

A COUNTEREXAMPLE TO A QUESTION
OF R. HAYDON, E. ODELL AND H. ROSENTHAL

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ABSTRACT. We give an example of a compact metric space K , an open dense subset U of K , and a sequence (f_n) in $C(K)$ which is pointwise convergent to a non-continuous function on K , such that for every $u \in U$ there exists $n \in \mathbf{N}$ with $f_m(u) = f_n(u)$ for all $m \geq n$, yet (f_n) is equivalent to the unit vector basis of the James quasi-reflexive space of order 1. Thus c_0 does not embed isomorphically in the closed linear span $[f_n]$ of (f_n) . This answers in the negative a question asked by H. Haydon, E. Odell and H. Rosenthal.

1. INTRODUCTION

A result of J. Elton [E], which was also proved later by R. Haydon, E. Odell and H. Rosenthal [HOR], states that if K is a compact metric space, and (f_n) is a uniformly bounded sequence in $C(K)$ such that

$$\sum_{n=1}^{\infty} |f_{n+1}(k) - f_n(k)| < \infty, \quad \forall k \in K,$$

and the pointwise limit of (f_n) on K is a non-continuous function, then c_0 embeds isomorphically in the closed linear span $[f_n]$ of (f_n) . Thus the following question was naturally raised by R. Haydon, E. Odell and H. Rosenthal:

Question 4.7 in [HOR]. Let K be a compact metric space, R be a residual subset of K (i.e. $K \setminus R$ is a first category set), and (f_n) be a sequence in $C(K)$ which converges pointwise on K to a non-continuous function, and

$$\sum_{n=1}^{\infty} |f_{n+1}(r) - f_n(r)| < \infty, \quad \text{for all } r \in R.$$

Does c_0 embed in the closed linear span $[f_n]$ of (f_n) ?

We will construct a compact metric space K , an open dense subset U of K and a sequence $(g_n) \subset C(K)$ such that

- (a) $(\sum_{i=1}^n g_i)_n$ is a uniformly bounded and pointwise convergent sequence on K to a non-continuous function;
- (b) For every $u \in U$ there exists $n \in \mathbf{N}$ such that $g_m(u) = 0$ for every $m \geq n$;

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(c) $[g_n]$ is isomorphic to the James quasi-reflexive of order 1 space J . Since, of course, c_0 does not embed isomorphically in J , this answers in the negative Question 4.7 of [HOR]. Our construction is very elementary and explicit, even though a shorter proof of the existence of a counterexample to Question 4.7 of [HOR] can be given along similar lines using more advanced machinery.

2. THE CONSTRUCTION

We recall the definition of the James space J and some simple facts. Let c_{00} denote the finitely supported sequences of real numbers. For $(x_n) \in c_{00}$ we define

$$\|(x_n)\|_J = \sup\{[x_{p_1}^2 + (x_{p_2} - x_{p_1})^2 + \dots + (x_{p_k} - x_{p_{k-1}})^2]^{1/2} : k \in \mathbf{N}, 1 \leq p_1 < p_2 < \dots < p_{k-1} < p_k\}.$$

Then the James space J is the completion of $(c_{00}, \|\cdot\|_J)$. If (e_n) is the unit vector basis of c_{00} , then (e_n) becomes the unit vector basis of J , which is monotone and shrinking. Also, $(\sum_{i=1}^n e_n)_n$ is a weak-Cauchy sequence which is not weakly convergent in J . If $(a_n) \in c_0$ is a monotone sequence of real numbers (i.e. non-increasing, or non-decreasing) then $\|(a_n)\|_J = |a_1|$ (this is because if $a, b \in \mathbf{R}$ with $ab \geq 0$, then $a^2 + b^2 \leq (a + b)^2$).

Notation. For $(a_n), (b_n) \in c_{00}$, we define $(a_n) \cdot (b_n) \in c_{00}$ by

$$(a_n) \cdot (b_n) = (a_n b_n).$$

Lemma 2.1. *For $(a_n), (b_n) \in c_{00}$ we have*

$$\|(a_n) \cdot (b_n)\|_J \leq \|(a_n)\|_J \|(b_n)\|_\infty + \|(a_n)\|_\infty \|(b_n)\|_J.$$

Proof. For some $k \in \mathbf{N}$ and some finite sequence of positive integers $1 \leq p_1 < p_2 < \dots < p_k$ we have

$$\begin{aligned} \|(a_n) \cdot (b_n)\|_J &= [(a_{p_1} b_{p_1})^2 + (a_{p_2} b_{p_2} - a_{p_1} b_{p_1})^2 + \dots + (a_{p_k} b_{p_k} - a_{p_{k-1}} b_{p_{k-1}})^2]^{1/2} \\ &= [(a_{p_1} b_{p_1})^2 + (a_{p_2} (b_{p_2} - b_{p_1}) + (a_{p_2} - a_{p_1}) b_{p_1})^2 \\ &\quad + \dots + (a_{p_k} (b_{p_k} - b_{p_{k-1}}) + (a_{p_k} - a_{p_{k-1}}) b_{p_{k-1}})^2]^{1/2}. \end{aligned}$$

