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ANALYTIC SUBGROUPS OF THE REALS

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ABSTRACT. We prove that every analytic proper subgroup of the reals can be covered by an F_σ null set. We also construct a proper Borel subgroup G of the reals that cannot be covered by countably many sets A_i such that $A_i + A_i$ is nowhere dense for every i.

Let G be an (additive) proper subgroup of the reals. It is well-known that if G is measurable then G is null, and if G has the Baire property then G is of the first category. This implies that if G is analytic, then G must be a first category null set. In the next theorem we prove a somewhat stronger statement.

Theorem 1. *Every analytic proper subgroup of the reals can be covered by an F_σ null set.*

The proof of Theorem 1 is based on two results. The first is due to Gy. Petruska [3], and states that if an analytic set $H \subset \mathbf{R}$ cannot be covered by an F_σ null set, then there is a closed set F such that each portion of F is of positive measure and H is residual (comeager) in F. (By a portion of a set $F \subset \mathbf{R}$ we mean a nonempty and relative open subset of F.) If A, B are arbitrary portions of F, then they are measurable sets of positive measure, and thus the set $A+B = \{x+y : x \in A, y \in B\}$ has nonempty interior. As we shall prove in the next lemma, this implies that whenever a set H is residual in F, then $H + H$ is residual in an interval. In our case, however, H is an analytic group, and hence $H + H = H$ is of first category. This contradiction proves the theorem.

Lemma 2. *Let F_1 and F_2 be closed subsets of $\mathbf{R}$, and suppose that whenever A_i is a portion of F_i $(i = 1, 2)$ then $\operatorname{int}(A_1 + A_2) \neq \emptyset$. If H_i is residual in F_i $(i = 1, 2)$, then $H_1 + H_2$ is residual in any interval contained in $F_1 + F_2$.*

Proof. Let I_0 be an interval contained in $F_1 + F_2$. We shall play the Banach-Mazur game in I_0 (as discribed in Chapter 6 of [2]), with the second player winning if the intersection of the intervals (moves) is a subset of $H_1 + H_2$. We give a winning strategy for the second player; by Theorem 6.1 of [2], this will prove that $H_1 + H_2$ is residual in I_0.

We may assume that H_i is a dense G_δ subset of F_i. Let $H_i = \bigcap_{n=1}^{\infty} G_i^n$ $(i = 1, 2)$, where G_i^n is open (in $\mathbf{R}$) and dense in F_i for every $i = 1, 2$ and $n = 1, 2, \dots$. Suppose

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that the first move of the first player is the interval I_1. Since $H_1 + H_2$ is dense in $F_1 + F_2$, there are points $x_1^1 \in H_1$, $x_2^1 \in H_2$ such that $x_1^1 + x_2^1 \in \operatorname{int} I_1$. Then there are open intervals J_i^1 such that $x_i^1 \in J_i^1$, $\operatorname{cl} J_i^1 \subset G_i^1$ $(i = 1, 2)$, and $J_1^1 + J_2^1 \subset I_1$. By assumption, the set

$$U_1 = (J_1^1 \cap F_1) + (J_2^1 \cap F_2)$$

has nonempty interior. Let the first move of the second player be any closed interval $I_2 \subset U_1$.

Let the second move of the first player be the interval I_3. Since $I_3 \subset I_2 \subset U_1$ and H_i is dense in F_i, there are points $x_i^2 \in H_i \cap J_i^1$ such that $x_1^2 + x_2^2 \in \operatorname{int} I_3$. Then there are open intervals J_i^2 such that $x_i^2 \in J_i^2$, $\operatorname{cl} J_i^2 \subset J_i^1 \cap G_i^2$ $(i = 1, 2)$, and $J_1^2 + J_2^2 \subset I_3$. By assumption, the set

$$U_2 = (J_1^2 \cap F_1) + (J_2^2 \cap F_2)$$

has nonempty interior. Let the second move of the second player be any closed interval $I_4 \subset U_2$, etc.

