ANALYTIC SUBGROUPS OF THE REALS

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Abstract. We prove that every analytic proper subgroup of the reals can be covered by an $F\sigma$ null set. We also construct a proper Borel subgroup $G$ of the reals that cannot be covered by countably many sets $A_i$ such that $A_i + A_i$ is nowhere dense for every $i$.

Let $G$ be an (additive) proper subgroup of the reals. It is well-known that if $G$ is measurable then $G$ is null, and if $G$ has the Baire property then $G$ is of the first category. This implies that if $G$ is analytic, then $G$ must be a first category null set. In the next theorem we prove a somewhat stronger statement.

Theorem 1. Every analytic proper subgroup of the reals can be covered by an $F\sigma$ null set.

The proof of Theorem 1 is based on two results. The first is due to Gy. Petruska [3], and states that if an analytic set $H \subset \mathbb{R}$ cannot be covered by an $F\sigma$ null set, then there is a closed set $F$ such that each portion of $F$ is of positive measure and $H$ is residual (comeager) in $F$. (By a portion of a set $F \subset \mathbb{R}$ we mean a nonempty and relative open subset of $F$.) If $A, B$ are arbitrary portions of $F$, then they are measurable sets of positive measure, and thus the set $A + B = \{x + y : x \in A, y \in B\}$ has nonempty interior. As we shall prove in the next lemma, this implies that whenever a set $H$ is residual in $F$, then $H + H$ is residual in an interval. In our case, however, $H$ is an analytic group, and hence $H + H = H$ is of first category. This contradiction proves the theorem.

Lemma 2. Let $F_1$ and $F_2$ be closed subsets of $\mathbb{R}$, and suppose that whenever $A_i$ is a portion of $F_i$ ($i = 1, 2$) then $\text{int}(A_1 + A_2) \neq \emptyset$. If $H_i$ is residual in $F_i$ ($i = 1, 2$), then $H_1 + H_2$ is residual in any interval contained in $F_1 + F_2$.

Proof. Let $I_0$ be an interval contained in $F_1 + F_2$. We shall play the Banach-Mazur game in $I_0$ (as described in Chapter 6 of [2]), with the second player winning if the intersection of the intervals (moves) is a subset of $H_1 + H_2$. We give a winning strategy for the second player; by Theorem 6.1 of [2], this will prove that $H_1 + H_2$ is residual in $I_0$.

We may assume that $H_i$ is a dense $G_\delta$ subset of $F_i$. Let $H_i = \bigcap_{n=1}^{\infty} G^n_i$ ($i = 1, 2$), where $G^n_i$ is open (in $\mathbb{R}$) and dense in $F_i$ for every $i = 1, 2$ and $n = 1, 2, \ldots$. Suppose

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that the first move of the first player is the interval $I_1$. Since $H_1 + H_2$ is dense in $F_1 + F_2$, there are points $x_1^1 \in H_1$, $x_2^1 \in H_2$ such that $x_1^1 + x_2^1 \in \text{int } I_1$. Then there are open intervals $J_1^1$ such that $x_1^1 \in J_1^1$, $\text{cl } J_1^1 \subset G_1^1$ ($i = 1, 2$), and $J_1^1 + J_1^2 \subset I_1$. By assumption, the set 

$$U_1 = (J_1^1 \cap F_1) + (J_1^2 \cap F_2)$$

has nonempty interior. Let the first move of the second player be any closed interval $I_2 \subset U_1$.

Let the second move of the first player be the interval $I_3$. Since $I_3 \subset I_2 \subset U_1$ and $H_i$ is dense in $F_i$, there are points $x_1^i \in H_i \cap J_1^i$ such that $x_1^2 + x_2^2 \in \text{int } I_3$. Then there are open intervals $J_2^i$ such that $x_1^2 \in J_2^i$, $\text{cl } J_2^i \subset I_1^i \cap G_1^i$ ($i = 1, 2$), and $J_2^1 + J_2^2 \subset I_3$. By assumption, the set 

$$U_2 = (J_2^1 \cap F_1) + (J_2^2 \cap F_2)$$

has nonempty interior. Let the second move of the second player be any closed interval $I_4 \subset U_2$, etc.

