A CHARACTERIZATION OF GORENSTEIN RINGS
IN CHARACTERISTIC $p > 0$

S. P. DUTTA

(Communicated by Wolmer V. Vasconcelos)

Abstract. A new characterization of Gorenstein rings in characteristic $p > 0$ is proved. It involves asymptotic behaviour of lengths of modules under the Frobenius map.

In this note we are going to study how the Gorenstein property for a local ring $(R, m, k = R/m)$ of characteristic $p > 0$, dimension $d$, is connected with the behaviour of $\lim_{n \to \infty} \ell(F^n(M))/p^{nd}$ and $\lim_{n \to \infty} \ell(F^n(M^v))/p^{nd}$, when $M$ is a module of finite length and finite projective dimension over $R$, $M^v = \text{Hom}_R(M, E)$, $E$ is the injective hull of $k$ over $R$ and $F^n(M)$ is obtained from $M$ by applying the Frobenius map $n$ times ($(M \otimes_R k^n)$; Notation). It is well known that for a module $M$ of finite length over any local ring $R$ of dim $d$ and ch. $p > 0$ $\lim_{n \to \infty} \ell(F^n(M))/p^{nd}$ is positive, and if $R$ is Cohen-Macaulay and $M = R/(x_1, \ldots, x_d)$, where $x_1, \ldots, x_d$ is a system of parameters in $R$, the above limit is $\ell(R/(x_1, \ldots, x_d))$. Recall that when $R$ is Gorenstein $M^v \simeq \text{Ext}_R^d(M, R)$ for any module $M$ of finite length. Moreover when $pd_RM < \infty$, we have $(F^n(M))^v \simeq \text{Ext}_R^d(F^n(M), R) \simeq F^n(\text{Ext}_R^d(M, R)) \simeq F^n(M^v)$. Thus $\ell(F^n(M)) = \ell(F^n(M^v))$, and obviously the above limits are same. This leads us to raise the same question for Cohen-Macaulay rings which are not Gorenstein. Our main theorem shows that the answer, in this case, is in the negative even for cyclic modules of the form $R/(x_1, \ldots, x_d)$, where $x_1, \ldots, x_d$ is a system of parameters of $R$. In fact our main theorem is the following:

Theorem. A Cohen-Macaulay local ring $R$ of dimension $d$ and ch. $p > 0$ is Gorenstein if and only if there exists a system of parameters $x_1, \ldots, x_d$ such that

$$\lim_{n \to \infty} \ell(F^n((R/\mathfrak{z})^v))/p^{nd} = \ell(R/\mathfrak{z}).$$

Motivation for the study of the above result came from the study of the Strong Intersection Conjecture due to Peskine and Szpiro [P-S]. In this conjecture they assert that, given a pair of finitely generated modules $M$ and $N$ over a local ring $R$ such that $\ell(M \otimes_R N) < \infty$ and $pd_RM < \infty$, dim$N \leq$ grade $M$. While studying the conjecture over a Cohen-Macaulay ring of dimension $d$ and positive characteristic, with a canonical module $\Omega \neq R$, this author was led to inquire whether $\lim_{n \to \infty} \ell(H^0_m(F^n(\Omega/(x_2, \ldots, x_d)\Omega)))/p^{nd}$ is $0$. Recall that, though $\Omega/(x_2, \ldots, x_d)\Omega$ is a Cohen-Macaulay module of dimension $1$, $F^n(\Omega/(x_2, \ldots, x_d)\Omega)$

Received by the editors March 15, 1996 and, in revised form, November 26, 1996.
1991 Mathematics Subject Classification. Primary 13D02, 13H10.
This research was partially supported by an NSF grant and an NSA grant.

©1998 American Mathematical Society

1637
is not necessarily Cohen-Macaulay. In the course of our proof of the main theorem, we show that the above limit is never zero. If the limit were zero, one could venture for the construction of a counterexample to the above conjecture.

I am thankful to the referee for comments and suggestions, some of which have been incorporated into the body of this paper.

