THE PRIMITIVE $p$-FROBENIUS GROUPS

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Abstract. Let $p$ be a fixed prime. A finite primitive permutation group $G$ with every two-point stabilizer $G_{\alpha,\beta}$ being a $p$-group is called a primitive $p$-Frobenius group. Using our earlier results on $p$-intersection subgroups, we give a complete classification of the primitive $p$-Frobenius groups.

1. Introduction

Let $\Sigma_\Omega$ denote the symmetric group on a finite set $\Omega$ and $G \leq \Sigma_\Omega$ a transitive permutation group. Suppose that every two-point stabilizer $G_{\alpha,\beta}$ is trivial; then a classical result of Frobenius states that $G$ is a semidirect product $K:G_{\alpha}$, where the normal subgroup $K$ consists precisely of the identity together with all elements $g \in G$ that do not fix any point of $\Omega$. Moreover the point stabilizer $G_{\alpha}$ acts on $K$ (via conjugation) in such a way that no nontrivial element $g \in G_{\alpha}$ has a nontrivial fixed point in $K$. A permutation group with these properties is called a Frobenius group with Frobenius kernel $K$.

As a consequence of a celebrated theorem of J. G. Thompson, the Frobenius kernel $K$ is nilpotent and $G_{\alpha}$ is a semiregular group, i.e. $G_{\alpha}$ has a faithful action on a suitable vector space $V$ such that $C_V(g) = 0$ for every $g \in G_{\alpha}\{id\}$ (notice that $V$ can be chosen as a characteristically simple subgroup of a Sylow group of $K$). If moreover $G$ is primitive, then $K \cong \mathbb{Z}_p^\ell =: V$ and $G_{\alpha}$ acts semiregularly on $V$.

In the course of his investigation of finite near fields, Zassenhaus [9] obtained a complete classification of finite semiregular groups. Putting all these results together, one has a complete picture of primitive Frobenius groups.

In this paper we finish our investigation of the following $p$-local variant of the situation above:

Definition 1.1. Let $p$ be a fixed prime. We define $\mathcal{F}(p)$ to be the set of all (faithful) finite primitive permutation groups with every 2-point stabilizer $G_{\alpha,\beta}$ being a $p$-group. An element of $\mathcal{F}(p)$ will be called a primitive $p$-Frobenius group.

Furthermore, $\mathcal{F}^a(p)$ denotes the subset of $\mathcal{F}(p)$ consisting of those groups with abelian socle, and we set $\mathcal{F}^{na}(p) := \mathcal{F}(p) \setminus \mathcal{F}^a(p)$.

In [3], we defined a proper subgroup $X < G$ to be a $p$-intersection subgroup of $G$ if and only if $X$ is not a $p$-group but for any $g \in G \setminus X$ the intersection $X \cap X^g$ is a $p$-group. The set of $p$-intersection subgroups of $G$ will be denoted by $I_p(G)$. The
The main result of [3] is a classification of the elements in $\mathcal{I}_p(G)$ for all almost simple groups. Now we will apply this result to obtain a complete classification of the primitive $p$-Frobenius groups.

Remarks. (i) It is clear that $\mathcal{F}(p)$ includes the class $\mathcal{F}^\infty$ of all finite primitive Frobenius groups, which is the intersection $\bigcap_{p \text{ prime}} \mathcal{F}(p)$.

(ii) Every finite group $G$ with $O_p(G) = 1$ admits a faithful transitive permutation representation with every $G_{\alpha, \beta}$ being a $p$-group: take the action of $G$ on $G/P$, $P \in \text{Syl}_p(G)$. Thus it is the primitivity condition that makes the class $\mathcal{F}(p)$ interesting.

(iii) It is immediate from the definition, that $G \in \mathcal{F}(p)$ implies that either $G_{\alpha}$ is a $p$-group or $G_{\alpha} \in \mathcal{I}_p(G)$.

