HAUSDORFF DIMENSION AND DOUBLING MEASURES
ON METRIC SPACES

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Abstract. Vol‘berg and Konyagin have proved that a compact metric space carries
a nontrivial doubling measure if and only if it has finite uniform metric
dimension. Their construction of doubling measures requires infinitely many
adjustments. We give a simpler and more direct construction, and also prove
that for any $\alpha > 0$, the doubling measure may be chosen to have full measure
on a set of Hausdorff dimension at most $\alpha$.

Let $(X, \rho)$ be a compact metric space. Vol‘berg and Konyagin proved in [VK]
that $(X, \rho)$ carries a nontrivial doubling measure $\mu$ (there exists $\Lambda \geq 1$ so that
$\mu(B(x, 2r)) \leq \Lambda \mu(B(x, r))$ for all $x \in X$ and $r > 0$) if and only if $(X, \rho)$ has
finite uniform metric dimension (in each ball $B(x, 2r)$, there exist at most $N$ points with
mutual distances at least $r$). Here $B(x, r) = \{y : \rho(x, y) < r\}$.

Assume that $(X, \rho)$ has finite uniform metric dimension. The construction of
doubling measures in [VK] requires infinitely many adjustments which cannot be
predicted in advance. In this note, we give a simpler and more direct construction,
and prove that given any $\alpha > 0$, there exists a doubling measure on $X$ that has
full measure on a set of Hausdorff dimension at most $\alpha$. Also we observe that a
doubling measure may be concentrated on a countable set even when $X$ is a set on
the real line of positive length. Some ideas have been adapted from [FKP], [VK]
and [T].

1. Theorems and examples

Assume, from now on, that $(X, \rho)$ is a compact metric space of finite uniform
metric dimension and that $\operatorname{diam} X < 1$.

For each $k \geq 0$, let $S_k = \{x_{k,j} : 1 \leq j \leq J(k)\}$ be a maximal $10^{-k}$-net on $X$
(points in $S_k$ having mutual distances at least $10^{-k}$, and points outside $S_k$ having
distances less than $10^{-k}$ to $S_k$), satisfying

$$S_0 \subseteq S_1 \subseteq \ldots \subseteq S_k \subseteq S_{k+1} \subseteq \ldots$$

Note that $S_0$ has only one point $x_{0,1}$.

For each $k \geq 0$, let $\{T_{k,j} : 1 \leq j \leq J(k)\}$ be a partition of $S_{k+1}$ satisfying

$$S_{k+1} \cap B(x_{k,j}, 10^{-k}/2) \subseteq T_{k,j} \subseteq S_{k+1} \cap B(x_{k,j}, 10^{-k}).$$
We call elements of $T_{k,j}$ branch points of $x_{k,j}$, the element $x_{k,j}$ an old branch point and the rest new branch points. Since $X$ has finite uniform metric dimension, $T_{k,j}$ has at most $N^4$ elements.

Let $M \geq N^4$, and let $w_{k,j}$ be weights at $x_{k,j}$, $k \geq 1$ so that

$$M - 1 \leq w_{k,j} \leq 1,$$

and

$$\sum_{x_{k+1,i} \in T_{k,j}} w_{k+1,i} = 1.$$

**Theorem 1.** Assume that $\mu_k$ ($k \geq 0$) are measures on $X$ with total mass concentrated on $S_k$, defined as follows: $\mu_0$ is the unit point measure at $x_{0,1}$; after $\mu_k$ is chosen, $\mu_{k+1}$ is defined by distributing the mass from $x_{k,j}$ to its branch points in $T_{k,j}$ so that

$$\mu_{k+1}(\{x_{k+1,i}\}) = w_{k+1,i} \mu_k(\{x_{k,j}\}), \quad x_{k+1,i} \in T_{k,j}.$$

Then $\{\mu_k\}$ converges in the weak star topology to a doubling measure $\mu$ on $(X, \rho)$ with

$$\mu(B(x, 2r)) \leq M^3 N^8 \mu(B(x, r))$$

for each $x \in X$ and $r > 0$.

This construction works because of (1.3)—the weight being a constant at all new branch points in any given generation. This allows us to compare measures of any two nearby branch points, regardless of their ancestors.

When $M$ is large, with a suitable choice of weights, the measure $\mu$ is concentrated on a small set. The next theorem extends a result of Tukia [T] on Euclidean space to metric spaces.

**Theorem 2.** Given $\alpha > 0$, there exists a doubling measure on $(X, \rho)$ that has full measure on a set of Hausdorff dimension at most $\alpha$.