**Notations.** Throughout this work \((R, m, k)\) denotes a local ring of characteristic \(p \ (> 0)\), dimension \(d\), \(m\) its maximal ideal; \(k = R/m\) and \(p\) is perfect. The Frobenius map \(f : R \to R\), given by \(f(x) = x^p\) for all \(x \in R\), is a ring homomorphism. We denote by \(f^n_R\) the bi-algebra \(R\), having the structure of an \(R\)-algebra from the left by \(f^n\) and from the right by the identity map, i.e. if \(\alpha \in R\), \(x \in f^n_R\), \(\alpha \cdot x = \alpha x^p x\) and \(x \cdot \alpha = x\alpha\). For any module \(M\), we write \(F^n_R(M)\) for \(M \otimes_R f^n_R\) (we usually drop \(R\) from the notation when there is no chance of confusion) and when \(\ell(M) = \) length of \(M < \infty\), \(M^v\) for \(\text{Hom}_R(M, E)\), where \(E\) is the injective hull of \(k\) over \(R\).

If \(pd_RM < \infty\), \(\chi_i(M, N)\) will denote \(\sum_{j=0}^{pdM} (-1)^j\ell(\text{Tor}^R_{i+j}(M, N))\). For any finitely generated module \(M\), \(\mu(M)\) denotes the minimal number of generators of \(M\) over \(R\). We write \(\text{lim} n \to -\infty\) and \(\Omega\) to denote a canonical module of \(R\).

For any system of parameters \(x_1, \ldots, x_d\) of \(R\) and for any module \(M\), \(x_iM\) denotes the submodule \((x_1, \ldots, x_d)M\) and \(\mathbb{M}\) denotes \(M/x_1M\). If \(\theta = (\theta_{ij})\) denotes an \(s \times r\) matrix, then \(\theta(x^n)\) stands for the \(s \times r\) matrix \((\theta_{ij}x^n)\). Also for any module \(M\), and for any element \(x \in R\), \((0 : x)M = \{\alpha \in M \mid x\alpha = 0\}\). If \(I\) denotes the ideal \((y_1, \ldots, y_t)\) in \(R\), then \(I^n\) denotes the ideal \((y_1^n, \ldots, y_t^n)\) in \(R\).

Now we state our main theorem.

**Theorem.** Let \(R\) be a Cohen-Macaulay local ring of dimension \(d\) and characteristic \(p > 0\). Then \(R\) is Gorenstein if and only if there exists a system of parameters \(x_1, \ldots, x_d\) such that

\[
\lim_{n \to -\infty} \ell(F^n((R/\mathfrak{m})^v))/p^{nd} = \ell(R/\mathfrak{m}).
\]

**Proof.** We note that without any loss of generality one can assume \(R\) is complete. Recall that for any system of parameters \(x_1, \ldots, x_d\) of \(R\)

\[
\lim \ell(F^n(R/\mathfrak{m}))/p^{nd} = \ell(R/\mathfrak{m}).
\]

Now let \(R\) be Gorenstein. Then we have

\[
(F^n(R/\mathfrak{m}^v))^v \simeq \text{Ext}_R^d(F^n(R/\mathfrak{m}), R) \simeq F^n(\text{Ext}_R^d(R/\mathfrak{m}, R)) \simeq F^n((R/\mathfrak{m})^v).
\]

Hence

\[
\lim \ell(F^n((R/\mathfrak{m})^v))/p^{nd} = \lim \ell((F^n(R/\mathfrak{m}^v))^v)/p^{nd} = \lim \ell(F^n(R/\mathfrak{m}))/p^{nd} = \ell(R/\mathfrak{m}).
\]

Next, let us assume that \(R\) is not Gorenstein. Our proof is completed in two parts.

In Part 1, we will show that it is possible to construct a system of parameters \(x_1, \ldots, x_d\) of \(R\) such that

\[
\lim \ell(F^n((R/\mathfrak{m})^v))/p^{nd} \neq \ell(R/\mathfrak{m});
\]

and in Part 2 we will prove the above result for any system of parameters.

**Part 1.** We first note a rather trivial fact in dimension 0, which will be used in our proof:
Fact. Let $R$ be a local ring of dimension zero. Let $M$ be a module of finite length. Let $\mu(M)$ denote the number of minimal generators of $M$. Then $\ell(F^n(M)) = \mu(M)\ell(R)$ for $n >> 0$.