We have to show that if the second player sticks to this strategy, then $\bigcap_{n=1}^{\infty} I_n \subset H_1 + H_2$ holds. We have $I_{2n} \subset J_1^n + J_2^n$ for every $n$, and $\text{cl } J_1^{n+1} \subset J_1^n \cap G_1^n$ for every $n$ and $i = 1, 2$. Let $x \in \bigcap_{n=1}^{\infty} I_{2n}$, and let $x = x_1^n + x_2^n$, where $x_1^n \in J_1^n$. If $(x_1, x_2)$ is the limit of a convergent subsequence of $(x_1^n, x_2^n)$, then $x_1 \in \bigcap_{n=1}^{\infty} \text{cl } J_1^n \subset H_i$ ($i = 1, 2$) and $x = x_1 + x_2 \in H_1 + H_2$.

We remark that the statement of Theorem 1 is true in every locally compact and second countable group with the Haar measure. Indeed, such a group is (homeomorphic to) a Polish space. Since Petruska’s theorem is valid for every Polish space (with any continuous Borel measure on it), it is easy to check that the proof of Theorem 1 works also in this more general setting.

The statement of Lemma 2 does not remain valid if we replace the condition on the sets $F_1, F_2$ by $\text{int } (F_1 + F_2) \neq \emptyset$. Consider the following example. Let $H \subset [0, 1]$ be a set containing exactly one point of each interval contiguous to the Cantor ternary set $C$. Let $F = C \cup H$. Then $F$ is closed, and $F + F = C \subset [0, 2]$. The set $H$ is a residual (moreover, dense open) subset of $F$, but $H + H$, being countable, is not residual in any interval.

Petruska’s theorem was generalized by S. Solecki as follows. Let $\mathcal{F}$ be an arbitrary system of closed subsets of $\mathbb{R}$, and let $\mathcal{F}_\omega$ denote the family of all sets that can be covered by countably many elements of $\mathcal{F}$. Solecki proved in [4, Theorem 1] that if $H$ is an analytic set with $H \notin \mathcal{F}_\omega$, then $H$ contains a $G_\delta$ set $U$ such that $U \notin \mathcal{F}_\omega$. Let $\mathcal{F}_2 = \{F \subset \mathbb{R} : F$ closed, and each portion of $F$ contains two portions, $A_1$ and $A_2$, such that $A_1 + A_2$ is nowhere dense$\}$.

If we combine Solecki’s theorem with Lemma 2, then we obtain the following statement: every proper analytic subgroup of $\mathbb{R}$ can be covered by countable many closed sets belonging to $\mathcal{F}$.

This result motivates the following question: is it true that every proper analytic subgroup of $\mathbb{R}$ can be covered by countably many closed sets, $F_1, F_2, \ldots$ such that $F_n + F_n$ is nowhere dense for every $n$? Our next aim is to show that the answer is negative, even for Borel subgroups.

**Theorem 3.** There exists a proper Borel subgroup $G \subset \mathbb{R}$ that cannot be covered by countably many sets $A_i$ such that $A_i + A_i$ is nowhere dense for every $i$. Consequently, if an $F_\sigma$ set $E$ covers $G$ then $\text{int } (E + E) \neq \emptyset$. 

We shall prove this through the following result.

**Theorem 4.** There exists a $G_δ$ set $A ⊂ \mathbb{R}$ such that the elements of $A$ are linearly independent over the rational numbers, and $A$ cannot be covered by countably many sets $A_i$, such that $A_i + A_i$ is nowhere dense for every $i$.

First we infer Theorem 3 from Theorem 4. Let $A$ be the set given by Theorem 4, and let $G$ be the group generated by $A$. It is enough to show that $G$ is Borel and $G \neq \mathbb{R}$. Let $k ∈ \mathbb{N}$ be fixed, and put

$$A_k = \{(x_1, \ldots, x_k) : x_1, \ldots, x_k \text{ are distinct elements of } A\}.$$

It is easy to see that $A_k$ is a $G_δ$ subset of $\mathbb{R}^k$. Let $n_1, \ldots, n_k$ be fixed nonzero integers, and let $f : A_k → \mathbb{R}$ be defined by

$$f(x_1, \ldots, x_k) = n_1x_1 + \ldots + n_kx_k \quad ((x_1, \ldots, x_k) ∈ A_k).$$

Then $f$ is continuous and, as the elements of $A$ are linearly independent over $\mathbb{Q}$, $f^{-1}(y)$ is finite for every $y ∈ f(A_k)$. This implies that the set $G_{n_1n_2...n_k} = f(A_k)$ is Borel (see [1, Corollary 5, p. 498]). Since

$$G = \{0\} ∪ \left(\bigcup \{G_{n_1n_2...n_k} : n_1, \ldots, n_k ∈ \mathbb{Z} \setminus \{0\}\right),$$

it follows that $G$ is Borel.

