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A UNIQUENESS THEOREM FOR HARMONIC FUNCTIONS

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ABSTRACT. The main result of this note is the following theorem:

Theorem 1. *Let $D = \{(x,t); |x|^2 + t^2 \leq r^2,\ t > 0\}$ be a half ball in R^{n+1} and $x \in R^n$. Assume that u is C^1 in $\overline{D}$ and harmonic in D, and that for every positive integer N there exists a constant C_N such that*

(1) $|\nabla u(x,0)| \leq C_N |x|^N$ *in a neighbourhood V of the origin in ∂D;*

(2) $$u(x,0) \geq u(0,0) \quad \text{in} \quad V.$$

Then $u \equiv u(0,0)$.

First we prove it for R^2, and then we show by induction that it holds for all $n \geq 3$.

INTRODUCTION

Theorem 1 stated in the Abstract is somewhat analogous to theorem 1 of Baouendi and Rothschild [1], which is a uniqueness theorem for holomorphic functions of one complex variable; they applied it to obtain results on unique continuation for functions of several complex variables. For $n = 2$, we show that our theorem is equivalent to theirs. In [2], Baouendi and Rothschild obtained results that are related to ours, but that neither imply nor are implied by ours. The main result of [2] is theorem 3, where they assume that u vanishes of infinite order at the origin in the normal direction and $u \geq 0$ in V, and deduce that u vanishes on V and also along the normal identically. In our case we assume that ∇u vanishes of infinite order at the origin restricted to ∂D, and deduce the same thing. We deduce uniqueness based on tangential behaviour, whereas they deduce uniqueness based on behaviour along the normal direction. But one might argue that ∇u includes the normal derivative. On the other hand it is not clear how, from the tangential decay of the normal derivative, one can deduce the decay of u along the normal direction unless one assumes u is C^∞ on $\overline{D}$. Also in the infinitely smooth case one can easily see the equivalence of our results to those of [2].

For completeness we quote theorem 1 of [1]:

Theorem BR. *Let $\Omega = \{z; |z| < r, y > 0\}$, a half disc in the complex plane C. Assume that h is continuous in $\overline{\Omega}$ and holomorphic in Ω, and that h vanishes up to infinite order at the origin, i.e., for every positive integer N there exists a constant*

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C_N such that

$$|h(z)| \le C_N|z^N|,\ z \in \overline{\Omega}; \tag{3}$$

$$\operatorname{Re} h(x) \ge 0,\ x \in \partial\Omega \cap R \qquad \textit{in a neighbourhood of the origin.} \tag{4}$$

Then $h \equiv 0$.

Condition (3) assumes that $h(z)$ vanishes up to infinite order from within Ω, but one does not really need that. Theorem BR is equivalent to the following:

Theorem 2. *In theorem BR, all else being equal, we replace* (3) *by the following: h vanishes up to infinite order at 0 on $\partial\Omega$, i.e., for every positive integer N there exists a constant C_N such that*

$$|h(x)| \le C_N|x|^N, \quad x \in \partial\Omega \cap R \quad \textit{in a neighbourhood of} \quad 0. \tag{3$'$}$$

Then $h \equiv 0$.

Proof. We show that ($3'$) implies (3). We note that $\ln|h(z)|$ is subharmonic in Ω; on $\partial\Omega$,

$$\ln|h(z)| \le N\ln|z| + \ln C_N \quad \text{in a neighbourhood of the origin,}$$

and outside that neighbourhood

$$\ln|h(z)| - N\ln|z| - \ln C_N \quad \text{is bounded above (say) by } C'_N > 0.$$

Therefore

$$\ln|h(z)| \le N\ln|z| + \ln C_N + C'_N \text{ on } \partial\Omega;$$

by the maximum principle,

$$\ln|h(z)| \le N\ln|z| + \ln C_N + C'_N \text{ on all of } \Omega,$$

and so

$$|h(z)| \le C_N e^{C'_N}|z|^N \quad \text{on } \Omega,$$

proving (3). □

In order to deal with harmonic functions in R^2, we reformulate theorem 1 as follows:

Theorem 3. *Suppose u is harmonic in Ω and C^1 in $\overline{\Omega}$. Assume also that for every positive integer N, there exists a constant C_N such that*

$$|\nabla u(z)| \le C_N|z|^N \textit{ for all } z \in \partial\Omega \textit{ in a neighbourhood of the origin;} \tag{5}$$

$$u(x) \ge u(0) \quad \textit{for all } x \textit{ in a neighbourhood of the origin on } \partial\Omega. \tag{6}$$

Then $u \equiv u(0)$.

Proof. Let $v(z)$ be the harmonic conjugate of u such that $v(0) = 0$. It is well-known that v is continuous in $\overline{\Omega}$. Let $f(z) = u(z) + iv(z) - u(0)$. Clearly $f(z)$ is continuous on $\overline{\Omega}$, holomorphic in Ω, and $\operatorname{Re} f \ge 0$ in a neighbourhood of the origin on $\partial\Omega$. We notice that

$$\frac{\partial u}{\partial x} - i\frac{\partial u}{\partial y} = f'(z),\ \int_0^z f'(s)\,ds = f(z).$$

Therefore, by (5), on $\partial\Omega$ in a neighbourhood of the origin we have

$$|f(z)| \leq C_N \int_0^{|z|} |s|^N |ds| = \frac{C_N}{N+1}|z|^{N+1}.$$

Now by theorem 2, $f \equiv 0$ and so $u \equiv u(0)$. □

We need the following lemma:

Lemma 4. *Suppose* u *is at least* C^2 *in a symmetric interval* $[-r, r]$, $u(x) \equiv u(-x)$, *and*

$$v(x) = I(u)(x) = \int_0^{\frac{\pi}{2}} u(x\cos\theta)x\cos\theta\, d\theta.$$

Then

$$v''(x) \equiv I(Lu)(x),$$

where $L(u)$ *is defined as follows:*

$$L(u)(x) = u''(x) + \frac{u'(x)}{x}.$$

This must be classical, but I do not know any good reference. Here is a short proof.