Recall that the $\beta$-dimensional Hausdorff content of a set $E$ in $X$ is the number $H_\beta(E) = \inf \sum_j r_j^\beta$, where the infimum is taken over all countable covers of $E$ by balls of radii $r_j$. The Hausdorff dimension of a set $E$ is $\inf \{\beta : H_\beta(E) = 0\}$.

A doubling measure on a ball in an Euclidean space cannot have full measure on a set of zero Hausdorff dimension. In contrast, the following examples exist for sets having no interiors.

**Example 1.** For each $\alpha \in [0, 1]$, there exists a compact set $X \subseteq \mathbb{R}^1$ of Hausdorff dimension $\alpha$ so that every doubling measure on $X$ is purely atomic.

**Example 2.** There exists a compact set $X \subseteq \mathbb{R}^1$ of positive length, so that some doubling measures on $X$ are purely atomic.

Both examples are essentially in [KW] and were constructed for another purpose. Let $E$ be the Cantor ternary set on the unit interval, $F$ be the midpoints of all complementary intervals and $X = E \cup F$. Then every doubling measure on $X$ is concentrated on $F$. A similar construction works for every $\alpha$ in $[0, 1)$. When $\alpha = 1$, we combine an appropriate sequence of such sets together with their limit points.
As for Example 2, let \( \nu \) be a doubling measure on \( \mathbb{R}^3 \) having full measure on a set of zero length as constructed in [BA], and let \( E \) be a compact subset contained in \([0,1]\) having positive length and zero \( \nu \)-measure. Let \( \mathcal{W} \) be a Whitney decomposition of \((-2,2) \setminus E\), and \( F \) be the collection of midpoints of the intervals in \( \mathcal{W} \). Let \( X = E \cup F \cup \{-2,2\} \), and let \( \mu \) be the measure on \( X \) with total mass on \( F \) so that at each \( x \in F \), \( \mu(\{x\}) \) is the \( \nu \)-measure of the corresponding Whitney interval. Then \( X \) and \( \mu \) have the properties required.

For details, see the examples \( X \) and \( Z \) in [KW].

2. Proof of Theorem 1

Define history \( h \) on \( \bigcup \limits_{k \geq 1} S_k \) as follows: \( h(x) = (x_{0,1}, x) \) on \( S_1 \); and for \( x \in T_{k,j} \subseteq S_{k+1} \), \( h(x) \) is the \((k+2)\)-tuple \((a_0, a_1, \ldots, a_k, x)\), where \((a_0, a_1, \ldots, a_k) = h(x_{k,j})\). We call \( a_m \) \((0 \leq m \leq k)\) the \( m \)-th generation ancestor of \( x \). These are well-defined because \( \{T_{k,j} : 1 \leq j \leq J(k)\} \) is a partition of \( S_{k+1} \).

There is a slight abuse of notation: when \( x_{k,j} \) and \( x_{\ell,i} \) are the same point in \( X \) while considered as branch points in two different generations, \( h(x_{k,j}) \) and \( h(x_{\ell,i}) \) have different numbers of components.

For \( \ell \geq k + 1 \), let

\[
T_{k,j}^\ell = \{ x \in S_\ell : \text{the } \ell \text{th generation ancestor of } x \text{ is } x_{k,j} \},
\]

and call elements of \( T_{k,j}^\ell \) the \( \ell \)-th generation branch points of \( x_{k,j} \). Note that \( T_{k,j}^{k+1} = T_{k,j} \),

\[
(2.1) \quad T_{k,j}^\ell \subseteq T_{k,j}^{\ell+1},
\]

and \( \{T_{k,j}^\ell : 1 \leq j \leq J(k)\} \) is a partition of \( S_\ell \). Denote by

\[
T_{k,j}^\infty = \bigcup \limits_{\ell \geq k+1} T_{k,j}^\ell
\]

all branch points of \( x_{k,j} \), and note that

\[
T_{k,j}^\infty \cap T_{m,i}^\infty = \emptyset
\]

if neither \( x_{k,j} \) nor \( x_{m,i} \) is an ancestor of the other.