Each $G_{n_1n_2...n_k}$ is a null set, since otherwise $G_{n_1n_2...n_k} + G_{n_1n_2...n_k}$ would contain an interval, contradicting the condition that the elements of $A$ are linearly independent over $\mathbb{Q}$. This proves that $G$ is also null, and thus $G \neq \mathbb{R}$.

We remark that $A + A$, being a subset of $G$, is of first category. Thus we obtain the following corollary.

**Corollary 5.** There exists a $G_δ$ set $A ⊂ \mathbb{R}$ such that $A + A$ is of first category, but $A$ cannot be covered by countably many sets $A_i$ such that $A_i + A_i$ is nowhere dense for every $i$.

We shall use the following notation. For every compact set $K ⊂ \mathbb{R}$ we shall denote by $\overline{K}$ the convex hull of $K$: that is, the interval $[\min K, \max K]$. Clearly, the components of $\overline{K} \setminus K$ are the bounded intervals contiguous to $K$. Let $\mathcal{K}$ be a system of nowhere dense perfect subsets of $\mathbb{R}$. We shall say that $\mathcal{K}$ is regular, if the elements of $\mathcal{K}$ are pairwise disjoint, and for every $K ∈ \mathcal{K}$ and for every component $I$ of $\overline{K} \setminus K$ there is $L ∈ \mathcal{K}$ with $L ⊂ I$.

**Lemma 6.** Let $\mathcal{K}$ be a countable and regular system of nowhere dense perfect subsets of $\mathbb{R}$, and put $A = \text{cl}(\bigcup \mathcal{K}) \setminus (\bigcup K)$. If $A = \bigcup_{i=1}^{∞} A_i$, then there is an index $i$ and a set $K ∈ \mathcal{K}$ such that $K ⊂ \text{cl} A_i$.

**Proof.** Let $K_1, K_2, \ldots$ be an enumeration of the elements of $\mathcal{K}$. Suppose that the statement of the lemma is false. Then $K_1 ⫅ \text{cl} A_1$, and we may choose a point $x_1 ∈ K_1 \setminus \text{cl} A_1$. Since $K_1$ is perfect, there is a component $I_1 = (a_1, b_1)$ of $\overline{K_1} \setminus K_1$ for which $I_1 ⊂ (x_1 - 1, x_1 + 1)$ and $\text{cl} I_1 ∩ A_1 = \emptyset$. By the regularity of $\mathcal{K}$ we can choose an index $n_1$ such that $K_{n_1} ⊂ I_1$. Then $K_{n_1} ⫅ \text{cl} A_2$ (since otherwise the statement of the lemma would be true) and we can choose a point $x_2 ∈ K_{n_1} \setminus \text{cl} A_2$. Since $K_{n_1}$ is nowhere dense and perfect, and the elements of $\mathcal{K}$ are pairwise disjoint, there is a component $I_2 = (a_2, b_2)$ of $\overline{K_{n_1}} \setminus K_{n_1}$ such that $I_2 ⊂ (x_2 - (1/2), x_2 + (1/2))$, etc.
\( \text{cl} I_2 \cap A_2 = \emptyset \), and

\[
(\text{cl} I_2) \cap \bigcup_{i=1}^{n_2-1} K_i = \emptyset.
\]

Then we choose an element \( K_{n_2} \in \mathcal{K} \) with \( K_{n_2} \subset I_2 \) and a component \( I_3 = (a_3, b_3) \) of \( \overline{K_{n_2}} \setminus K_{n_2} \) such that \( |I_3| \leq 2/3 \) and \( \text{cl} I_3 \) is disjoint from the sets \( A_3 \) and \( \bigcup_{i=1}^{n_2-1} K_i \).

