}(1 - \text{Re} \rho(x))^2 + \frac{1}{2}(1 - (\text{Re} \rho(x))^2 - (\text{Im} \rho(x))^2) \geq 0,$$

we see from (5) that $a(x) \geq 0$ for every $x \in S$.

The fact that the function

$$\rho \mapsto (1 - \text{Re} \rho(x))(1 - \text{Re} \rho(y))$$
We have proved that

\[ (9) \]

Taking \( b \) satisfies

we obtain

We have because the function

is in \( E \) we see, as in [1, p. 103], that the function

implication (ii) \( \Rightarrow \)

The dominated convergence theorem implies

\[ (8) \]

Using (7) and the equality

\[ (7) \]

is in \( E \) \( (\text{ii}). \) Let

\[ (\ell \in \mathcal{A}(S)) \text{ defined by} \]

\[ \ell(x) = \Im \varphi(x) - \int_{\Omega} ((-\Im \rho(x)) + L(x, \rho))d\mu(\rho) \]

satisfies

\[ \ell(x^*) = -\ell(x). \]

We have

\[ \Im \varphi(x) + \Im \varphi(y) - \Im \varphi(x + y) = \int_{\Omega} (\Im \rho(x + y) - \Im \rho(x) - \Im \rho(y))d\mu(\rho) \]

because the function

\[ \rho \mapsto \Im \rho(x) + \Im \rho(y) - \Im \rho(x + y) = (1 - \Re \rho(x))\Im \rho(y) + (1 - \Re \rho(y))\Im \rho(x) \]

is in \( E \). Consequently the function \( \ell \) also satisfies

\[ \ell(x + y) = \ell(x) + \ell(y). \]

Taking \( b(x, y) = \frac{1}{2} B(x, y) \), we finish the proof of the implication (i) \( \Rightarrow \) (ii).

(iii) \( \Rightarrow \) (ii). Let \( b \in \mathcal{B}(S) \). Using the relation

\[ b(x + y^*, x + y^*) = b(x, x) + b(y, y) - 2b(x, y) \]

we see, as in [1, p. 103], that the function \( x \mapsto b(x, x) \) is negative definite. Now the implication (ii) \( \Rightarrow \) (i) is clear.

If we have a representation as in (ii), we obtain

\[ (8) \]

\[ \phi(n(x + x^*)) = \phi(0) + 2\pi \alpha(x) + n^2 b(x + x^*, x + x^*) 
+ \int_{\Omega} (1 - \rho(n(x + x^*)))d\mu(\rho), \quad x \in S, n \in \mathbb{N}^*. \]

The dominated convergence theorem implies

\[ (9) \]

\[ \lim_{n \to \infty} \frac{1}{2n} \int_{\Omega} (1 - \rho(x)^{2n})d\mu(\rho) = 0, \]

because \( \frac{1 - |\rho(x)|^{2n}}{2n} \leq 1 - |\rho(x)|^2 \).
The functions $Q$ and the definition of $[3, p. 636, Theorem 8].$

From the equality (7) we obtain
\[ 2b(nx, nx^*) = \varphi(0) - \varphi(nx) - \varphi(nx^*) + \varphi(n(x + x^*)) \]
\[ + \int \Omega (1 - (\rho(x))^n)(1 - (\rho(x^*))^n) d\mu(\rho) \]

We have, using again the dominated convergence theorem,
\[ \lim_{n \to \infty} \frac{1}{n^2} \int \Omega (1 - (\rho(x))^n)(1 - (\rho(x^*))^n) d\mu(\rho) = 0 \]
because $\frac{1}{n^2}|(1 - (\rho(x))^n)(1 - (\rho(x^*))^n)| \leq |1 - \rho(x)|^2$.

The relations $b(x, x^*) = -b(x, x)$, (10), (11) and (12) now give
\[ b(x, x) = \lim_{n \to \infty} \frac{1}{n^2} \text{Re} \varphi(nx). \]

The representation from (ii) implies that
\[ \varphi(x + y) + \varphi(x + y^*) - 2\varphi(x) \]
\[ = 2(a(y) + b(y, y) + \int \Omega \rho(x)(1 - \text{Re} \rho(y)) d\mu(\rho)), \]
which proves the unicity of the measure $\mu$. This completes the proof of the theorem. \hfill \Box

**Corollary.** A function $q : S \to [0, \infty]$ is an element of $Q(S)$ if and only if there are $a \in \mathcal{A}(S)$ and $b \in \mathcal{B}(S)$ such that $q(x) = a(x) + b(x, x)$.