We have to show that if the second player sticks to this strategy, then $\bigcap_{n=1}^{\infty} I_n \subset H_1 + H_2$ holds. We have $I_{2n} \subset J_1^n + J_2^n$ for every n, and $\operatorname{cl} J_i^{n+1} \subset J_i^n \cap G_i^n$ for every n and $i = 1, 2$. Let $x \in \bigcap_{n=1}^{\infty} I_{2n}$, and let $x = x_1^n + x_2^n$, where $x_i^n \in J_i^n$. If (x_1, x_2) is the limit of a convergent subsequence of (x_1^n, x_2^n), then $x_i \in \bigcap_{n=1}^{\infty} \operatorname{cl} J_i^n \subset H_i$ $(i = 1, 2)$ and $x = x_1 + x_2 \in H_1 + H_2$. □

We remark that the statement of Theorem 1 is true in every locally compact and second countable group with the Haar measure. Indeed, such a group is (homeomorphic to) a Polish space. Since Petruska's theorem is valid for every Polish space (with any continuous Borel measure on it), it is easy to check that the proof of Theorem 1 works also in this more general setting.

The statement of Lemma 2 does not remain valid if we replace the condition on the sets F_1, F_2 by $\operatorname{int}(F_1 + F_2) \neq \emptyset$. Consider the following example. Let $H \subset [0, 1]$ be a set containing exactly one point of each interval contiguous to the Cantor ternary set C. Let $F = C \cup H$. Then F is closed, and $F + F = C + C = [0, 2]$. The set H is a residual (moreover, dense open) subset of F, but $H + H$, being countable, is not residual in any interval.

Petruska's theorem was generalized by S. Solecki as follows. Let $\mathcal{F}$ be an arbitrary system of closed subsets of $\mathbf{R}$, and let $\mathcal{F}_\omega$ denote the family of all sets that can be covered by countably many elements of $\mathcal{F}$. Solecki proved in [4, Theorem 1] that if H is an analytic set with $H \notin \mathcal{F}_\omega$, then H contains a G_δ set U such that $U \notin \mathcal{F}_\omega$. Let $\mathcal{F} = \{F \subset \mathbf{R} : F$ is closed, and each portion of F contains two portions, A_1 and A_2, such that $A_1 + A_2$ is nowhere dense$\}$.

If we combine Solecki's theorem with Lemma 2, then we obtain the following statement: *every proper analytic subgroup of* $\mathbf{R}$ *can be covered by countable many closed sets belonging to* $\mathcal{F}$.

This result motivates the following question: is it true that every proper analytic subgroup of $\mathbf{R}$ can be covered by countably many closed sets, $F_1, F_2, \dots$ such that $F_n + F_n$ is nowhere dense for every n? Our next aim is to show that the answer is negative, even for Borel subgroups.

Theorem 3. *There exists a proper Borel subgroup* $G \subset \mathbf{R}$ *that cannot be covered by countably many sets* A_i *such that* $A_i + A_i$ *is nowhere dense for every* i. *Consequently, if an* F_σ *set* E *covers* G *then* $\operatorname{int}(E + E) \neq \emptyset$.

We shall prove this through the following result.

Theorem 4. *There exists a G_δ set $A \subset \mathbf{R}$ such that the elements of A are linearly independent over the rational numbers, and A cannot be covered by countably many sets A_i such that $A_i + A_i$ is nowhere dense for every i.*

First we infer Theorem 3 from Theorem 4. Let A be the set given by Theorem 4, and let G be the group generated by A. It is enough to show that G is Borel and $G \neq \mathbf{R}$. Let $k \in \mathbf{N}$ be fixed, and put

$$A_k = \{(x_1, \dots, x_k) : x_1, \dots, x_k \text{ are distinct elements of } A\}.$$

It is easy to see that A_k is a G_δ subset of $\mathbf{R}^k$. Let $n_1, \dots, n_k$ be fixed nonzero integers, and let $f : A_k \to \mathbf{R}$ be defined by

$$f(x_1, \dots, x_k) = n_1 x_1 + \dots + n_k x_k \quad ((x_1, \dots, x_k) \in A_k).$$

Then f is continuous and, as the elements of A are linearly independent over $\mathbf{Q}$, $f^{-1}(y)$ is finite for every $y \in f(A_k)$. This implies that the set $G_{n_1 n_2 \dots n_k} = f(A_k)$ is Borel (see [1, Corollary 5, p. 498]). Since

$$G = \{0\} \cup \bigcup \{G_{n_1 n_2 \dots n_k} : n_1, \dots, n_k \in \mathbf{Z} \setminus \{0\}\},$$

it follows that G is Borel.