We claim that for \( \ell \geq k + 1 \),

\[
(2.2) \quad S_\ell \cap B(x_{k,j}, 10^{-k}/3) \subseteq T_{k,j}^\ell \subseteq S_\ell \cap B(x_{k,j}, 10^{-k+1}/9);
\]

thus

\[
\bigcup \limits_{k+1}^\infty S_\ell \cap B(x_{k,j}, 10^{-k}/3) \subseteq T_{k,j}^\infty \subseteq B(x_{k,j}, 10^{-k+1}/9).
\]

Therefore, any point in \( \bigcup \limits_{k+1}^\infty S_\ell \) which is sufficiently close to \( x_{k,j} \) is a branch point of \( x_{k,j} \), and all branch points of \( x_{k,j} \) are not far from \( x_{k,j} \). To prove (2.2) let \( x \in T_{k,j}^\ell \) and follow along its ancestors since \( x_{k,j} \); we have \( \rho(x_{k,j}, x) < 10^{-k} + 10^{-k-1} + \cdots + 10^{-\ell+1} < 10^{-k+1}/9 \); this proves the second inclusion in (2.2). If \( x_{\ell,i} \in S_\ell \cap B(x_{k,j}, 10^{-k}/3) \), then either \( x_{\ell,i} = x_{k+1,p} \) or \( x_{\ell,i} \in T_{k+1,p}^\ell \) for some \( p \). Apply the second inclusion to \( x_{k+1,p} \); we have \( \rho(x_{\ell,i}, x_{k+1,p}) < 10^{-k}/9 \), and hence \( \rho(x_{k+1,p}, x_{k,j}) < 10^{-k}/9 + 10^{-k}/3 < 10^{-k}/2 \). In view of (1.1), \( x_{k+1,p} \in T_{k,j} \) and hence \( x_{\ell,i} \in T_{k,j}^\ell \); this proves the first inclusion in (2.2).
The convergence of \( \{\mu_k\} \) is now clear.

We note from (1.3), (1.4), (1.5) and (2.1) that for \( \ell \geq k + 1 \),

\[
\mu_\ell(T_{k,j}^\ell) = \mu_k(\{x_{k,j}\}),
\]

and

\[
\mu_\ell(\{x_{\ell,i}\}) = \left( \prod_{k=1}^{\ell} w_m \right) \mu_k(\{x_{k,j}\}),
\]

provided that \( x_{\ell,i} \in T_{k,j}^\ell \), and \( x_{\ell,i} \) and all ancestors since the \((k+1)\)st generation are new branch points.

The main idea of the proof is contained in the following lemma.

**Lemma 1.** If \( k \geq 1 \) and \( \rho(x_{k,i}, x_{k,j}) < \frac{2}{9} 10^{-k+3} \), then

\[
\mu_k(\{x_{k,i}\})/\mu_k(\{x_{k,j}\}) \leq M^3.
\]

**Proof.** For \( k = 1 \), the estimate follows from (1.2) and (1.5). Assume \( k \geq 2 \) and let

\[
h(x_{k,i}) = (a_0, a_1, \ldots, a_{k-1}, x_{k,i}), h(x_{k,j}) = (b_0, b_1, \ldots, b_{k-1}, x_{k,j}).
\]

Denote by \( k_0 \) the largest index for which \( a_{k_0} = b_{k_0} \).

If \( k_0 < k-3 \), we claim that \( a_m \) and \( b_m \) are new branch points in \( S_m \) for each \( m \in [k_0 + 2, k-2] \). Otherwise, assume that \( a_m \) is an old branch point in \( S_m \); thus \( a_m \) and \( a_{m-1} \) are the same point in \( X \). Because \( a_m \) is an ancestor of \( x_{k,i} \), it follows from (2.2) that \( \rho(x_{k,i}, a_m) < 10^{-m+1}/9 \). Because \( a_{m-1} \neq b_{m-1} \), \( a_{m-1} \) is not an ancestor of \( x_{k,j} \); from (2.2) again, we have \( \rho(x_{k,j}, a_{m-1}) > 10^{-m+1}/3 \). Thus \( \rho(x_{k,i}, x_{k,j}) > 10^{-m+1}/3 - 10^{-m+1}/9 > \frac{2}{9} 10^{-k+3} \), which is a contradiction. Therefore \( a_m \), and similarly \( b_m \), is a new branch point. In view of (2.4),

\[
\mu_{k-2}(\{a_{k-2}\}) = \left( \prod_{k_0+2}^{k-2} w_\ell \right) \mu_{k_0+1}(\{a_{k_0+1}\})
\]

and

\[
\mu_{k-2}(\{b_{k-2}\}) = \left( \prod_{k_0+2}^{k-2} w_\ell \right) \mu_{k_0+1}(\{b_{k_0+1}\}).
\]

As \( a_{k_0+1} \) and \( b_{k_0+1} \) are branch points of \( a_{k_0} = b_{k_0} \), \( \mu_{k_0+1}(\{a_{k_0+1}\})/\mu_{k_0+1}(\{b_{k_0+1}\}) \leq M \) by (1.2) and (1.5); similarly \( M^{-2} \leq \mu_k(\{x_{i,j}\})/\mu_k(\{x_{k-2}\}) \leq 1 \) and \( M^{-2} \leq \mu_k(\{x_{k,j}\})/\mu_{k-2}(\{b_{k-2}\}) \leq 1 \). From these, (2.5) follows.