Our notation will be as follows: $\Omega$ always denotes a finite set and $G$ a finite group with $\pi(G)$ the set of prime divisors of $|G|$. For $\alpha \in \Omega$ and $G \leq \Sigma_\Omega$ we write $G_{\alpha}$ for $\text{Stab}_G(\alpha)$. For any subgroup $S \leq G$ we set $\text{Aut}_G(S) := N_G(S)/C_G(S)$. Also, $X < Y$ means that $X$ is a maximal subgroup of a group $Y$. The socle of $G$, $\text{soc}(G)$, is the product of all minimal normal subgroups of $G$. A subgroup $G \leq \Sigma_\Omega$ is called primitive (resp. regular) if it is transitive and $G_{\alpha} < G$ (resp. $G_{\alpha} = 1$).

The symmetric, resp. alternating, group on $n$ symbols is denoted by $\Sigma_n$, resp. $\mathcal{A}_n$.

Observe that if $G \in \mathcal{F}^a(p)$ then $\text{soc}(G) = V = \mathbb{F}_p^\ell$ is elementary abelian (the prime $\ell$ may differ from $p$) and $G$ is the semi-direct product of $V$ and any point stabilizer $G_{\alpha}$, $a \in \Omega$. In this case, one can identify $\Omega$ with $V$, $a \in \Omega$ with the zero vector of $V$, then embed $G_0$ in $GL(V)$ so that the action of $G_{\alpha} = G_0$ on $V$ and the linear action of $G_0$ on $V$ are compatible.

Let us recall the following definition of [2]: A pair $(G, V)$ consisting of a finite group $G$ and a finite-dimensional $\mathbb{F}G$-module $V$ over some field $\mathbb{F}$ is called $p'$-semiregular if every nontrivial $p'$-element of $G$ acts without any fixed points on $V \setminus \{0\}$. $G$ is called $p'$-semiregular if $(G, V)$ is $p'$-semiregular for a suitable $V$.

Now the following statement is immediate:

**Proposition 1.2.** Suppose $\text{soc}(G) = V$ is elementary abelian. Then $G \in \mathcal{F}^a(p)$ if and only if the pair $(G_0, V)$ is $p'$-semiregular, and $V$ is a faithful irreducible $G_0$-module.

In [2] all $p'$-semiregular pairs $(G_0, V)$ have been determined.

The present paper provides a proof of the following complete classification of the groups in $\mathcal{F}(p)$:

**Theorem 1.3.** Let $G$ be an element of $\mathcal{F}(p)$ and put $S := \text{soc}(G)$. Then precisely one of the following three cases occurs:

(i) $G$ has a regular normal subgroup $V$. In this case $V = S$ is elementary abelian, $G \in \mathcal{F}^a(p)$ and $G = V : G_0$ with $p'$-semiregular pair $(G_0, V)$ as described in [2] (and $V$ is a faithful irreducible $G_0$-module).

(ii) $G$ has no regular normal subgroup and $G_\alpha$ is nilpotent. In this case $p = 2$, $G \in \mathcal{F}^n_a(2)$, $G_\alpha \in \text{Syl}_2(G)$ and $S = O_2^2(G) = S_1 \times \cdots \times S_k$ is the unique minimal normal subgroup of $G$; moreover, $S_1 \cong \cdots \cong S_k \cong L_2(q)$ with $q = 2^a \pm 1 > 5$ a prime or $q = 9$. Furthermore, $[G : S]$ and $k$ are powers of $2$.