Therefore by the triangle inequality in ℓ_2 we have that

$$\begin{aligned} \|(a_n) \cdot (b_n)\|_J &\leq [a_{p_1}^2 b_{p_1}^2 + (a_{p_2} - a_{p_1})^2 b_{p_1}^2 + \dots + (a_{p_k} - a_{p_{k-1}})^2 b_{p_{k-1}}^2]^{1/2} \\ &\quad + [a_{p_2}^2 (b_{p_2} - b_{p_1})^2 + \dots + a_{p_k}^2 (b_{p_k} - b_{p_{k-1}})^2]^{1/2} \\ &\leq [a_{p_1}^2 + (a_{p_2} - a_{p_1})^2 + \dots + (a_{p_k} - a_{p_{k-1}})^2]^{1/2} \|(b_n)\|_\infty \\ &\quad + \|(a_n)\|_\infty [(b_{p_2} - b_{p_1})^2 + \dots + (b_{p_k} - b_{p_{k-1}})^2]^{1/2} \\ &\leq \|(a_n)\|_J \|(b_n)\|_\infty + \|(a_n)\|_\infty \|(b_n)\|_J, \end{aligned}$$

which finishes the proof of the lemma. □

Now we are ready to see the counterexample. Let $K := \{(a, b) \in \mathbf{R}^2 : \mathbf{0} \leq \mathbf{a} \leq \mathbf{1}, \mathbf{0} \leq \mathbf{b} \leq \mathbf{1}\}$. Since $C[0, 1]$ is universal for the class of separable spaces, there exist a sequence $(f_n) \subset C[0, 1]$ and $M > 0$ such that (f_n) is M -equivalent to the unit vector basis of J . For $n \in \mathbf{N}$ set $K_n := \{(a, b) \in \mathbf{R}^2 : \mathbf{0} \leq \mathbf{a} \leq \mathbf{1}, \mathbf{1}/2^n \leq \mathbf{b} \leq \mathbf{1}\}$, $L_n := \{(a, b) \in \mathbf{R}^2 : \mathbf{0} \leq \mathbf{a} \leq \mathbf{1}, \mathbf{b} = \mathbf{1}/2^n\}$, $L := \{(a, 0) : 0 \leq a \leq 1\}$ and $U = K \setminus L$. Now, for $n \in \mathbf{N}$ define $g_n : K \rightarrow \mathbf{R}$ by

- $g_n | K_n \equiv 0$,

- for every $0 \leq a \leq 1$, g_n restricted to the segment connecting the points $(a, 1/2^n)$ and $(a, 0)$, is linear,
- $g_n \upharpoonright L \equiv f_n$,
- g_n is continuous on K .

We will show that (g_n) is equivalent to the unit vector basis (e_i) of the James space. This will imply that $(\sum_{i=1}^n g_i)_n$ is a weak Cauchy sequence which is not weakly convergent, which will finish the proof. Let $n \in \mathbf{N}$ and $(\lambda_i)_{i=1}^n \subset \mathbf{R}$. We want to estimate $\|\lambda_1 g_1 + \dots + \lambda_n g_n\|_\infty$. For $(a, b), (c, d) \in K$, let $[(a, b), (c, d)]$ denote the linear segment connecting the points (a, b) and (c, d) . For every $0 \leq a \leq 1$ we have that

- $(\lambda_1 g_1 + \dots + \lambda_n g_n) \upharpoonright [(a, 1), (a, 1/2)] \equiv 0$,
- $(\lambda_1 g_1 + \dots + \lambda_n g_n) \upharpoonright [(a, 1/2^i), (a, 1/2^{i+1})]$ is linear, for every $i = 1, \dots, n-1$,
- $(\lambda_1 g_1 + \dots + \lambda_n g_n) \upharpoonright [(a, 1/2^n), (a, 0)]$ is linear,
- $\lambda_1 g_1 + \dots + \lambda_n g_n$ is continuous on K .