Repeating this process, we can define the sets \( K_n \) and the intervals \( I_j = (a_j, b_j) \) for every \( j = 1, 2, \ldots \). Let \( \bigcap_{j=1}^{\infty} \text{cl} I_j = \{x\} \). Then \( x \in \text{cl} (\bigcup \mathcal{K}) \), since \( a_j \in K_{n_{j-1}} \subset \bigcup \mathcal{K} \) and \( a_j \to x \). On the other hand, \( x \notin \bigcup \mathcal{K} \), since \( \text{cl} I_j \cap \bigcup_{i=1}^{n_j-1} K_i = \emptyset \) for every \( j \). Therefore \( x \in \text{cl} (\bigcup \mathcal{K}) \setminus (\bigcup \mathcal{K}) = A \), and hence \( x \in A_i \) for some \( i \). However, \( \text{cl} I_i \cap \text{cl} A_i = \emptyset \) implies \( x \notin A_i \), a contradiction. \( \square \)

**Lemma 7.** Let \( \mathcal{K} \) be a countable and regular system of nowhere dense perfect subsets of \( \mathbb{R} \), and suppose that \( \text{int}(K + K) \neq \emptyset \) for every \( K \in \mathcal{K} \). Let

\[
(1) \quad A = \text{cl}(\bigcup \mathcal{K}) \setminus (\bigcup \mathcal{K}).
\]

If \( A = \bigcup_{i=1}^{\infty} A_i \), then there is an index \( i \) such that \( A_i + A_i \) is dense in an interval.

**Proof.** Since

\[
\text{cl}(A_i + A_i) \supset (\text{cl} A_i) + (\text{cl} A_i)
\]

for every \( i \), the statement is immediate from the previous lemma. \( \square \)

In the sequel we shall construct a system \( \mathcal{K} \) satisfying the conditions of Lemma 7 such that the elements of the set \( A \) defined by (1) are independent over the rationals. Since \( A \) is \( G_6 \), this will prove Theorem 4.

We shall say that a set \( H \subset \mathbb{R} \) is a figure, if \( H \) is the union of finitely many closed segments.

**Lemma 8.** Let \( I \) and \( J \) be closed intervals, and let \( F \subset \mathbb{R} \) be a nowhere dense closed set not containing the endpoints of \( I \) and \( J \). Then there are figures \( A \subset I \setminus F \) and \( B \subset J \setminus F \) such that \( A + B = I + J \).

**Proof.** Let \( I = [a, b] \) and \( J = [c, d] \). First we show that for every \( x \in I + J \) there are numbers \( x_1 \in I \setminus F \), \( x_2 \in J \setminus F \) with \( x = x_1 + x_2 \). This is clear if \( x = a + c \) or \( x = b + d \), since \( a, b, c, d \notin F \). If \( x \in I + J \) is not of this form, then \( a + c < x < b + d \).

In this case \( (x - b, x - a) \cap (c, d) \) is a nonempty open interval, and thus the set

\[
D = [(x - b, x - a) \cap (c, d)] \setminus [((-F) + x) \cup F]
\]

is nonempty. If \( x_2 \in D \) then \( x_2 \in (c, d) \setminus F \subset J \setminus F \) and \( x_1 = x - x_2 \in (a, b) \setminus F \subset I \setminus F \).

Let

\[
A_k = \{x \in I : \text{dist}(x, F) > 1/k\}, \quad B_k = \{x \in J : \text{dist}(x, F) > 1/k\}.
\]

By the preceding argument we have

\[
I + J = \bigcup_{k=1}^{\infty} (A_k + B_k).
\]

Since the sets \( A_k + B_k \) are relatively open in the compact interval \( I + J \), this implies that \( A_k + B_k = I + J \) for some \( k \). Fix a \( k \) with this property, and put

\[
A = \{x \in I : \text{dist}(x, F) \geq 1/k\}, \quad B = \{x \in J : \text{dist}(x, F) \geq 1/k\}.
\]

Then \( A + B = I + J \), and it is easy to see that \( A \) and \( B \) are figures. \( \square \)
Lemma 9. Suppose that $H \subset \mathbb{R}$ is a figure and $F \subset \text{int} H$ is nowhere dense and closed. Then there is a figure $H' \subset H \setminus F$ such that $H' + H' = H + H$.