(iii) $G$ has no regular normal subgroup and a point stabilizer $G_\alpha$ is an element of $\mathcal{I}_q(G)$. In this case $S$ is simple, i.e. $G$ is almost simple. Furthermore, the tuples $(S, G, p, G_\alpha)$ are as listed in Table I below.
Table I. $p$-intersection maximal subgroups in almost simple groups

<table>
<thead>
<tr>
<th>$S$</th>
<th>$G$</th>
<th>$p$</th>
<th>$G_\alpha$</th>
</tr>
</thead>
<tbody>
<tr>
<td>$A_4$</td>
<td>$G = S$</td>
<td>3</td>
<td>$\cdot$</td>
</tr>
<tr>
<td>$A_5$</td>
<td>$A_5 \leq G \leq \Sigma_5$</td>
<td>2</td>
<td>$N_G(\mathbb{Z}_3)$</td>
</tr>
<tr>
<td>$A_6$</td>
<td>$A_6 \leq G \leq \text{Aut}(A_6)$</td>
<td>2</td>
<td>$N_G(3^2)$</td>
</tr>
<tr>
<td>$A_6$</td>
<td>$\text{PGL}<em>2(9), M</em>{10}, \text{Aut}(A_6)$</td>
<td>2</td>
<td>$N_G(\mathbb{Z}_5)$</td>
</tr>
<tr>
<td>$A_{n,n}$, $n$ prime $\notin {7, 11, 17, 23}$</td>
<td>$G = S$ with $\frac{n-1}{2} = p^j$</td>
<td>$Z_p : Z_{2n-1}$</td>
<td></td>
</tr>
<tr>
<td>$A_{n,5}$, $5 \leq n \in \mathcal{F}$</td>
<td>$Z_n$</td>
<td>2</td>
<td>$Z_n : Z_{2n-1}$</td>
</tr>
<tr>
<td>$L_2(7) \cong L_3(2)$</td>
<td>$G = S$</td>
<td>2</td>
<td>$\Sigma_3^{(i)}, i = 1, 2$</td>
</tr>
<tr>
<td>$L_2(7)$</td>
<td>$G = \text{PGL}_2(7)$</td>
<td>2</td>
<td>$N(X), N(T_1)$</td>
</tr>
<tr>
<td>$L_2(11)$</td>
<td>$G = S$</td>
<td>3</td>
<td>$B$</td>
</tr>
<tr>
<td>$L_2(11)$</td>
<td>$G = S$</td>
<td>2</td>
<td>$N(T_{35})$</td>
</tr>
<tr>
<td>$L_2(11)$</td>
<td>$G = S$</td>
<td>2</td>
<td>$N(T_1) \cong D_{20}$</td>
</tr>
<tr>
<td>$L_2(11)$</td>
<td>$G = S$</td>
<td>2</td>
<td>$N(T_{35}) \cong D_{24}$</td>
</tr>
<tr>
<td>$L_2(2^a), 2^a &gt; 4$</td>
<td>$G = S$</td>
<td>5</td>
<td>$B$</td>
</tr>
<tr>
<td>$L_2(2^a)$, $q \notin \mathcal{F} \cup {4, 7, 9, 11}$</td>
<td>$G = S$</td>
<td>$p = 2^a - 1 \in \mathcal{M}$</td>
<td></td>
</tr>
<tr>
<td>$L_2(2^a)$, $q \notin \mathcal{M} \cup {4, 7, 9, 11}$</td>
<td>$G = S$</td>
<td>$p = 2^a - 1 \in \mathcal{M}$</td>
<td></td>
</tr>
<tr>
<td>$L_2(2^a)$</td>
<td>$G = S$</td>
<td>2</td>
<td>$B$</td>
</tr>
<tr>
<td>$L_2(3^m), {3^m = 2p^a + 1 m \text{ an odd prime}}$</td>
<td>$G = S$</td>
<td>$p &gt; 2$</td>
<td></td>
</tr>
<tr>
<td>$L_2(3^m)$</td>
<td>$G = S$</td>
<td>2</td>
<td>$B$</td>
</tr>
<tr>
<td>$P S U_3(q), q \notin {3, 5, 9}$</td>
<td>$G \leq S \leq \Sigma_3$</td>
<td>2</td>
<td>$B, N_G(B)$</td>
</tr>
<tr>
<td>$^3G_2(3^n) \cong L_2(3^n)$</td>
<td>$G \leq S \leq \Sigma_3$</td>
<td>2</td>
<td>$B, N_G(B)$</td>
</tr>
<tr>
<td>$^3B_2(q), q = 2^l, l = 2a + 1 &gt; 1$</td>
<td>$G = S$</td>
<td>$p = 2^a - 1 \in \mathcal{M}$</td>
<td></td>
</tr>
<tr>
<td>$^3B_2(q)$</td>
<td>$G = S$</td>
<td>2</td>
<td>$B$</td>
</tr>
<tr>
<td>$^3B_2(q)$</td>
<td>$G = S$</td>
<td>2</td>
<td>$B$</td>
</tr>
<tr>
<td>$^3D_4(q)$</td>
<td>$\pi(G/S) \subseteq {2}$</td>
<td>2</td>
<td>$N(T_5)$</td>
</tr>
<tr>
<td>$^3D_4(q)$</td>
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<td>2</td>
<td>$N(T_5)$</td>
</tr>
<tr>
<td>$^3D_4(q)$</td>
<td>$\pi(G/S) \subseteq {p}$</td>
<td>2</td>
<td>$N_{T_{35}}(q)$</td>
</tr>
<tr>
<td>$M_{11}$</td>
<td>$G = S$</td>
<td>2</td>
<td>$3^2 : Q_8 \cdot 2$</td>
</tr>
<tr>
<td>$J_1$</td>
<td>$G = S$</td>
<td>2</td>
<td>$\Sigma_3 \times D_{10}$</td>
</tr>
<tr>
<td>$M_{23}$</td>
<td>$G = S$</td>
<td>11</td>
<td>$23 : 11$</td>
</tr>
<tr>
<td>$BM$</td>
<td>$G = S$</td>
<td>23</td>
<td>$47 : 23$</td>
</tr>
<tr>
<td>$M$ (?)</td>
<td>$G = S$</td>
<td>29</td>
<td>$59 : 29$</td>
</tr>
</tbody>
</table>

