PROCEEDINGS OF THE
AMERICAN MATHEMATICAL SOCIETY
Volume 126, Number 6, June 1998, Pages 1669–1671
S 0002-9939(98)04236-1

# A VERSION OF STRASSEN'S THEOREM FOR VECTOR-VALUED MEASURES

A. HIRSHBERG AND R. M. SHORTT

(Communicated by Clifford J. Earle, Jr.)

Abstract. A formulation of Strassen's Theorem is given for measures taking values in a Banach lattice. The main result (Theorem 2) corrects earlier work of the second author.

In joint work of M. März and the second author [5, Theorem 3.7], a version of the theorem known in probability theory as "Strassen's Theorem" (see [8], [2, 11.6]) was generalized to the context of measures assuming values in a reflexive Banach lattice. In [7, Theorem 3.2], the second author announced an extension of the result to all order-complete Banach lattices. However, the proof given in [7, p.816] contains an error, leaving the validity of the result an open question.

In Theorem 2 below, a result of this general type is proved for measures taking values in Banach lattices of a certain type: the so-called KB-spaces. The KB-spaces occupy a position between the reflexive and the complete Banach lattices (reflexive ⇒ KB ⇒ complete), so that Theorem 2 is a generalization of [5, Theorem 3.7], but is not as strong as what was asserted in [7]. Our technique makes use of a Strassen-type result (given below as Theorem 1 and proved in [3]) for finitely additive measures with values in a complete Banach lattice. As a general reference for basic facts about vector measures, we recommend the text of Diestel and Uhl [1]. For Banach lattices, see [4], [6].

If $\mathcal{A}$ and $\mathcal{B}$ are fields on sets $X$ and $Y$, respectively, then $\mathcal{A} \times \mathcal{B}$ is the field on $X \times Y$ generated by all rectangles $E \times F$ for $E \in \mathcal{A}$ and $F \in \mathcal{B}$. The following result is Theorem 2.1 in [3]. It replaces the imprecisely stated and incompletely proved Theorem 2.2 of [7].

**Theorem 1.** *Let $\mathcal{A}$ and $\mathcal{B}$ be countable fields on sets $X$ and $Y$ respectively and let $\mu : \mathcal{A} \to G^+$ and $\nu : \mathcal{B} \to G^+$ be finitely additive measures taking values in the positive cone of a divisible, $\sigma$-complete, partially ordered group $G$. We assume that $\mu(X) = \nu(Y) = \alpha$ for some $\alpha \in G^+$. Let $S$ be an arbitrary subset of $X \times Y$ and let $\mathcal{C}$ be the field on $X \times Y$ generated by $S$ and the sets in $\mathcal{A} \times \mathcal{B}$. For an element $v \in G$ with $0 \leq v \leq \alpha$, we consider the following conditions:*

*i) There is a finitely additive measure $\rho : \mathcal{C} \to G^+$ such that $\rho(E \times Y) = \mu(E)$ and $\rho(X \times F) = \nu(F)$ for all $E \in \mathcal{A}$ and $F \in \mathcal{B}$ (i.e. $\rho$ has* marginals *$\mu$ and $\nu$) and such that $\rho(S) = v$.*

*ii) Whenever $E \times F \subseteq S$ for $E \in \mathcal{A}$ and $F \in \mathcal{B}$, then $\mu(E) + \nu(F) \leq \alpha + v$.*

---

Received by the editors July 24, 1996 and, in revised form, October 28, 1996.
1991 *Mathematics Subject Classification.* Primary 28B05; Secondary 30C62, 46B42.

*iii) Whenever $E \times F \subseteq S^c$ for $E \in \mathcal{A}$ and $F \in \mathcal{B}$, then $\mu(E) + \nu(F) \leq 2\alpha - v$.*
*Then i) is equivalent to the conjunction of ii) and iii).*

Let $\mathcal{A}$ be a field of subsets of a set $X$ and suppose that $\mu : \mathcal{A} \to B$ is a finitely additive measure taking values in a Banach space $B$. Then $\mu$ is called *strongly additive* if the series $\sum \mu(E_n)$ converges in $B$ whenever $(E_n)_n$ is a sequence of pairwise disjoint sets drawn from $\mathcal{A}$. For further information on strong additivity, see [1]. Every non-negative real-valued measure is strongly additive, but non-negative measures taking values in a partially ordered Banach space may fail to be strongly additive, even if they are countably additive.