Proof. Let $H = \bigcup_{i=1}^{n} [a_i, b_i]$. By Lemma 8, for every $1 \leq i, j \leq n$ there are figures $A_{i,j} \subset [a_i, b_i] \setminus F$ and $B_{i,j} \subset [a_j, b_j] \setminus F$ such that $A_{i,j} + B_{i,j} = [a_i, b_i] + [a_j, b_j]$.

Let

$$H' = \bigcup_{i=1}^{\infty} \bigcup_{j=1}^{\infty} (A_{i,j} \cup B_{i,j}).$$

Then $H'$ is a figure, $H' \subset H \setminus F$, and $H' + H' = H + H$. \hfill \Box

Lemma 10. If $A \subset (a, b)$ is a set of first category, then there is a nowhere dense perfect set $K \subset [a, b] \setminus A$ such that $K + K = [2a, 2b]$.

Proof. We may assume that $A$ is dense in $(a, b)$, and $A = \bigcup_{n=1}^{\infty} F_n$, where $F_n \subset (a, b)$ is nowhere dense and closed for every $n$. We define a sequence of figures $H_n$ as follows. Put $F_0 = \emptyset$ and $H_0 = [a, b]$. Let $n \geq 0$ and suppose that the figure $H_n$ has been defined in such a way that the endpoints of the components of $H_n$ do not belong to $A$. Applying the previous lemma, we can find a figure $B \subset H_n \setminus F_n$ such that $B + B = H_n + H_n$. By enlarging the components of $B$, we may assume that the endpoints of the components of $B$ do not belong to $A$. Also, by adding new intervals to $B$ if necessary, we can suppose that each component of $H_n$ contains at least two components of $B$. Then we put $H_{n+1} = B$. In this way we define the figures $H_n$ for every $n$ so that $H_{n+1} \subset H_n \setminus F_n$ ($n = 1, 2, \ldots$), and

$$[2a, 2b] = H_0 + H_0 = H_1 + H_1 + \ldots$$

Let $K = \bigcap_{n=1}^{\infty} H_n$. It is easy to see, using the fact that each $H_n$ is compact, that $K + K = [2a, 2b]$. Since $A$ is dense in $(a, b)$ and $K \cap A = \emptyset$, it follows that $K$ is nowhere dense. Finally, the condition that each component of $H_n$ contains at least two components of $H_{n+1}$ implies that $K$ is perfect. \hfill \Box

Let $\mathcal{A}$ be a system of subsets of $\mathbb{R}$. Suppose that, whenever $A_1, \ldots, A_n$ are distinct elements of $\mathcal{A}$, and $r_1, \ldots, r_n$ are nonzero rational numbers, then

$$0 \notin r_1 A_1 + \ldots + r_n A_n.$$

In this case we shall say that the system $\mathcal{A}$ is linearly independent over $\mathbb{Q}$.

Lemma 11. Let $K_1, K_2, \ldots$ be nonempty compact sets such that the system $(K_1, K_2, \ldots)$ is linearly independent over $\mathbb{Q}$. Then for every sequence of intervals $[a_i, b_i]$ ($i = 1, 2, \ldots$) there are nowhere dense perfect sets $P_i \subset [a_i, b_i]$ ($i = 1, 2, \ldots$) such that $\text{int}(P_i + P_i) \neq \emptyset$ for every $i$, and the system $(K_1, K_2, \ldots, P_1, P_2, \ldots)$ is also linearly independent over $\mathbb{Q}$.

Proof. Clearly, $0 \notin K_i$ for every $i$. Also, if $i_1, \ldots, i_n$ are distinct indices and $r_1, \ldots, r_n \in \mathbb{Q} \setminus \{0\}$, then the set $F = r_1 K_{i_1} + \ldots + r_n K_{i_n}$ is nowhere dense. Indeed, $F$ is compact, and thus, if $F$ is not nowhere dense, then its interior is nonempty. In this case we can choose an index $i_{n+1}$ distinct from $i_1, \ldots, i_n$, and a nonzero rational number $r_{n+1}$ such that

$$0 \in r_1 K_{i_1} + \ldots + r_n K_{i_n} + r_{n+1} K_{i_{n+1}}.$$
since $K_{i_{n+1}}$ contains a nonzero element. This, however, contradicts the condition that the system $\{K_1, K_2, \ldots\}$ is linearly independent over $\mathbb{Q}$, and thus $F$ must be nowhere dense.