Each $G_{n_1 n_2 \dots n_k}$ is a null set, since otherwise $G_{n_1 n_2 \dots n_k} + G_{n_1 n_2 \dots n_k}$ would contain an interval, contradicting the condition that the elements of A are linearly independent over $\mathbf{Q}$. This proves that G is also null, and thus $G \neq \mathbf{R}$.

We remark that $A + A$, being a subset of G, is of first category. Thus we obtain the following corollary.

Corollary 5. *There exists a G_δ set $A \subset \mathbf{R}$ such that $A + A$ is of first category, but A cannot be covered by countably many sets A_i such that $A_i + A_i$ is nowhere dense for every i.*

We shall use the following notation. For every compact set $K \subset \mathbf{R}$ we shall denote by $\overline{K}$ the convex hull of K; that is, the interval $[\min K, \max K]$. Clearly, the components of $\overline{K} \setminus K$ are the bounded intervals contiguous to K. Let $\mathcal{K}$ be a system of nowhere dense perfect subsets of $\mathbf{R}$. We shall say that $\mathcal{K}$ is *regular*, if the elements of $\mathcal{K}$ are pairwise disjoint, and for every $K \in \mathcal{K}$ and for every component I of $\overline{K} \setminus K$ there is $L \in \mathcal{K}$ with $L \subset I$.

Lemma 6. *Let $\mathcal{K}$ be a countable and regular system of nowhere dense perfect subsets of $\mathbf{R}$, and put $A = \mathrm{cl}(\bigcup \mathcal{K}) \setminus (\bigcup \mathcal{K})$. If $A = \bigcup_{i=1}^{\infty} A_i$, then there is an index i and a set $K \in \mathcal{K}$ such that $K \subset \mathrm{cl}\, A_i$.*

Proof. Let $K_1, K_2, \dots$ be an enumeration of the elements of $\mathcal{K}$. Suppose that the statement of the lemma is false. Then $K_1 \not\subset \mathrm{cl}\, A_1$, and we may choose a point $x_1 \in K_1 \setminus \mathrm{cl}\, A_1$. Since K_1 is perfect, there is a component $I_1 = (a_1, b_1)$ of $\overline{K_1} \setminus K_1$ for which $I_1 \subset (x_1 - 1, x_1 + 1)$, and $\mathrm{cl}\, I_1 \cap A_1 = \emptyset$. By the regularity of $\mathcal{K}$ we can choose an index n_1 such that $K_{n_1} \subset I_1$. Then $K_{n_1} \not\subset \mathrm{cl}\, A_2$ (since otherwise the statement of the lemma would be true) and we can choose a point $x_2 \in K_{n_1} \setminus \mathrm{cl}\, A_2$. Since K_{n_1} is nowhere dense and perfect, and the elements of $\mathcal{K}$ are pairwise disjoint, there is a component $I_2 = (a_2, b_2)$ of $\overline{K_{n_1}} \setminus K_{n_1}$ such that $I_2 \subset (x_2 - (1/2), x_2 + (1/2))$,

$\operatorname{cl} I_2 \cap A_2 = \emptyset$, and

$$(\operatorname{cl} I_2) \cap \bigcup_{i=1}^{n_1-1} K_i = \emptyset.$$

Then we choose an element $K_{n_2} \in \mathcal{K}$ with $K_{n_2} \subset I_2$ and a component $I_3 = (a_3, b_3)$ of $\overline{K_{n_2}} \setminus K_{n_2}$ such that $|I_3| \le 2/3$ and $\operatorname{cl} I_3$ is disjoint from the sets A_3 and $\bigcup_{i=1}^{n_2-1} K_i$. Repeating this process, we can define the sets K_{n_j} and the intervals $I_j = (a_j, b_j)$ for every $j = 1, 2, \ldots$. Let $\bigcap_{j=1}^{\infty} \operatorname{cl} I_j = \{x\}$. Then $x \in \operatorname{cl}(\bigcup \mathcal{K})$, since $a_j \in K_{n_{j-1}} \subset \bigcup \mathcal{K}$ and $a_j \to x$. On the other hand, $x \notin \bigcup \mathcal{K}$, since $\operatorname{cl} I_j \cap \bigcup_{i=1}^{n_{j-1}-1} K_i = \emptyset$ for every j. Therefore $x \in \operatorname{cl}\,(\bigcup \mathcal{K}) \setminus (\bigcup \mathcal{K}) = A$, and hence $x \in A_i$ for some i. However, $\operatorname{cl} I_i \cap \operatorname{cl} A_i = \emptyset$ implies $x \notin A_i$, a contradiction. □