Conversely, each of the groups mentioned in (i), (ii) and (iii) is a member of $\mathcal{F}(p)$, where in case (ii) we have to assume in addition that $A_5$ if $q = 7$ and $A_6$ if $q = 9$. In Table I $B$ denotes a Borel subgroup: $T_1$, $T_{35}$ denote split- and Coxeter tori, respectively, $L_2(q) : = \text{PSL}_2(q)$, $N(H) : = N_G(H)$; if $G > S$, $N(X) : = \{N_G(X) \mid X \in \mathcal{I}_p(S), \text{maximal in } S\}$. Here $??$ means either $59 : 29 < L_2(59) < \cdot M$.
or $59 : 29 < \cdot M$. The existence of $L_2(59)$ in $M$ is not settled yet. $\mathcal{F}$, resp. $\mathcal{M}$, denotes the set of Fermat, resp. Mersenne, primes.

Due to the isomorphisms $A_5 \cong L_2(4) \cong L_2(5)$ and $A_6 \cong L_2(9)$ these groups and their automorphic decorations are listed only in the alternating groups’ section. For $L_3(4)$, $F$ (resp. $\Delta \cong \mathbb{Z}_2$) is generated by field (resp. graph) automorphisms.

Note that for any $X \in \mathcal{I}_p(G)$ there is a subgroup $H \leq G$ such that $X \in \mathcal{I}_p(H)$ and $X$ is maximal in $H$. Hence the results above give a complete classification of $p$-intersection subgroups occurring in arbitrary finite groups. In particular we obtain the following result which can be viewed as the $p$-local version of Zassenhaus’ classification of Frobenius complements:

\begin{theorem}
Let $G$ be a finite group, $p$ a prime and $X < G$ a $p$-intersection subgroup. Then either $X$ is $p'$-semiregular or $X$ is a solvable group (occurring as $G_{\alpha}$ in Table I up to a suitable normal $p$-subgroup of $X$).
\end{theorem}

2. Prerequisites

The reader is referred to [2] for a complete list of $p'$-semiregular groups; here we restrict ourselves to listing the perfect ones. $\mathcal{R}$ is the set of all primes $r$ such that $r = 2^a \cdot 3^b + 1$ for $a \geq 2$, $b \geq 0$, and $(r+1)/2$ is a prime.