Let $A$ denote the union of all sets of the form $r_1K_{i_1} + \ldots + r_nK_{i_n}$, where $r_1, \ldots, r_n \in \mathbb{Q} \setminus \{0\}$ and $i_1, \ldots, i_n$ are distinct indices. Then $A$ is of the first category. Let $c, d \in [a_1, b_1] \setminus A$. By the previous lemma, there is a nowhere dense perfect set $P_1 \subset [c, d] \setminus A$ such that $P_1 + P_1 = [2c, 2d]$. It is easy to see, using $P_1 \cap A = \emptyset$, that the system $\{K_1, K_2, \ldots, P_1\}$ is linearly independent over $\mathbb{Q}$. Repeating this argument, we find a nowhere dense perfect set $P_2 \subset [a_2, b_2]$ such that $\text{int}(P_2 + P_2) \neq \emptyset$, and the system $\{K_1, K_2, \ldots, P_1, P_2\}$ is linearly independent over $\mathbb{Q}$. Continuing this process, we find the sets $P_i$ with the required properties. \hfill \Box

Now we turn to the proof of Theorem 4. We shall construct a system $K$ with the properties described in Lemma 7 such that the elements of the set $A$ defined by (1) are linearly independent over the rational numbers. Let $\Sigma$ denote the set of those finite sequences

$$(n_0, n_1, \ldots, n_j, r_1, \ldots, r_j)$$

in which $n_0, \ldots, n_j \in \mathbb{N}$, $n_1, \ldots, n_j$ are distinct, and $r_1, \ldots, r_j \in \mathbb{Q} \setminus \{0\}$. Let $\sigma_1, \sigma_2, \ldots$ be an enumeration of the elements of $\Sigma$. By Lemma 11, we can find nowhere dense perfect sets $K^n_i \subset [0, 1]$ ($n = 1, 2, \ldots$) such that $\text{int}(K^n_i + K^n_i) \neq \emptyset$ for every $n$, the system $\{K^n_1, K^n_2, \ldots\}$ is linearly independent over $\mathbb{Q}$, and the convex hulls $K^n_i$ are pairwise disjoint. We also put $S_1 = \emptyset$.

Let $k \geq 1$, and suppose that we have defined the finite set $S_k \subset \Sigma$ and the nowhere dense perfect sets $K^n_i$ ($i = 1, \ldots, k$, $n = 1, 2, \ldots$) with the following properties:

(i) $\text{int}(K^n_i + K^n_i) \neq \emptyset$ for every $i \leq k$ and $n = 1, 2, \ldots$;
(ii) the system $K_k = \{K^n_i : i = 1, \ldots, k, n = 1, 2, \ldots\}$ is linearly independent over $\mathbb{Q}$;
(iii) for every $i \leq k$, the intervals $K^n_i$ ($n = 1, 2, \ldots$) are pairwise disjoint and are shorter than $1/i$;
(iv) whenever $i < k$, $n \in \mathbb{N}$ and $I$ is a component of $K^n_i \setminus K^n_i$, there is exactly one index $m$ such that $K^m_i \subset I$;
(v) for every $i < k$ and $n \in \mathbb{N}$, there is $m \in \mathbb{N}$ such that $K^{i+1}_n$ is contained in one of the components of $K^m_i \setminus K^m_i$; finally,
(vi) if $(n_0, n_1, \ldots, n_j, r_1, \ldots, r_j) \in S_k$, then $n_0 < k$ and

$0 \notin r_1F^{n_0, k}_{n_1} + \ldots + r_jF^{n_0, k}_{n_j},$

where

$$F^{n_0, k}_{n_j} = \bigcup\{K^n_i : K^n_i \subset K^m_{n_j} \} \quad (m \in \mathbb{N}).$$