Lemma 7. *Let $\mathcal{K}$ be a countable and regular system of nowhere dense perfect subsets of $\mathbf{R}$, and suppose that $\operatorname{int}(K + K) \neq \emptyset$ for every $K \in \mathcal{K}$. Let*

$$A = \operatorname{cl}(\bigcup \mathcal{K}) \setminus (\bigcup \mathcal{K}). \tag{1}$$

If $A = \bigcup_{i=1}^{\infty} A_i$, then there is an index i such that $A_i + A_i$ is dense in an interval.

Proof. Since

$$\operatorname{cl}(A_i + A_i) \supset (\operatorname{cl} A_i) + (\operatorname{cl} A_i)$$

for every i, the statement is immediate from the previous lemma. □

In the sequel we shall construct a system $\mathcal{K}$ satisfying the conditions of Lemma 7 such that the elements of the set A defined by (1) are indepentent over the rationals. Since A is G_δ, this will prove Theorem 4.

We shall say that a set $H \subset \mathbf{R}$ is a *figure*, if H is the union of finitely many closed segments.

Lemma 8. *Let I and J be closed intervals, and let $F \subset \mathbf{R}$ be a nowhere dense closed set not containing the endpoints of I and J. Then there are figures $A \subset I \setminus F$ and $B \subset J \setminus F$ such that $A + B = I + J$.*

Proof. Let $I = [a, b]$ and $J = [c, d]$. First we show that for every $x \in I + J$ there are numbers $x_1 \in I \setminus F,\ x_2 \in J \setminus F$ with $x = x_1 + x_2$. This is clear if $x = a + c$ or $x = b + d$, since $a, b, c, d \notin F$. If $x \in I + J$ is not of this form, then $a + c < x < b + d$. In this case $(x - b, x - a) \cap (c, d)$ is a nonempty open interval, and thus the set

$$D = [(x - b, x - a) \cap (c, d)] \setminus [((-F) + x) \cup F]$$

is nonempty. If $x_2 \in D$ then $x_2 \in (c, d) \setminus F \subset J \setminus F$ and $x_1 \stackrel{\text{def}}{=} x - x_2 \in (a, b) \setminus F \subset I \setminus F$. Let

$$A_k = \{x \in I : \operatorname{dist}(x, F) > 1/k\}, \quad B_k = \{x \in J : \operatorname{dist}(x, F) > 1/k\}.$$

By the preceding argument we have

$$I + J = \bigcup_{k=1}^{\infty} (A_k + B_k).$$

Since the sets $A_k + B_k$ are relatively open in the compact interval $I + J$, this implies that $A_k + B_k = I + J$ for some k. Fix a k with this property, and put

$$A = \{x \in I : \operatorname{dist}(x, F) \ge 1/k\}, \quad B = \{x \in J : \operatorname{dist}(x, F) \ge 1/k\}.$$

Then $A + B = I + J$, and it is easy to see that A and B are figures. □

Lemma 9. *Suppose that $H \subset \mathbf{R}$ is a figure and $F \subset \operatorname{int} H$ is nowhere dense and closed. Then there is a figure $H' \subset H \setminus F$ such that $H' + H' = H + H$.*