The functions $a$ and $b$ are uniquely determined by $q$ according to the relations
\[ a(x) = \frac{1}{2} q(x + x^*) \quad \text{and} \quad b(x, x) = \lim_{n \to \infty} \frac{q(nx)}{n^2}, \quad x \in S. \]

The corollary is an immediate consequence of the unicity results in the theorem of this section and [1, p. 108, Theorem 3.19]. It also results from [1, p. 102], because if we denote by $a : S \to \mathbb{R}$ the function defined by
\[ a(x) = q(x) - b(x, x), \]
where $b(x, y) = \frac{1}{2} (-q(x) - q(y) + q(x + y))$, we have
\[ a(x + y) = a(x) + a(y), a(x^*) = a(x), \]
and the definition of $Q(S)$ gives
\[ a(x) = q(x) - \frac{1}{2} (-2q(x) + q(2x)) = \frac{1}{2} q(x + x^*). \]

**Remark 1.** Using the method of Bloom and Ressel from [2], we deduce from the representation given in this paper that the quadratic form on some commutative hypergroups may be written as a sum of two quadratic forms (see [2, p. 248, Theorem 2.4, and p. 250, Theorem 2.6]).

**Remark 2.** The relation $q(x) = a(x) + b(x, x)$ of the corollary is also a consequence of [3, p. 636, Theorem 8].
3. An application

Proposition. Consider on the semigroup \((\mathbb{Z}^2, +)\) the involution \((m, n)^* = (n, m)\). For a function \(\varphi : \mathbb{Z}^2 \to \mathbb{C}\) the following conditions are equivalent:

(i) the function \(\varphi\) is negative definite and has real part bounded below;
(ii) there are real numbers \(C, \alpha, \beta, \gamma\), such that \(\alpha, \beta \geq 0\), and a positive Radon measure \(\mu\) on \(\{z \in \mathbb{C}||z| = 1, z \neq 1\}\) such that the function \(z \mapsto 1 - \Re z\) is \(\mu\)-integrable, which satisfy

\[
\varphi(m, n) = C + (m + n)\alpha + (m - n)^2\beta + i(m - n)\gamma \\
+ \int_{T\setminus\{1\}} (1 - z^m\bar{z}^n + (m - n)\Im z) d\mu(z),
\]

where \(T = \{z \in \mathbb{C}||z| = 1\}\).

\(C, \alpha, \beta, \gamma\) and \(\mu\) are uniquely determined by \(\varphi\).

Proof. We note that the function

\[z \mapsto ((m, n) \mapsto z^m\bar{z}^n)\]

is a homeomorphism of \(T\) onto the space of bounded characters of \(\mathbb{Z}^2\).

It is easy to see that \(a \in \mathcal{A}(\mathbb{Z}^2)\) if and only if there is \(\alpha \in [0, \infty]\) such that \(a(m, n) = (m + n)\alpha\), that \(b \in \mathcal{B}(\mathbb{Z}^2)\) if and only if there is \(\beta \in [0, \infty]\) such that \(b((m, n), (p, q)) = (mp + nq - mq - np)\beta\), and that \(\ell \in T(\mathbb{Z}^2)\) if and only if there is a real number \(\gamma\) such that \(\ell(m, n) = (m - n)\gamma\).

Let \(\mu\) be a positive Radon measure on \(T \setminus \{1\}\) such that the function \(z \mapsto 1 - \Re z\) is \(\mu\)-integrable. We note that the function \(z \mapsto (1 - z)^2\) is also \(\mu\)-integrable.

We show that the functions \(z \mapsto 1 - z^m\bar{z}^n + m(z - 1) + n(\bar{z} - 1)\) and \(z \mapsto 1 - \Re z^m\bar{z}^n\) are \(\mu\)-integrable for every \((m, n) \in \mathbb{Z}^2\). Take, for example, \(m < 0\) and \(n \geq 0\). Using the binomial theorem, we obtain that the function

\[
(\frac{1}{z})^{-m} \bar{z}^n + m(\frac{1}{z} - 1) - n(\bar{z} - 1) - 1
\]

is \(\mu\)-integrable.

We have

\[
z^m\bar{z}^n - m(z - 1) - n(\bar{z} - 1) - 1 \\
= \left(\frac{1}{z}\right)^{-m}\bar{z}^n + m\left(\frac{1}{z} - 1\right) - n(\bar{z} - 1) - 1 - m\frac{(1 - z)^2}{z},
\]

which means that the functions

\[z \mapsto 1 - z^m\bar{z}^n + m(z - 1) + n(\bar{z} - 1)\]

and \(z \mapsto 1 - \Re z^m\bar{z}^n\) are \(\mu\)-integrable.

The other cases are proved in a similar way. It follows that we can choose the function \(L : \mathbb{Z}^2 \times (T \setminus \{1\}) \to \mathbb{C}\) defined by \(L((m, n), z) = (m - n)\Im z\) as a Lévy function for \(\mathbb{Z}^2\), and that \(\mathcal{M}(\mathbb{Z}^2) = \{\mu \text{ positive Radon measure on } T \setminus \{1\}\text{ such that the function } z \mapsto 1 - \Re z \text{ is } \mu\text{-integrable}\}\).

Now the proposition is a particular case of the theorem given in Section 2. \(\square\)
REFERENCES


Department of Mathematics, Chalmers University of Technology and Göteborg University, S-412 96 Göteborg, Sweden

E-mail address: dragu@math.chalmers.se