Let $I_1, I_2, \ldots$ be an enumeration of those intervals that are components of any of the sets $K^n_i \setminus K^n_i$ ($n = 1, 2, \ldots$). By Lemma 11, there are nonempty sets $P_j \subset I_j$ such that the system $\{K^n_i \cup \{P_1, P_2, \ldots\}\}$ is linearly independent over $\mathbb{Q}$. Let $y_j \in P_j$ for every $j$. Let $\sigma_p = (n_0, n_1, \ldots, n_j, r_1, \ldots, r_j)$ be the first element of the sequence $\sigma_1, \sigma_2, \ldots$ for which $n_0 \leq k$ and $\sigma_p \notin S_k$. Then we define $S_{k+1} = S_k \cup \{\sigma_p\}$. Let

$$H^{n_0}_{n_j} = \bigcup\{K^n_i : n_0 \leq i \leq k, K^n_i \subset K^m_{n_j}\} \cup \{y_j : y_j \in K^m_{n_j}\}.$$
It is easy to check, using (iv) and (v), that $H^{n_0}_m$ is compact for every $m$. This implies that the set

$$H = r_1 H^{n_0}_m + \ldots + r_j H^{n_0}_{n_j}$$

is also compact. Since $K_k \cup \{ P_1, P_2, \ldots \}$ is linearly independent over $\mathbb{Q}$, it follows that $0 \notin H$, and thus there is $\delta > 0$ such that $H \cap (-\delta, \delta) = \emptyset$. Let

$$\eta = \min \left( \frac{1}{k+1}, \frac{\delta}{|r_1| + \ldots + |r_j|} \right).$$

Applying Lemma 11, we obtain nowhere dense perfect sets $K^{k+1}_n \subset (y_n, y_n + \eta) \cap I_n$ such that $\text{int}(K^{k+1}_n + K^{k+1}_n) \neq \emptyset$ for every $n$, and the system

$$K_{k+1} = \{ K_n^i : i = 1, \ldots, k+1, n = 1, 2, \ldots \}$$

is linearly independent over $\mathbb{Q}$. Then

$$0 \notin r_1 F_{n_1}^{n_0, k+1} + \ldots + r_j F_{n_j}^{n_0, k+1}$$

follows from the choice of $\eta$. Since $F_m^{i, k+1} \subset F_m^{i, k}$ for every $i < k$ and $m \in \mathbb{N}$, (vi) remains valid for $k+1$. It is clear from the construction that (i)-(v) are also satisfied by $K_{k+1}$.

In this way we have defined, by induction, the sets $S_k$ and $K^k_n$ for every $k, n \in \mathbb{N}$. Clearly, $\mathcal{K} = \{ K_n^i : i, n \in \mathbb{N} \}$ is a countable and regular system of nowhere dense perfect sets such that $\text{int}(K + K) \neq \emptyset$ for every $K \in \mathcal{K}$. In order to complete the proof, we have to show that the elements of the set $A$ defined by (1) are linearly independent over the rationals. First we remark that the set

$$M^k = \bigcup_{n=1}^{\infty} K_n^k$$

contains $A$ for every $k = 1, 2, \ldots$. Indeed, the set

$$L^k = M^k \cup \bigcup_{i=1}^{k-1} \bigcup_{n=1}^{\infty} K_n^i$$

is closed and contains $\bigcup \mathcal{K}$. Thus we have $\text{cl}(\bigcup \mathcal{K}) \subset L^k$ and

$$A = \text{cl}(\bigcup \mathcal{K}) \setminus \bigcup \mathcal{K} \subset L^k \setminus \bigcup \mathcal{K} \subset M^k.$$

Let $x_1, \ldots, x_j$ be distinct elements of $A$ and let $r_1, \ldots, r_j \in \mathbb{Q} \setminus \{0\}$. We show that $r_1 x_1 + \ldots + r_j x_j \neq 0$. Choose a positive integer $n_0$ with

$$1/n_0 < \min \{|x_{i_2} - x_{i_1}| : 1 \leq i_1 < i_2 \leq j\}.$$ 

Since $A \subset M^{n_0}$, there are indices $n_1, \ldots, n_j$ such that $x_i \in \overline{K_{n_i}^{n_0}}$ ($i = 1, \ldots, j$). By the choice of $n_0$ and by (iii), the indices $n_1, \ldots, n_j$ are distinct. Let

$$(n_0, n_1, \ldots, n_j, r_1, \ldots, r_j) = \sigma_p.$$ 

We put $k = n_0 + p$; then it follows from the construction that $\sigma_p \in S_k$. We have

$$x_i \in \overline{K_{n_i}^{n_0}} \cap A \subset \overline{K_{n_i}^{n_0}} \cap M^k = F_{n_i}^{n_0, k}$$

for every $i = 1, \ldots, j$. Therefore, by (vi) we obtain $r_1 x_1 + \ldots + r_j x_j \neq 0$, which completes the proof. \qed
References


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