In [7], a Strassen Theorem for countably additive vector measures taking values in the positive cone of a complete Banach lattice was announced [7, Theorem 3.2]. Unfortunately, there was a gap in the proof of that result, *viz.* the appeal to the extension theorem of Kluvanek [1, Theorem 2, p.27] made at the top of p. 816 is improper, there being no guarantee that the measure $\rho_0$ is strongly additive. The authors have not been able to repair this breach; neither has anyone discovered a counter-example. Thus, no answer has yet been found to the open

*Question.* Is the formulation of Strassen's Theorem given in [7, Theorem 3.2] valid?

In view of all this, we now seek further conditions on a Banach lattice $(B, \leq)$ sufficient to yield a reasonable Strassen Theorem for countably additive $B$-valued measures. Our approach here is to restrict attention to the so-called KB-spaces: a Banach lattice $(B, \leq)$ is a *KB-space* if each norm bounded increasing sequence $(x_n)_n$ in $B$ is convergent. For information on these spaces, see the books of Vulikh [9, p.188ff.], Schaefer [6, p. 92 *et seq.*] and Zaanen [10, Chapter 15]. In these sources are to be found the following results.

**Fact.** *The KB-spaces are precisely the Banach lattices that contain no subspace isomorphic (as a Banach space) to $c_0$, the space of all real sequences converging to zero.*

**Fact.** *Every KB space $(B, \leq)$ is complete. In fact, for any $A \subseteq B$ having a supremum $x$, there is a countable $A_0 \subseteq A$ such that $x = \sup(A_0)$. (The latter property is called* super Dedekind completeness.*)*

**Fact.** *Every reflexive Banach lattice is a KB-space; so too is every Banach lattice $(B, \leq)$ of type $(L)$, i.e. such that $\|x + y\| = \|x\| + \|y\|$ for all $x, y \geq 0$.*

Thus, spaces of the form $L^1(X, \mathcal{A}, m)$ (e.g. $\ell^1$) are KB-spaces, but are not reflexive unless they are finite-dimensional.

**Lemma.** *Let $\mathcal{A}$ be a field of subsets of a set $X$. Then every finitely additive measure $\mu : \mathcal{A} \to B^+$ taking values in the positive cone of a KB-space $(B, \leq)$ is strongly additive.*

*Proof.* If $(E_n)_n$ is a sequence of pairwise disjoint sets in $\mathcal{A}$, then the sequence of partial sums $s_N = \sum_{n=1}^{N} \mu(E_n)$ is increasing and is bounded above by $\mu(X)$. Thus the infinite series $\sum \mu(E_n)$ converges. □

It is this result that enables us to patch, though not fully repair, the error in [7]. If $\mathcal{A}$ and $\mathcal{B}$ are $\sigma$-fields of subsets of sets $X$ and $Y$, respectively, then $\mathcal{A} \otimes \mathcal{B}$ is the $\sigma$-field on $X \times Y$ generated by $\mathcal{A} \times B$.

**Theorem 2.** *Let $\mathcal{A}$ and $\mathcal{B}$ be $\sigma$-fields of subsets of sets $X$ and $Y$, respectively, and let $\mu : \mathcal{A} \to B^+$ and $\nu : \mathcal{B} \to B^+$ be countably additive vector measures taking values in the positive cone of a KB-space $(B, \leq)$ and such that $\mu(X) = \nu(Y) = \alpha$. Suppose that $\mu$ is a perfect measure (see* [7]*) and that $S \in \mathcal{A} \otimes \mathcal{B}$ is a countable intersection of sets in $\mathcal{A} \times B$. For any $v \in B^+$, the following are equivalent:*

*i) There is a $\sigma$-additive measure $\rho : \mathcal{A} \otimes \mathcal{B} \to B^+$ with marginals $\mu$ and $\nu$ such that $\rho(S) \geq v$.*

*ii) For all $E \in \mathcal{A}$ and $F \in \mathcal{B}$, we have $\mu(E)+\nu(F) \leq 2\alpha - v$ whenever $E \times F \subseteq S^c$.*

*Proof. i) $\Longrightarrow$ ii):* This is just as in Theorem 1.