Proof. Differentiating under the integral sign twice, we obtain

$$v''(x) = \int_0^{\frac{\pi}{2}} u''(x\cos\theta)x\cos^3\theta + 2u'(x\cos\theta)x\cos^2\theta\ d\theta \tag{7}$$

and further

$$I(L(u))(x) = \int_0^{\frac{\pi}{2}} u''(x\cos\theta)x\cos\theta + u'(x\cos\theta)\ d\theta. \tag{8}$$

Hence

$$\begin{aligned} v''(x) - I(L(u))(x) &= \int_0^{\frac{\pi}{2}} (-u''(x\cos\theta)x\cos\theta\sin^2\theta + u'(x\cos\theta)(2\cos^2\theta - 1))\ d\theta \\ &= \int_0^{\frac{\pi}{2}} \frac{d}{d\theta}\{u'(x\cos\theta)\sin\theta\cos\theta\}\ d\theta \\ &= u'(x\cos\theta)\sin\theta\cos\theta\ |_0^{\frac{\pi}{2}} \\ &= 0. \end{aligned}$$

This proves the lemma. □

Proof of theorem 1. For simplicity of notation we shall deal with R^3, but the method easily generalizes to R^n. Without loss of generality we may assume that $u(0,0,0) = 0$. Let

$$u_\theta(x, y, t) = u(x\cos\theta - y\sin\theta, x\sin\theta + y\cos\theta, t).$$

Then u_θ is harmonic for any fixed θ in D, and is C^1 in $\overline{D}$. Let

$$m(x, y, t) = \frac{1}{2\pi}\int_0^{2\pi} u_\theta(x, y, t)d\theta.$$

Then $m(x,y,t)$ is harmonic in D, and is C^1 in $\overline{D}$. But also $p(x,t)=m(x,0,t)$ is C^1 in Ω [the half-disc in theorem BR] and satisfies the differential equation

$$\frac{\partial^2 p}{\partial x^2}+\frac{1}{x}\frac{\partial p}{\partial x}+\frac{\partial^2 p}{\partial t^2}=0.$$

Therefore by Lemma 4, the function

$$v(x,t)=\int_0^{\frac{\pi}{2}} p(x\cos\theta,t)x\cos\theta d\theta$$

is harmonic in Ω and C^1 in $\overline{\Omega}$. It is clear that

$$|\nabla m|\leq \max_{0\leq\theta\leq 2\pi}|\nabla u_\theta|,\ |\nabla p(x,t)|\leq|\nabla m(x,0,t)|.$$

Therefore from (1) we have the estimate

(9) $|\nabla p(x,0)|\leq C_N|x|^N$in a neighbourhood of the origin on the x-axis.

Also

$$\frac{\partial v}{\partial x}=\int_0^{\frac{\pi}{2}}\frac{\partial p}{\partial x}(x\cos\theta,t)x\cos^2\theta+p(x\cos\theta,t)\cos\theta d\theta,$$

(10)

$$\frac{\partial v}{\partial t}=\int_0^{\frac{\pi}{2}}\frac{\partial p}{\partial t}(x\cos\theta,t)x\cos\theta d\theta$$

and further we notice that $|p(x,0)-p(0,0)|=|p(x,0)|\leq|x|\max_{|s|\leq|x|}|\nabla p(s,0)|\leq C_N|x|^{N+1}$. Combining (9) and (10), we get that in a neighbourhood of the origin on the x-axis,

$$|\nabla v(x,0)|\leq C_N|x|^{N+1},\ v(x,0)\geq 0.$$

Now applying theorem 3 to v, we have $v(x,t)\equiv 0$. Because p is non-negative on the x-axis in a neighbourhood of the origin, this gives $p(x,0)=0$ in a neighbourhood of the origin, and this in turn leads to the conclusion that $m(x,y,0)=0$ and hence $u(x,y,0)=0$ in a neighbourhood of the origin. But this would imply that u and hence ∇u are real-analytic at the origin, and thus (1) implies $u\equiv 0$. □

Remark 5. We can reduce the general R^n to the case of R^{n-1} in the same way we reduced from R^3 to R^2.

Open problem. Suppose u is continuously differentiable on the closed upper half of the unit disk Ω in R^2 and ∇u restricted to $\partial\Omega$ vanishes up to order N at the origin. Then it is easy to show that it vanishes up to order N at the origin from within Ω also. Is the same result true in higher dimensions?

References

[1] M. S. Baouendi and L. P. Rothschild, *Unique continuation and a Schwarz reflection principle for analytic sets*, Comm. P. D. E. 18(11) (1993), pp. 1961-1970. MR **94i:**32014

[2] M. S. Baouendi and L. P. Rothschild, *A local Hopf lemma and unique continuation for harmonic functions*, Int. Math. Res. Notices, 1993, no. 8, pp. 245–251. MR **94i:**31008

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