Proof. Let $H = \bigcup_{i=1}^n [a_i, b_i]$. By Lemma 8, for every $1 \le i, j \le n$ there are figures $A_{i,j} \subset [a_i, b_i] \setminus F$ and $B_{i,j} \subset [a_j, b_j] \setminus F$ such that

$$A_{i,j} + B_{i,j} = [a_i, b_i] + [a_j, b_j].$$

Let

$$H' = \bigcup_{i=1}^{\infty} \bigcup_{j=1}^{\infty} (A_{i,j} \cup B_{i,j}).$$

Then H' is a figure, $H' \subset H \setminus F$, and $H' + H' = H + H$. □

Lemma 10. *If $A \subset (a, b)$ is a set of first category, then there is a nowhere dense perfect set $K \subset [a, b] \setminus A$ such that $K + K = [2a, 2b]$.*

Proof. We may assume that A is dense in (a, b), and $A = \bigcup_{n=1}^{\infty} F_n$, where $F_n \subset (a, b)$ is nowhere dense and closed for every n. We define a sequence of figures H_n as follows. Put $F_0 = \emptyset$ and $H_0 = [a, b]$. Let $n \ge 0$ and suppose that the figure H_n has been defined in such a way that the endpoints of the components of H_n do not belong to A. Applying the previous lemma, we can find a figure $B \subset H_n \setminus F_n$ such that $B + B = H_n + H_n$. By enlarging the components of B, we may assume that the endpoints of the components of B do not belong to A. Also, by adding new intervals to B if necessary, we can suppose that each component of H_n contains at least two components of B. Then we put $H_{n+1} = B$. In this way we define the figures H_n for every n so that $H_{n+1} \subset H_n \setminus F_n$ $(n = 1, 2, \dots)$, and

$$[2a, 2b] = H_0 + H_0 = H_1 + H_1 = \dots .$$

Let $K = \bigcap_{n=1}^{\infty} H_n$. It is easy to see, using the fact that each H_n is compact, that $K + K = [2a, 2b]$. Since A is dense in $[a, b]$ and $K \cap A = \emptyset$, it follows that K is nowhere dense. Finally, the condition that each component of H_n contains at least two components of H_{n+1} implies that K is perfect. □

Let $\mathcal{A}$ be a system of subsets of $\mathbf{R}$. Suppose that, whenever $A_1, \dots, A_n$ are distinct elements of $\mathcal{A}$, and $r_1, \dots, r_n$ are nonzero rational numbers, then

$$0 \notin r_1 A_1 + \dots + r_n A_n.$$

In this case we shall say that the system $\mathcal{A}$ is linearly independent over $\mathbf{Q}$.

Lemma 11. *Let $K_1, K_2, \dots$ be nonempty compact sets such that the system $\{K_1, K_2, \dots\}$ is linearly independent over $\mathbf{Q}$. Then for every sequence of intervals $[a_i, b_i]$ $(i = 1, 2, \dots)$ there are nowhere dense perfect sets $P_i \subset [a_i, b_i]$ $(i = 1, 2, \dots)$ such that $\operatorname{int}(P_i + P_i) \ne \emptyset$ for every i, and the system $\{K_1, K_2, \dots, P_1, P_2, \dots\}$ is also linearly independent over $\mathbf{Q}$.*

Proof. Clearly, $0 \notin K_i$ for every i. Also, if $i_1, \dots, i_n$ are distinct indices and $r_1, \dots, r_n \in \mathbf{Q} \setminus \{0\}$, then the set $F = r_1 K_{i_1} + \dots + r_n K_{i_n}$ is nowhere dense. Indeed, F is compact, and thus, if F is not nowhere dense, then its interior is nonempty. In this case we can choose an index i_{n+1} distinct from $i_1, \dots, i_n$, and a nonzero rational number r_{n+1} such that

$$0 \in r_1 K_{i_1} + \dots + r_n K_{i_n} + r_{n+1} K_{i_{n+1}},$$

since $K_{i_{n+1}}$ contains a nonzero element. This, however, contradicts the condition that the system $\{K_1, K_2, \dots\}$ is linearly independent over $\mathbf{Q}$, and thus F must be nowhere dense.