If \( k_0 \geq k-3 \), (2.5) holds because of (1.2) and (1.5). \( \square \)

Given \( x \in X \) and \( r > 0 \), we shall prove (1.6). Assume that \( 10^{-k} \leq r \leq 10^{-k+1} \) for some \( k \geq 1 \). Because \( S_{k+1} \) is a maximal net, \( \rho(x, x_{k+1,p}) \leq 10^{-k-1} \) for some \( p \) and \( T_{k+1,p} \subseteq B(x_{k+1,p}, 10^{-k-1}/9) \subseteq B(x, r/4) \). Therefore, by (2.3),

\[
\mu(B(x, r/2)) \geq \mu\left( T_{k+1,p}^\infty \right) \geq \mu_{k+1}(\{x_{k+1,p}\}).
\]

Let \( \mathcal{J} \) be the set of \( j \)'s so that \( x_{k+1,j} \in B(x, 2r) \); then \( \mathcal{J} \) contains at most \( N^8 \) elements. We claim that

\[
S_\ell \cap B(x, 3r/2) \subseteq \bigcup_{\mathcal{J}} T_{k+1,j}^\ell \quad \text{for each} \quad \ell \geq k + 2.
\]
In fact, given \( x_{t,i} \in B(x, 3r/2) \), \( x_{t,i} \) is contained in \( T^k_{k+1,q} \) for some \( q \). Since \( T^k_{k+1,q} \subseteq B(x_{k+1,q}, 10^{-k}/9) \), we have \( \rho(x_{k+1,q}, x) \leq \rho(x_{k+1,q}, x_{t,i}) + \rho(x_{t,i}, x) < 10^{-k}/9 + 3r/2 < 2r \). Thus \( q \in J \). This proves (2.7). Therefore
\[
\mu_\ell(B(x, 3r/2)) \leq \sum_{j} \mu_\ell(T^k_{k+1,j}) = \sum_{j} \mu_{k+1}(\{x_{k+1,j}\})
\]
for each \( \ell \geq k + 2 \). Since \( \rho(x_{k+1,p}, x_{k+1,j}) \leq \rho(x_{k+1,p}, x) + \rho(x, x_{k+1,j}) < 10^{-k-1} + 2r < \frac{2}{3}10^{-k+1} \), we deduce from (2.5) and (2.6) that
\[
\mu_\ell(B(x, 3r/2)) \leq M^{k}N^8\mu(B(x, r/2)).
\]
From this, (1.6) follows. And this proves Theorem 1. \( \square \)

3. Proof of Theorem 2

For \( x \in S_k \), recall that \( h(x) \) has the form \((x_{0,1}, a_1, a_2, \ldots, a_{k-1}, a_k)\) and that the first element \( x_{0,1} \) is not a branch point. For \( k \geq 1 \) and \( 0 \leq p \leq k \), denote by
\[
S_k(p) = \{ x \in S_k : h(x) \text{ contains exactly } p \text{ old branch points} \}.
\]
There are exactly \( \binom{k}{p} \) different ways to position \( p \) old branch points in \( h(x) \); afterwards there are at most \((N-1)^{k-p}\) different ways to place new branch points in the remaining slots. Therefore \( S_k(p) \) has at most \( \binom{k}{p}(N-1)^{k-p} \) elements. Thus the set
\[
\sigma_k(p) = \{ x \in S_k : h(x) \text{ contains at least } p \text{ old branch points} \}
\]
has at most \( \sum_{m=p}^{k} \binom{k}{m} (N-1)^{k-m} \) elements.

Denoting \( \frac{N-1}{M^k} \) by \( \gamma \), we prove the following.