Proof of the Fact. Consider a minimal representation of $M$:
\[ R^r \xrightarrow{\theta} R^s \rightarrow M \rightarrow 0 \]
where the entries of $\theta$ are in $m$. Applying $\otimes f_R^p$, we get an exact sequence
\[ R^r \xrightarrow{\phi[p^n]} R^s \rightarrow F^n(M) \rightarrow 0. \]
Since $\dim R = 0$, for $n >> 0$, entries of $\phi[p^n]$ are all 0. Hence the result.

We note that $(R/\mathfrak{a})^\ell \simeq \text{Ext}^d(R/\mathfrak{a}, \Omega) = \mathfrak{a}\Omega$. So we need to establish that for a certain system of parameters $x_1, \ldots, x_d$
\[ \ell(R/\mathfrak{a}) = \lim_{n \to \infty} \ell(F^n(R/\mathfrak{a}))/p^n. \]
(2)

Thus the above limits are equal if and only if $\mu(\Omega) = 1$ i.e., $R$ is Gorenstein.

Now suppose $x_1, \ldots, x_d (d > 0)$ is a system of parameters of $R$ such that
\[ (1) \quad \lim_{n \to \infty} \ell(F^n(R/\mathfrak{a}))/p^n = \lim_{n \to \infty} \ell(F^n(\Omega/\mathfrak{a}\Omega))/p^n. \]
(1) holds if and only if $\Omega \subset R$ and $\chi_1(R/\mathfrak{a}^n, F^n(\Omega))/p^n \to 0$ as $n \to \infty$.

Step 1. (1) holds if and only if $\Omega \subset R$ and $\chi_1(R/\mathfrak{a}^n, F^n(\Omega))/p^n \to 0$ as $n \to \infty$.

Since support $\Omega = \text{support } R$ and $F^n(\Omega)_p = F^n_R(\Omega_p)$ [P-S], we have
\[ \chi(R/\mathfrak{a}^n, F^n(\Omega)) = p^n\chi(R/\mathfrak{a}, F^n(\Omega)). \]
We have
\[ \chi(R/\mathfrak{a}^n, F^n(\Omega)) = p^n \sum_{P \in \text{Ass}(R)} \mu_R(P)\ell(R_P)\chi(R/\mathfrak{a}, R/P). \]
for $n >> 0$ (Fact).

Recall ([L]) that for any finitely generated module $T$ such that $\ell(T/\mathfrak{a}T) < \infty$, $\chi(R/\mathfrak{a}, T) \geq 0$ and equality holds if and only if $\dim T < d$. Moreover $\chi_1(R/\mathfrak{a}, T) \geq 0$. Thus, if $\mu_R(P) > 1$ for any prime $P \in \text{Ass}(R)$, it follows from the above that
\[ \lim_{n \to \infty} \chi(R/\mathfrak{a}^n, F^n(\Omega))/p^n > \sum_{P \in \text{Ass}(R)} \ell(R_P)\chi(R/\mathfrak{a}, R/P). \]
But the right-hand side of the above inequality is nothing but $\chi_1(R/\mathfrak{a}, R) = \ell(R/\mathfrak{a})$.

Since $\chi_1(R/\mathfrak{a}^n, F^n(\Omega)) \geq 0$, when $\ell(F^n(\Omega/\mathfrak{a}\Omega))/p^n = \ell(R/\mathfrak{a})$, the above inequality forces us to have
\[ (2) \quad \mu_R(P) = 1, \forall P \in \text{Ass}(R) \quad \text{and} \quad \lim_{n \to \infty} \chi_1(R/\mathfrak{a}^n, F^n(\Omega))/p^n = 0. \]
This can easily be checked to be both necessary and sufficient in order that (1) may hold. Note that \( \Omega_p \) is a canonical module of \( R_p \), and thus (2) implies that \( R \) is generically Gorenstein. So, we can assume that \( \Omega \) is an ideal of height 1 in \( R \). Recall that in such a case \( R/\Omega \) is Gorenstein.

Now we need to study what it means to have \( \lim \chi_1(R/x^n, F^n(\Omega))/p^{nd} = 0 \).