Therefore we obtain

$$\begin{aligned} & \|\lambda_1 g_1 + \dots + \lambda_n g_n\|_\infty \\ &= \max_{2 \leq k \leq n} \|(\lambda_1 g_1 + \dots + \lambda_n g_n) \upharpoonright L_k\|_\infty \vee \|(\lambda_1 g_1 + \dots + \lambda_n g_n) \upharpoonright L\|_\infty \\ &= \max_{2 \leq k \leq n} \|(\lambda_1 g_1 + \dots + \lambda_{k-1} g_{k-1}) \upharpoonright L_k\|_\infty \vee \|\lambda_1 f_1 + \dots + \lambda_n f_n\|_\infty. \end{aligned}$$

Therefore we immediately obtain the lower estimate

$$\begin{aligned} \|\lambda_1 g_1 + \dots + \lambda_n g_n\|_\infty &\geq \|\lambda_1 f_1 + \dots + \lambda_n f_n\|_\infty \\ &\geq \frac{1}{M} \|\lambda_1 e_1 + \dots + \lambda_n e_n\|_J. \end{aligned}$$

For the upper estimate we need to estimate $\|(\lambda_1 g_1 + \dots + \lambda_n g_n \upharpoonright L_k)\|_\infty$ for $2 \leq k \leq n$. Note that for $0 \leq a \leq 1$ and $2 \leq k \leq n$ we have

$$\begin{aligned} & (\lambda_1 g_1 + \dots + \lambda_n g_n)(a, 1/2^k) \\ &= \lambda_1 \frac{\frac{1}{2} - \frac{1}{2^k}}{\frac{1}{2}} f_1(a) + \lambda_2 \frac{\frac{1}{2^2} - \frac{1}{2^k}}{\frac{1}{2^2}} f_2(a) + \dots + \lambda_{k-1} \frac{\frac{1}{2^{k-1}} - \frac{1}{2^k}}{\frac{1}{2^{k-1}}} f_{k-1}(a) \\ &= \lambda_1 \frac{2^{k-1} - 1}{2^{k-1}} f_1(a) + \lambda_2 \frac{2^{k-2} - 1}{2^{k-2}} f_2(a) + \dots + \lambda_{k-1} \frac{2 - 1}{2} f_{k-1}(a). \end{aligned}$$

Therefore we have that

$$\begin{aligned} & \|\lambda_1 g_1 + \dots + \lambda_{k-1} g_{k-1} \upharpoonright L_k\|_\infty \\ &= \|\lambda_1 \frac{2^{k-1} - 1}{2^{k-1}} f_1 + \lambda_2 \frac{2^{k-2} - 1}{2^{k-2}} f_2 + \dots + \lambda_{k-1} \frac{2 - 1}{2} f_{k-1}\|_\infty \\ &\leq M \|\lambda_1 \frac{2^{k-1} - 1}{2^{k-1}} e_1 + \lambda_2 \frac{2^{k-2} - 1}{2^{k-2}} e_2 + \dots + \lambda_{k-1} \frac{2 - 1}{2} e_{k-1}\|_J \\ &= M \|(\lambda_1, \lambda_2, \dots, \lambda_{k-1}, 0, \dots) \\ &\quad \cdot (\frac{2^{k-1} - 1}{2^{k-1}}, \frac{2^{k-2} - 1}{2^{k-2}}, \dots, \frac{2 - 1}{2}, 0, \dots)\|_J \\ &\leq M \|\lambda_1 e_1 + \dots + \lambda_{k-1} e_{k-1}\|_J \cdot 1 \\ &\quad + M \|(\lambda_i)_{i=1}^{k-1}\|_\infty \|(\frac{2^{k-1} - 1}{2^{k-1}}, \dots, \frac{2 - 1}{2}, 0, \dots)\|_J \quad (\text{by Lemma 2.1}) \end{aligned}$$

$$\begin{aligned} &\leq M\|\lambda_1 e_1 + \cdots + \lambda_{k-1} e_{k-1}\|_J + M\|(\lambda_i)\|_\infty \frac{2^{k-1} - 1}{2^{k-1}} \quad (\text{since the} \\ &\quad \text{sequence } (\frac{2^{k-1} - 1}{2^{k-1}}, \frac{2^{k-2} - 1}{2^{k-2}}, \dots, \frac{2 - 1}{2}, 0, \dots) \text{ is decreasing}) \\ &\leq 2M\|\lambda_1 e_1 + \cdots + \lambda_{k-1} e_{k-1}\|_J \quad (\text{since } \|(\lambda_i)_{i=1}^{k-1}\|_\infty \leq \|(\lambda_i)_{i=1}^{k-1}\|_J). \end{aligned}$$

Also, since $\|\lambda_1 f_1 + \cdots + \lambda_n f_n\|_J \leq M\|\lambda_1 e_1 + \cdots + \lambda_n e_n\|_J$, we obtain

$$\|\lambda_1 g_1 + \cdots + \lambda_n g_n\|_\infty \leq 2M\|\lambda_1 e_1 + \cdots + \lambda_n e_n\|_J.$$

This finishes the proof. □

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