**Lemma 2.** If \( k \geq 1 \), then
\[
\mu_k(\sigma_k(p)) \geq \sum_{m=p}^{k} \binom{k}{m} (1-\gamma)^m \gamma^{k-m} \quad \text{for} \quad 0 \leq p \leq k.
\]

**Proof.** If \( k = 1 \) and \( p = 0 \), then \( \sigma_1(0) = S_1 \) and \( \mu_1(\sigma_1(0)) = 1 \). If \( k = 1 \) and \( p = 1 \), then \( \sigma_1(1) = \{ \text{the old branch point in } S_1 \} \) and \( \mu_1(\sigma_1(1)) \geq 1 - \gamma \). Hence (3.1) holds for \( k = 1 \).

Assume that (3.1) is true for some \( k \geq 1 \). We shall prove the inequality for \( k+1 \) and all \( p \) in \([0, k+1]\). If \( p = 0 \), then \( \mu_{k+1}(\sigma_{k+1}(0)) = 1 \). If \( p = k+1 \), then \( \mu_{k+1}(\sigma_{k+1}(k+1)) \geq (1-\gamma)^{k+1} \).

Let \( 1 \leq p \leq k \). For \( x \in \sigma_{k+1}(p) \), denote by \( a_1(x) \) the first generation ancestor of \( x \). Then either \( a_1(x) \) is an old branch point and there are at least \( p-1 \) old branch points in the remaining \( k \) slots in \( h(x) \), or \( a_1(x) \) is a new branch point and there are at least \( p \) old branch points in the remaining \( k \) slots. From the induction
hypothesis, it follows that

\[
\mu_{k+1}(\sigma_{k+1}(p)) = \mu_1(\sigma_1(1)) \sum_{m=p-1}^{k} \binom{k}{m} (1-\gamma)^m \gamma^{k-m} + (1-\mu_1(\sigma_1(1))) \sum_{m=p}^{k} \binom{k}{m} (1-\gamma)^m \gamma^{k-m} \\
\geq (1-\gamma) \sum_{m=p-1}^{k} \binom{k}{m} (1-\gamma)^m \gamma^{k-m} + \gamma \sum_{m=p}^{k} \binom{k}{m} (1-\gamma)^m \gamma^{k-m} \\
= \sum_{n=p}^{k+1} \binom{k+1}{n} (1-\gamma)^n \gamma^{k+1-n}.
\]

The inequality follows from the fact that \(\lambda A + (1-\lambda)a \geq (1-\gamma)A + \gamma a\) provided that \(\lambda \geq 1-\gamma\) and \(A \geq a > 0\). Therefore (3.1) holds for \(k+1\). The lemma is proved.

Assume that \(M\) is large enough so that \(\gamma = \frac{N-1}{M} < \frac{1}{5}\) and

\[
(1-2\gamma)^{-1-2\gamma} (2\gamma)^{-2\gamma} (2N)^{2\gamma} 10^{-\alpha} < 2^{-\alpha}.
\]

Choose \(p\) to be \([1-2\gamma]k\) in the remaining part of the proof, and let

\[
\tau_k = \bigcup \{T_{k,j}^\infty : x_{k,j} \in \sigma_k(p)\}.
\]

Then for large \(k\),

\[
H_\alpha(\tau_k) \leq \sum_{m=p}^{k} \binom{k}{m} (N-1)^{k-m} (10^{-k+1})^\alpha \\
\leq 10k \binom{k}{p} N^{k-p} 10^{-ka} \\
\leq (1-2\gamma)^{-1-2\gamma} (1-2\gamma)^{-2\gamma} (2\gamma)^{-2\gamma} (2N)^{2\gamma} 10^{-\alpha k} \\
< 2^{-\alpha k}.
\]

The third inequality follows from Stirling’s formula \((k! \approx k^{k+1/2}e^{-k}\sqrt{2\pi})\). Note from (3.1) that, for large \(k\),

\[
\mu(\tau_k) \geq \mu_k(\sigma_k(p)) \\
= \sum_{m=p}^{k} \binom{k}{m} (1-\gamma)^m \gamma^{k-m} \\
= 1 - \sum_{m=0}^{p-1} \binom{k}{m} (1-\gamma)^m \gamma^{k-m} \\
> 1 - p \binom{k}{p} (1-\gamma)^p \gamma^{k-p} \\
> 1 - 10 \left(\frac{e}{4}\right) \gamma^k.
\]

Here Stirling’s formula is again used in the last estimate.
Let
\[ \tau = \bigcap_{K \geq 5} \bigcup_{k \geq K} \tau_k. \]

It follows from (3.2) and (3.3) that
\[ H_\alpha(\tau) = 0 \quad \text{and} \quad \mu(\tau) = 1. \]
This proves Theorem 2. \hfill \Box

REFERENCES


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