**Step 2.** We can choose \( x_1 \) so that \( \lim \chi_1(R/x^n, F^n(\Omega))/p^{nd} = 0 \) if and only if

\[
\lim \ell(H^0_m(F^n(\Omega/\mathcal{L}_2\Omega)))/p^{nd} = 0.
\]

From Step 1, \( \Omega \subset R \) is a height one ideal of \( R \). Take \( x_1 \in \Omega \).

Consider the short exact sequence

\[
0 \to \Omega \to R \to R/\Omega \to 0.
\]

Applying \( \otimes f^n_R \), we obtain the following short exact sequences:

\[
0 \to \text{Tor}_1(R/\Omega, f^n_R) \to F^n(\Omega) \to \Omega[p^n] \to 0, \quad 0 \to \Omega[p^n] \to R \to R/\Omega[p^n] \to 0.
\]

Note that \( \ell(R/\Omega + \mathcal{L}_2) \) \( \langle \infty \rangle \) and hence \( \ell(\text{Tor}_i(R/\mathcal{L}_2, F^n(\Omega)) \langle \infty \rangle \). Now consider the exact sequence

\[
0 \to R/x^n \xrightarrow{x_1} R/x^n \to R/x^n \to 0.
\]

Applying \( \otimes F^n(\Omega) \) to the above sequence, from the long exact sequence of Tor’s we obtain

\[
\chi_1(F^n(\Omega), R/x^n) = \ell((0 : x_1^n)F^n(\Omega/\mathcal{L}_2\Omega)).
\]

Again from the exact sequence

\[
0 \to \Omega/\mathcal{L}_2\Omega \to R/\mathcal{L}_2 \to R/(\Omega + \mathcal{L}_2) \to 0,
\]

applying \( \otimes f^n_R \), we get the following exact sequence:

\[
0 \to \text{Tor}_1(R/(\Omega + \mathcal{L}_2), f^n_R) \to F^n(\Omega/\mathcal{L}_2\Omega) \to R/\mathcal{L}_2^n \to R/(\Omega + \mathcal{L}_2)[p^n] \to 0.
\]

Since \( x_1 \in \Omega \), it follows from the above sequence that

\[
(0 : x_1^n)F^n(\Omega/\mathcal{L}_2\Omega) = \text{Tor}_1(R/(\Omega + \mathcal{L}_2), f^n_R) = H^0_m(F^n(\Omega/\mathcal{L}_2\Omega)).
\]

Thus \( \lim \chi_1(R/x^n, F^n(\Omega))/p^{nd} = 0 \) if and only if

\[
(3) \quad \lim \ell(H^0_m(F^n(\Omega/\mathcal{L}_2\Omega)))/p^{nd} = 0.
\]

We have now two cases to consider:

(a) \( \ell(H^0_m(F^n(\Omega/\mathcal{L}_2\Omega))) \leq Kp^{n(d-2)} \), where \( K \) is a constant.

(b) \( \ell(H^0_m(F^n(\Omega/\mathcal{L}_2\Omega))) = O(p^{n(d-1)}) \).

**Step 3.** Case (a). We consider the case when \( \ell(H^0_m(F^n(\Omega/\mathcal{L}_2\Omega))) \leq Kp^{n(d-2)} \). We use induction on \( d \) \( \geq 0 \). For \( d = 0 \), see **Step 0**. Consider the exact sequence

\[
0 \to H^0_m(F^n(\Omega/\mathcal{L}_2\Omega)) \to F^n(\Omega/\mathcal{L}_2\Omega) \to N_n \to 0.
\]