Let A denote the union of all sets of the form $r_1 K_{i_1} + \dots + r_n K_{i_n}$, where $r_1, \dots, r_n \in \mathbf{Q} \setminus \{0\}$ and $i_1, \dots, i_n$ are distinct indices. Then A is of the first category. Let $c, d \in [a_1, b_1] \setminus A,\ c < d$. By the previous lemma, there is a nowhere dense perfect set $P_1 \subset [c, d] \setminus A$ such that $P_1 + P_1 = [2c, 2d]$. It is easy to see, using $P_1 \cap A = \emptyset$, that the system $\{K_1, K_2, \dots, P_1\}$ is linearly independent over $\mathbf{Q}$. Repeating this argument, we find a nowhere dense perfect set $P_2 \subset [a_2, b_2]$ such that $\operatorname{int}(P_2 + P_2) \neq \emptyset$, and the system $\{K_1, K_2, \dots, P_1, P_2\}$ is linearly independent over $\mathbf{Q}$. Continuing this process, we find the sets P_i with the required properties. $\square$

Now we turn to the proof of Theorem 4. We shall construct a system $\mathcal{K}$ with the properties described in Lemma 7 such that the elements of the set A defined by (1) are linearly independent over the rational numbers. Let Σ denote the set of those finite sequences

$$(n_0, n_1, \dots, n_j, r_1, \dots, r_j)$$

in which $n_0, \dots, n_j \in \mathbf{N}$, $n_1, \dots, n_j$ are distinct, and $r_1, \dots, r_j \in \mathbf{Q} \setminus \{0\}$. Let $\sigma_1, \sigma_2, \dots$ be an enumeration of the elements of Σ. By Lemma 11, we can find nowhere dense perfect sets $K_n^1 \subset [0, 1]$ $(n = 1, 2, \dots)$ such that $\operatorname{int}(K_n^1 + K_n^1) \neq \emptyset$ for every n, the system $\{K_1^1, K_2^1, \dots\}$ is linearly independent over $\mathbf{Q}$, and the convex hulls $\overline{K_n^1}$ are pairwise disjoint. We also put $S_1 = \emptyset$.

Let $k \geq 1$, and suppose that we have defined the finite set $S_k \subset \Sigma$ and the nowhere dense perfect sets K_n^i $(i = 1, \dots, k,\ n = 1, 2, \dots)$ with the following properties:

(i) $\operatorname{int}(K_n^i + K_n^i) \neq \emptyset$ for every $i \leq k$ and $n = 1, 2, \dots$;
(ii) the system $\mathcal{K}_k = \{K_n^i : i = 1, \dots, k,\ n = 1, 2, \dots\}$ is linearly independent over $\mathbf{Q}$;
(iii) for every $i \leq k$, the intervals $\overline{K_n^i}$ $(n = 1, 2, \dots)$ are pairwise disjoint and are shorter than $1/i$;
(iv) whenever $i < k,\ n \in \mathbf{N}$ and I is a component of $\overline{K_n^i} \setminus K_n^i$, there is exactly one index m such that $K_m^{i+1} \subset I$;
(v) for every $i < k$ and $n \in \mathbf{N}$, there is $m \in \mathbf{N}$ such that K_n^{i+1} is contained in one of the components of $\overline{K_m^i} \setminus K_m^i$; finally,
(vi) if $(n_0, n_1, \dots, n_j, r_1, \dots, r_j) \in S_k$, then $n_0 < k$ and

$$0 \notin r_1 F_{n_1}^{n_0,k} + \dots + r_j F_{n_j}^{n_0,k},$$

where

$$F_m^{n_0,k} = \bigcup \{\overline{K_n^k} : \ \overline{K_n^k} \subset \overline{K_m^{n_0}}\} \quad (m \in \mathbf{N}).$$

Let $I_1, I_2, \dots$ be an enumeration of those intervals that are components of any of the sets $\overline{K_n^k} \setminus K_n^k$ $(n = 1, 2, \dots)$. By Lemma 11, there are nonempty sets $P_j \subset I_j$ such that the system $\mathcal{K}_k \cup \{P_1, P_2, \dots\}$ is linearly independent over $\mathbf{Q}$. Let $y_j \in P_j$ for every j. Let $\sigma_p = (n_0, n_1, \dots, n_j, r_1, \dots, r_j)$ be the first element of the sequence $\sigma_1, \sigma_2, \dots$ for which $n_0 \leq k$ and $\sigma_p \notin S_k$. Then we define $S_{k+1} = S_k \cup \{\sigma_p\}$. Let

$$H_m^{n_0} = \bigcup \left\{ K_n^i : n_0 \leq i \leq k,\ \overline{K_n^i} \subset \overline{K_m^{n_0}} \right\} \cup \{y_j : y_j \in \overline{K_m^{n_0}}\}.$$