R. Guralnick and R. Wiegand [5] also classified $p'$-semiregular groups $G$ such that the underlying field of the corresponding $G$-modules has characteristic $p$, and pointed out a very interesting connection of these groups with multiplicative structures of Galois field extensions. The authors’ proof in [2] is independent of that in [5] and in addition also describes the relevant $G$-modules.

\begin{theorem}
Let $G$ be a perfect finite group and $(G, V)$ a $p'$-semiregular pair for a faithful irreducible $\mathbb{F} G$-module $V$. Then one of the following holds:
\begin{enumerate}[(i)]
    \item $G \cong SL_2(p^a)$ for some $a \geq 1$ with $p^a > 3$.
    \item $G \cong 2B_2(2^{2a+1})$ for some $a \geq 1$ with $p = 2$.
    \item $G \cong 2B_2(2^{2a+1}) \times SL_2(2^{2b+1})$ with $a, b \geq 1$, $\gcd(2a + 1, 2b + 1) = 1$ and $p = 2$.
    \item $G \cong SL_2(r)$ with $r \in \mathcal{R} \cup \{7, 17\}$ and $p = 3$.
    \item $G \cong SL_2(5)$ and $p \geq 7$.
    \item $G = ES$, where $E = O_2(G) \cong 2^{1+4}$, $S \cong SL_2(5)$, $E \cap S = Z(G) \cong \mathbb{Z}_2$ and $p = 2$.
\end{enumerate}

Conversely, if $(G, p)$ satisfies any of the conditions (i) – (vi), then there exists a faithful absolutely irreducible $G$-module $V$ such that $(G, V)$ is $p'$-semiregular.
\end{theorem}

\begin{proof}
Let $(G, V)$ be $p'$-semiregular for a faithful irreducible $\mathbb{F} G$-module $V$ and $\text{char} \mathbb{F} = \ell$. First suppose that $\ell$ divides $|G|$. Then it is clear that $\ell = p$ and the irreducibility of $V$ forces $O_p(G) = 1$. By Theorem 4.1 of [2] (cf. also [5]), $(G, p)$ is as listed in (i) – (iv). Next suppose that $\ell$ does not divide $|G|$. Then Theorem 5.6 of [2] and the irreducibility of $V$ force $(G, p)$ to satisfy one of the conditions (iv) – (vi) or (i) with $p^a = 4, 5, 9$. The existence of $p'$-semiregular pairs for the groups $G$ listed has also been established in [2].
\end{proof}

\begin{example}
(i) (Zassenhaus) $\ell^2 : SL_2(5) \in \mathcal{F}^\infty$ for all primes $\ell \equiv \pm 1 (\text{mod } 10)$ and $\ell^4 : SL_2(5) \in \mathcal{F}^\infty$ for all primes $\ell \equiv \pm 3 (\text{mod } 10)$.

(ii) $p^a : SL_2(p^a) \in \mathcal{F}(p)$ for any prime $p$.

(iii) $3^a : SL_2(13) \in \mathcal{F}(3)$ (Hering’s group).

(iv) $7^a : (2^{1+4} \setminus A_5) \in \mathcal{F}(2)$ and $7^4 : SL_2(9), 5^{12} : SL_2(13) \in \mathcal{F}(3)$.
\end{example}
The following result will be used in the next section.

**Theorem 2.3.** Let $G$ be a finite group with a nilpotent maximal subgroup $H$.

(i) (Thompson; see [4], Thm. 10.3.2) If $H$ has odd order then $G$ is solvable.