Because of (1), it follows that \( \lim \ell(N_n/x_1^nN_n)/p^{nd} = \ell(R/x_1) \). But \( x_1 \) is not a zero-divisor on \( N_n \), so we have \( \lim \ell(N_n/x_1N_n)/p^{n(d-1)} = \ell(R/x_1) \). Tensoring (4) by \( R/x_1R \), we get the following exact sequence:

\[
0 \to H^0_m(F^n(\Omega/\mathcal{L}_2\Omega)) \to F^n_R(\Omega/\mathcal{L}_2\Omega) \to N_n \to 0.
\]
(For any module $M$, we write $\overline{M}$ to denote $M/x_1M$.) Since $\ell(H^0_m(F^n(\Omega/x_2\Omega))) \leq Kp^{n(d-2)}$, \( \ell(H^0_m(F^n(\Omega/x_2\Omega)))/p^n = 0 \). This shows that
\[
\lim \ell(F^n(\overline{\Omega}/x_2\overline{\Omega}))/p^{n(d-1)} = \lim \ell(\overline{\Omega}/p^{n(d-1)}) = \ell(R/p) = \ell(\overline{R}/x_2\overline{R}).
\]
Since the dimension of $\overline{R}$ is $d - 1$, we are done by induction on $d = \dim R$ ($\overline{\Omega} = \Omega/x_1\Omega$ is a canonical module for $\overline{R} = R/x_1R$).

**Step 4. Case (b).** Now we consider the case when $\ell(H^0_m(F^n(\Omega/x_2\Omega)))$ is strictly $O(p^{n(d-1)})$. If $\ell(H^0_m(F^n(\Omega/x_2\Omega))) = O(p^{n(d-2)})$, we are done by Step 3. So assume that $\ell(H^0_m(F^n(\Omega/x_2\Omega))) = O(p^{n(d-1)})$. Recall from Step 2, that $H^0_m(F^n(\Omega/x_2\Omega)) = \text{Tor}_1(R/(\Omega + x_2), f^m_{\overline{R}})$. We have
\[
\ell(H^0_m(F^n(\Omega/x_2\Omega))) \leq c p^{n(d-1)},
\]
where $c$ is a constant. Suppose that
\[
\ell(\overline{H}^0_m(F^n(\Omega/x_2\Omega))) = c_1 p^{n(d-1)} + O(p^{n(d-2)})
\]
(the existence of $c_1$ follows from (5)) and
\[
\ell(H^0_m(F^n(\Omega/x_2\Omega))) \otimes R/x_1^2R = c_1 p^{n(d-1)} + O(p^{n(d-2)}).
\]
Let $t$ be an integer such that $t > [c/c_1]$. Applying $\otimes R/x_1^2R$ to (4), we obtain the following exact sequence:
\[
0 \rightarrow H^0_m(F^n(\Omega/x_2\Omega)) \otimes R/x_1^2R \rightarrow F^n(\Omega/x_2\Omega) \otimes R/x_1^2R \rightarrow R/x_1^2R \rightarrow 0.
\]
Since $x_1$ is a non-zero divisor on $N_n$, $\ell(N_n/x_1^2N_n) = i\ell(\overline{N}_n)$. Again, from the exact sequence
\[
0 \rightarrow R/x_1R \rightarrow R/x_1^2R \rightarrow R/x_1^2R \rightarrow 0
\]
we get
\[
\ell(F^n(\Omega/x_1) \otimes R/x_1^2R) = 2\ell(F^n(\Omega/x_2\Omega) \otimes R/x_1R) + O(p^{n(d-2)})
\]
(by applying Proposition 1, [S]). Hence for any fixed $i$,
\[
\ell(F^n(\Omega/x_1) \otimes R/x_1R) = i\ell(F^n(\overline{\Omega}/x_2\overline{\Omega})) + O(p^{n(d-2)}).
\]
Now choose an $i > t$. It then follows from (8) and the above equality that, for $n >> 0$,
\[
\ell(H^0_m(F^n(\Omega/x_2\Omega)) \otimes R/x_1^2R) = i\ell(F^n(\overline{\Omega}/x_2\overline{\Omega})) - i\ell(\overline{\Omega}/p^{n(d-2)}) = i\ell(H^0_m(F^n(\Omega/x_2\Omega)))/p^{n(d-2)} + O(p^{n(d-2)})
\]
from (5)
\[
\ell(H^0_m(F^n(\Omega/x_2\Omega)))/p^{n(d-2)} + O(p^{n(d-2)}).
\]