*ii) $\Longrightarrow$ i):* Define $I = \inf\{2\alpha - \mu(E) - \nu(F) : E \times F \subset S^c\}$ and

$$\Sigma = \sup\{\mu(E) + \nu(F) - \alpha : E \times F \subset S\}.$$

We now show that $\Sigma \leq I$: suppose that $E_1 \times F_1 \subset S$ and $E_2 \times F_2 \subset S^c$. Note that either $E_1 \cap E_2 = \emptyset$ or $F_1 \cap F_2 = \emptyset$; it follows that $\mu(E_1)+\nu(F_1)+\mu(E_2)+\nu(F_2) \leq 3\alpha$, and hence that $\mu(E_1)+\nu(F_1)-\alpha \leq 2\alpha - \mu(E_2) - \nu(F_2)$, as desired. Let $v_0 = v \vee \Sigma$. It can be seen that $\Sigma \leq v_0 \leq I$, so that conditions *ii)* and *iii)* of Theorem 1 hold with $v_0$ in place of $v$. Let $\mathcal{C}$ be the field generated by $\mathcal{A} \times B$ and the set $S$. By Theorem 1, there is a finitely additive measure $\rho_0 : \mathcal{C} \to B^+$ with marginals $\mu$ and $\nu$ and such that $\rho_0(S) = v_0$. Now the restriction of $\rho_0$ to $\mathcal{A} \times B$ has countably additive marginals, one of which is perfect, so that [7, Theorem 3.1] implies that $\rho_0$ is $\sigma$-additive on $\mathcal{A} \times \mathcal{B}$. Using Lemma 3.3 and Kluvanek's Theorem [1, p. 27], we find a $\sigma$-additive measure $\rho : \mathcal{A} \otimes \mathcal{B} \to B^+$ such that $\rho = \rho_0$ on $\mathcal{A} \times B$. Choose sets $C_n \in \mathcal{A} \times B$ such that $C_n \downarrow S$ as $n \to \infty$. We reckon

$$\rho(S) = \lim \rho(C_n) = \lim \rho_0(C_n) \geq \rho_0(S) = v_0 \geq v,$$

establishing the theorem. ☐

## REFERENCES

1. J. Diestel and J. J. Uhl, Jr., *Vector Measures*, Mathematical Surveys, No. 15, Amer. Math. Soc., Providence, RI, 1977. MR **56:**12216
2. R. M. Dudley, *Real Analysis and Probability*, Wadsworth & Brooks/Cole, Pacific Grove, 1989. MR **91g:**60001
3. A. Hirshberg and R. M. Shortt, *Strassen's Theorem for group-valued charges*, pre-print.
4. J. L. Kelley, J. Namioka, et al., *Linear Topological Spaces*, Van Nostrand, Princeton, Reprinted by Springer-Verlag, New York, 1976. MR **52:**14890
5. M. März and R. M. Shortt, *Weak convergence of vector measures*, Publicationes Math. **45** (1994), 71–92. MR **96g:**28015
6. H. H. Schaefer, *Banach Lattices and Positive Operators*, Springer-Verlag, Berlin, 1974. MR **54:**11023
7. R. M. Shortt, *Strassen's Theorem for vector measures*, Proc. Amer. Math. Soc. **122** (1994), 811–820; *Correction*, Proc. Amer. Math. Soc., this number. MR **95a:**28005
8. V. Strassen, *The existence of probability measures with given marginals*, Ann. Math. Stat. **36** (1965), 423–439. MR **31:**1693
9. B. C. Vulikh, *Introduction to the Theory of Partially Ordered Spaces*, Wolters-Noordhoff, Groningen, 1967. MR **37:**121
10. A. C. Zaanen, *Riesz Spaces II*, North-Holland, Amsterdam, 1983. MR **86b:**46001

DEPARTMENT OF MATHEMATICS, WESLEYAN UNIVERSITY, MIDDLETOWN, CONNECTICUT 06459-0128