(ii) (Baumann; see [1]) If $G$ is non-solvable then $O^2(G/F(G))$ is a direct product of simple groups isomorphic to $L_2(q)$ with primes $q$ of the form $2^n \pm 1$ or $q = 9$.

We will also need the following result, which is an easy consequence of the classification of finite simple groups:

**Lemma 2.4.** Let $E$ be a nonabelian finite simple group and let $\alpha \in \text{Aut}(E)$ be an element whose order is coprime to $|E|$. Then $C_E(\alpha)$ is not nilpotent.

**Proof.** See [3].

3. **Reduction to the simple socle case**

**Proposition 3.1.** Let $G \in \mathcal{F}^{na}(p)$. Then $G$ does not contain any regular normal subgroup. In particular, $S := \text{syl}(G)$ is the unique minimal normal subgroup of $G$, $C_G(S) = 1$, and $G/S \cong G_a/S_a$.

Moreover, for any point stabilizer $G_\alpha$ one of the following is true:

(i) $p = 2$ and $G_\alpha \in \text{Syl}2(G)$;

(ii) $G_\alpha \in \mathcal{I}_p(G)$.

**Proof.**

1) Suppose first that $G_\alpha$ is a $p$-group. As $G_\alpha$ is maximal and nilpotent, 2.3 implies $p = 2$ and so $G_\alpha \in \text{Syl}2(G)$.

2) Suppose next that $1 \neq R < G$ with $R \cap G_\alpha = 1$. We can assume $R \leq \text{syl}(G)$ and $R$ is not solvable (since $G \in \mathcal{F}^{na}(p)$). Suppose in addition that $G_\alpha$ is not a $p$-group. Then we can find $x \in G_\alpha$ of prime order $q \neq p$. Now for any $1 \neq y \in R$, $G_\alpha \cap G_y(\alpha)$ is a $p$-group. In particular, $x^y \notin G_\alpha$, so $y^x \neq y$ and $x$ acts fixed-point-freely on $R \setminus \{1\}$. By a well-known theorem of Thompson (cf. [4], Thm. 10.2.1) $R$ must be nilpotent, a contradiction. Thus $G_\alpha$ must be a $p$-group. By 1), $G_\alpha \in \text{Syl}2(G)$, and $|R| = [G : G_\alpha]$ is odd. So $R$ (and $G = R \cdot G_\alpha$) is solvable, again a contradiction.

3) The claims concerning $S$ now follow immediately. Furthermore, if $G_\alpha$ is a $p$-group, then (i) is fulfilled due to 1); otherwise one arrives at (ii).

**Corollary 3.2.** Let $G \in \mathcal{F}(p)$. Then $G \in \mathcal{F}_a(p)$ if and only if $G$ contains a regular normal subgroup.

**Proposition 3.3.** Suppose that $G \in \mathcal{F}^{na}(p)$ and $\text{syl}(G)$ is not simple. Then conclusion (ii) of Theorem 1.3 holds.

**Proof.** A basic tool for studying finite permutation groups is the reduction theorem first stated by O’Nan and Scott (see [8]). Here we are using an expanded version of this theorem given in [7]. Because of 3.2, the primitive permutation group $G$ under question has no regular normal subgroups (and $\text{syl}(G)$ is not simple). In this case, the O’Nan-Scott theorem says that $G$ is either a simple diagonal action or a product action group; cf. [7]. We shall use the notation given there. In particular, $B = \text{syl}(G) = S_1 \times \ldots \times S_k$ with $S_1 \cong \ldots \cong S_k \cong T$ for a non-abelian finite simple group $T$.