Thus, (6), (7), (8) and the above inequality imply that
\[
\ell(H^0_m(F^n(\Omega/x_2\Omega)) \otimes R/x_1^2R) > \ell(H^0_m(F^n(\Omega/x_2\Omega)))/p^{n(d-2)} + O(p^{n(d-2)}).
\]
for $n >> 0$, which is impossible. (The above arguments also show that $c$ cannot be equal to $c_1$.) Hence, by Step 3 and Step 4, $\lim \ell(F^n(H^0_m(\Omega/x_2\Omega)))/p^{n(d-2)}$ cannot be 0. Thus for any system of parameters $x_1, \ldots, x_d$ of $R$, with $x_1 \in \Omega$,
\[
\ell(R/p) = \lim \ell(F^n(R/p))/p^{n(d-2)} \neq \lim \ell(F^n((R/p)^n))/p^{n(d-2)}.
\]
Part 2. If possible, let there be a system of parameters $x_1, \ldots, x_d$ such that
\[
\ell(R/x) = \lim \ell(F^n(R/x))/p^{nd} = \lim \ell(F^n((R/x)^v))/p^{nd}.
\]
Then as in Part 1, we conclude that $Ω$ is a height 1 ideal of $R$ and
\[
\lim \chi_1(R/x^{pn}, F^n(Ω))/p^{nd} = 0.
\]
If any $x_i \in Ω$, we arrive at a contradiction by Theorem 1. So we can assume that
\[
\text{not a single } x_i \text{ is in } Ω.\text{ Now consider } x_1^t, \ldots, x_d^t, \text{ where } t_1, \ldots, t_d \text{ are positive integers. Then}
\]
\[
\chi_1(R/(x_1^{t_1}p^n, \ldots, x_d^{t_d}p^n), F^n(Ω)) \leq t_1t_2, \ldots, t_d\chi_1(R/x^{pn}, F^n(Ω)).
\]
Hence
\[
\lim \chi_1(R/(x_1^{t_1}p^n, \ldots, x_d^{t_d}p^n), F^n(Ω))/p^{nd} = 0.
\]
Thus $x_1', \ldots, x_d'$ also satisfy the above equality, i.e.
\[
\ell(R/(x_1', \ldots, x_d')) = \lim \ell(F^n(R/(x_1', \ldots, x_d')))/p^{nd} \quad \text{ (Step 1, Theorem 1).}
\]
Now, since $\ell(R/(Ω+x)) < \infty$, we can reorganize the ideal $(x_1, \ldots, x_d)$ in such a way
\[
\text{that } x_1, \ldots, x_{d-1} \text{ is a system of parameters on } R/Ω \text{ (Theorem 124, [K]). Choose}
\]
$t$ so that $x_d' \notin Ω + (x_1, \ldots, x_{d-1})$. Then, by our observation above, the system of
\[
\text{parameters } x_1, \ldots, x_{d-1}, x_d' \text{ also satisfies the following:}
\]
\[
\ell(R/(x_1, \ldots, x_{d-1}, x_d')) = \lim \ell(F^n(Ω/(x_1, \ldots, x_{d-1}, x_d')))/p^{nd}.
\]
But this contradicts the result proved in Part 1, since
\[
(x_1, \ldots, x_{d-1}, x_d') = (x_1, \ldots, x_{d-1}, x_d'),
\]
where $x_d' \notin Ω$.

Remark. On a complete Cohen-Macauay local ring $R$ of dimension $d$, if $M$ is a
module of finite length and finite projective dimension, $ℓ(M)$ may be different from
$ℓ(M)$, where $M = \text{Ext}^d(M, R)$. (See [R], for such an example.) Still, one can show,
by using Theorem 1.5 of [D2], that
\[
\lim ℓ(F^n(M))/p^{nd} = \lim ℓ(F^n(\hat{M}))/p^{nd}.
\]
Actually, a more general statement is valid: for any module $M$ of finite length
\[
\lim ℓ(\text{Ext}^d(F^n(M), R))/p^{nd} = \lim ℓ(F^n(M))/p^{nd}.
\]

References

220–226. MR 32:5688
Paris 42 (1973), 47–119. MR 51:10330


DEPARTMENT OF MATHEMATICS, UNIVERSITY OF ILLINOIS, 1409 WEST GREEN STREET, URBANA, ILLINOIS 61801
E-mail address: dutta@math.uiuc.edu