It is easy to check, using (iv) and (v), that $H_m^{n_0}$ is compact for every m. This implies that the set

$$H = r_1 H_{n_1}^{n_0} + \ldots + r_j H_{n_j}^{n_0}$$

is also compact. Since $\mathcal{K}_k \cup \{P_1, P_2, \ldots\}$ is linearly independent over $\mathbf{Q}$, it follows that $0 \notin H$, and thus there is $\delta > 0$ such that $H \cap (-\delta, \delta) = \emptyset$. Let

$$\eta = \min\left(\frac{1}{k+1}, \frac{\delta}{|r_1| + \ldots + |r_j|}\right).$$

Applying Lemma 11, we obtain nowhere dense perfect sets $K_n^{k+1} \subset (y_n, y_n + \eta) \cap I_n$ such that $\operatorname{int}(K_n^{k+1} + K_n^{k+1}) \neq \emptyset$ for every n, and the system

$$\mathcal{K}_{k+1} = \{K_n^i : i = 1, \ldots, k+1,\ n = 1, 2, \ldots\}$$

is linearly independent over $\mathbf{Q}$. Then

$$0 \notin r_1 F_{n_1}^{n_0, k+1} + \ldots + r_j F_{n_j}^{n_0, k+1}$$

follows from the choice of η. Since $F_m^{i,k+1} \subset F_m^{i,k}$ for every $i < k$ and $m \in \mathbf{N}$, (vi) remains valid for $k+1$. It is clear from the construction that (i)-(v) are also satisfied by $\mathcal{K}_{k+1}$.

In this way we have defined, by induction, the sets S_k and K_n^k for every $k, n \in \mathbf{N}$. Clearly, $\mathcal{K} = \{K_n^i : i, n \in \mathbf{N}\}$ is a countable and regular system of nowhere dense perfect sets such that $\operatorname{int}(K + K) \neq \emptyset$ for every $K \in \mathcal{K}$. In order to complete the proof, we have to show that the elements of the set A defined by (1) are linearly independent over the rationals. First we remark that the set

$$M^k = \bigcup_{n=1}^{\infty} \overline{K_n^k}$$

contains A for every $k = 1, 2, \ldots$. Indeed, the set

$$L^k = M^k \cup \bigcup_{i=1}^{k-1} \bigcup_{n=1}^{\infty} K_n^i$$

is closed and contains $\bigcup \mathcal{K}$. Thus we have $\operatorname{cl}(\bigcup \mathcal{K}) \subset L^k$ and

$$A = \operatorname{cl}(\bigcup \mathcal{K}) \setminus \bigcup \mathcal{K} \subset L^k \setminus \bigcup \mathcal{K} \subset M^k.$$

Let $x_1, \ldots, x_j$ be distinct elements of A and let $r_1, \ldots, r_j \in \mathbf{Q} \setminus \{0\}$. We show that $r_1 x_1 + \ldots + r_j x_j \neq 0$. Choose a positive integer n_0 with

$$1/n_0 < \min\{|x_{i_2} - x_{i_1}| : 1 \leq i_1 < i_2 \leq j\}.$$

Since $A \subset M^{n_0}$, there are indices $n_1, \ldots, n_j$ such that $x_i \in \overline{K_{n_i}^{n_0}}$ $(i = 1, \ldots, j)$. By the choice of n_0 and by (iii), the indices $n_1, \ldots, n_j$ are distinct. Let

$$(n_0, n_1, \ldots, n_j, r_1, \ldots, r_j) = \sigma_p.$$

We put $k = n_0 + p$; then it follows from the construction that $\sigma_p \in S_k$. We have

$$x_i \in \overline{K_{n_i}^{n_0}} \cap A \subset \overline{K_{n_i}^{n_0}} \cap M^k = F_{n_i}^{n_0, k}$$

for every $i = 1, \ldots, j$. Therefore, by (vi) we obtain $r_1 x_1 + \ldots + r_j x_j \neq 0$, which completes the proof. $\square$

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