1) Suppose $G$ is a simple diagonal action group, that is, case III(a) of [7] occurs. Then for some $\alpha \in \Omega$ one has $B_\alpha = \{(a, a, \ldots, a) \mid a \in T\}$. In particular, taking $g := (a, a, \ldots, a)$, $h := (a, 1, \ldots, 1)$ for a non-identity $p'$-element $a \in T$, one sees
that \( g = g^h \in B_\alpha \cap B_{h(\alpha)} \). Meanwhile \( h \notin B_\alpha \), contrary to the assumption that \( G \in \mathcal{F}(p) \).

We have shown that \( G \) is a product action group, i.e., case III(b) of [7] occurs. In this case, \( \Omega \) can be identified with \( \Gamma^i \) for some finite set \( \Gamma \) and some \( i \) dividing \( k \). Furthermore, if \( H \) denotes \( Aut_G(S_1 \times S_2 \times \ldots \times S_{k/\ell}) \) (after suitably reindexing the \( S_i \)'s if necessary), then \( H \) acts primitively on \( \Gamma \), with socle \( K \) isomorphic to \( T^{k/\ell} \). Moreover, \( H \) is of type II or III(a) (in the notation of [7]). Finally, \( G \) can be embedded in \( W = H \wr \Sigma_\ell \), and the action of \( G \) on \( \Omega \) is induced by the natural product action of \( W \) on \( \Omega \) (cf. [7]).

2) Next we consider the situation when \( H \) is of type III(a). Then \( \ell < k \). Arguing as in 1), one sees that \( K_\gamma \cap K_{\gamma'} \) is not a \( p \)-group for some \( \gamma, \gamma' \in \Gamma \), \( \gamma \neq \gamma' \). For the distinct points \( \alpha := (\gamma, \ldots, \gamma), \alpha' := (\gamma', \ldots, \gamma') \) in \( \Omega \), one has \( B_\alpha = (K_\gamma)^\ell \), \( B_{\alpha'} = (K_\gamma')^\ell \). In particular, \( B_\alpha \cap B_{\alpha'} \) is not a \( p \)-group, a contradiction.

Thus \( H \) must be of type II, i.e., \( k = \ell \).

3) At this point we show that \( H_\gamma \) is a \( p \)-group.

First observe that \( B_\alpha \) is a \( p \)-group, with \( \alpha = (\gamma, \ldots, \gamma) \). Indeed, suppose \( s = (s_1, \ldots, s_k) \in B_\alpha \) is not a \( p \)-element. Without loss we may suppose \( s_1 \in T \) is not a \( p \)-element. Then \( s = s^t \in B_\alpha \cap B_{t(\alpha)} \) with \( t := (s_1, 1, \ldots, 1) \). This implies that \( t \in B_\alpha \). Now taking \( u := (1, g_2, \ldots, g_k) \), one has \( t = t^u \in B_\alpha \cap B_{u(\alpha)} \) for any \( g_i \in T \). We conclude that all \((1, g_2, \ldots, g_k)\) are contained in \( B_\alpha \), contradicting the equality \( B_\alpha = (K_\gamma)^k \).

Suppose there is an element \( x \in H_\gamma \), whose order is a prime \( r \) different from \( p \). Every element in \( W \) can be canonically written in the form \( (g_1, \ldots, g_k) \pi \) for \( g_i \in \Sigma_\Gamma \) and \( \pi \in \Sigma_k \). Then \( G_\alpha \) contains an element \( g = (x, g_2, \ldots, g_k) \pi \) with \( \pi(1) = 1 \). For any \( y \in C_K(x) \) and \( h := (y, 1, \ldots, 1) \), we have \( g = g^h \in G_\alpha \cap G_{h(\alpha)} \), yielding \( h \in B_\alpha \). Due to the above observation \( C_K(x) \) is a \( p \)-group. Choose \( Q \in Syl_r(H) \) with \( x \in Q \). Then \( Q \cap K \lhd Q \) and \( Z(Q) \cap (Q \cap K) \leq C_K(x) \). So \( 1 = Z(Q) \cap (Q \cap K) \), which implies that \( 1 = Q \cap K \). But \( Q \cap K \in Syl_r(K) \), hence \( (r, |K|) = 1 \). Now 2.4 applied to \( K \) and \( x \) provides a contradiction.

4) We have proved that \( H_\gamma \) is a \( p \)-group. If \( p \) is odd, then the maximality of \( H_\gamma \) in \( H \) together with 2.3 implies that \( H \) is solvable, a contradiction. So we conclude that \( p = 2 \). We claim that \( G_\alpha \) is a 2-group. For, suppose that \( g = (g_1, \ldots, g_k) \pi \in G_\alpha \) has order \( r \), an odd prime, and \( \alpha = (\gamma, \ldots, \gamma) \). Observe that \( G_\alpha \leq W_\alpha = H_\gamma \cap \Sigma_k \). Since \( g_i \in H_\gamma \) has order a power of \( 2 \), we conclude that \( \pi \) has order \( r \). In particular, we may suppose that \( \pi \) permutes the groups \( S_1, \ldots, S_r \) cyclically. Choose \( c \in K \setminus K_\gamma \). Then for \( y := (c, 1, \ldots, 1) \in B \) we have \( [y^g, y^\pi] = 1 \) for all \( i, j = 1, 2, \ldots, r \). Hence \( \tilde{y} := yy^g y^{\pi^2} \cdots y^{\pi^{r-1}} \in C_B(g) \). From this it follows that \( g = g^\tilde{y} \in G_\alpha \cap G_{\tilde{y}(\alpha)} \), yielding \( \tilde{y} \in B_\alpha \). But in this case \( c \) belongs to \( K_\gamma \), contrary to the choice of \( c \). Consequently, \( G_\alpha \) is a 2-group, and so \( G_\alpha \in Syl_2(G) \).

Applying 2.3 to the maximal subgroup \( G_\alpha \) of \( G \), we come to conclusion (ii) of Theorem 1.3. \( \diamond \)

The following is one of the main results in [3]; here it classifies the elements in \( \mathcal{F}^{wa}(p) \) with simple socle:

**Theorem 3.4.** Let \( S \leq G \leq Aut(S) \) with \( S \) a finite nonabelian simple group, and suppose that \( X \in \mathcal{T}_p(G) \) is maximal in \( G \). Then \( (S, G, p, X = G_\alpha) \) is one of the tuples listed in Table I.

**Proof.** See [3]. \( \diamond \)
4. Proof of Theorem 1.3

By 3.2 and the remarks in the introduction we can assume that $G \in \mathcal{F}^{na}(p)$.

First we suppose that $G_{\alpha}$ is nilpotent. Applying 2.3, we see that $F(G) = 1$ (as $G \in \mathcal{F}^{na}(p)$) and $S := soc(G) = O^2(G)$ is the direct product of $k$ isomorphic (since $S$ is characteristically simple) simple groups $S_1 \cong \cdots \cong S_k \cong L_2(q)$ with $q = 2^n \pm 1$ a prime or $q = 9$. If $k = 1$, then a direct inspection of maximal subgroups of $G$ with $L_2(q) \leq G \leq Aut(L_2(q))$ shows that $p = 2$ and $G_{\alpha} \in Syl_2(G)$, that is, conclusion (ii) of 1.3 holds. If $k > 1$, then, applying 3.3, one again obtains that $p = 2$ and $G_{\alpha} \in Syl_2(G)$. Moreover, due to 3.1, $G/S \cong G_{\alpha}/S_{\alpha}$ and hence $[G : S]$ is a 2-power. But $G/S$ acts transitively on the set $\{S_1, \ldots, S_k\}$; therefore $k$ is a 2-power. Thus (ii) is fulfilled.

Now suppose that (ii) does not hold. Then $G_{\alpha} \in I_p(G)$. In this case 3.3 shows that $soc(G)$ is simple. Applying 3.4, we arrive at (iii).  

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