ON ASYMMETRY OF TOPOLOGICAL CENTERS
OF THE SECOND DUALS OF BANACH ALGEBRAS

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Abstract. Let \( A \) be a Banach algebra with a bounded approximate identity
and let \( Z_1(A^{**}) \) and \( Z_2(A^{**}) \) be the left and right topological centers of \( A^{**} \).
It is shown that i) \( A^*A = AA^* \) is not sufficient for \( Z_1(A^{**}) = Z_2(A^{**}) \); ii) the
inclusion \( \hat{A}Z_1(A^{**}) \subseteq \hat{A} \) is not sufficient for \( Z_2(A^{**}) \hat{A} \subseteq \hat{A} \); iii) \( Z_1(A^{**}) = Z_2(A^{**}) = \hat{A} \) is not sufficient for \( A \) to be weakly sequentially complete. These
results answer three questions of Anthony To-Ming Lau and Ali Ülger.

Suppose that \( A \) is a Banach algebra and let \( A^* \) be the dual space of \( A \). Then \( A^* \)
can be made into a Banach \( A \)-bimodule as follows: for \( f \in A^*, a \in A \), \( fa \) and \( af \)
are defined by
\[
\langle fa, b \rangle = \langle f, ab \rangle, \\
\langle af, b \rangle = \langle f, ba \rangle 
\]
where \( \langle , \rangle \) is used for the dual pairing between elements of \( A^* \) and \( A \).

Let \( A^*A = \{ fa : f \in A^*, a \in A \} \) and \( AA^* = \{ af : a \in A, f \in A^* \} \). The
dual space \( A^* \) is said to factor on the left (resp. right) if \( A^* = A^*A \) (resp. \( A^* = AA^* \)). In a recent paper [5] Anthony To-Ming Lau and Ali Ülger have obtained
various necessary and sufficient conditions for factoring of \( A^* \). Here we answer three
questions left open in [5].

The second dual space \( A^{**} \) of a Banach algebra \( A \) admits two Banach algebra
products known as first (left) and second (right) Arens products. Each of these
products extends the product of \( A \) as canonically embedded in \( A^{**} \) (see [1] or [3]).
We briefly recall the definition of these products. For \( m, n \in A^{**} \), their first (left)
Arens product indicated by \( m \Box n \) is given by
\[
\langle m \Box n, f \rangle = \langle m, nf \rangle \quad (f \in A^*),
\]
where \( nf \in A^* \) is defined by
\[
\langle nf, a \rangle = \langle n, fa \rangle \quad (a \in A),
\]
where \( fa \) is as defined earlier.

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The second Arens product of elements \( m, n \in \mathfrak{A}^{**} \) indicated by \( m \diamond n \) is defined by
\[
\langle m \diamond n, f \rangle = \langle n, fm \rangle \quad (f \in \mathfrak{A}^{*}),
\]
where \( fm \) is an element of \( \mathfrak{A}^{*} \) defined by
\[
\langle fm, a \rangle = \langle m, af \rangle \quad (a \in \mathfrak{A}).
\]
Again we note that \( af \) is as defined earlier.

For each \( n \in \mathfrak{A}^{**} \), the mapping \( m \mapsto m \Box n \) (resp. \( m \mapsto n \Box m \)) is weak*-weak* continuous. However for certain \( n \), the mapping \( m \mapsto n \Box m \) (resp. \( m \mapsto m \Box n \)) may fail to be weak* continuous. Due to this lack of symmetry the left (resp. right) topological center \( Z_2(\mathfrak{A}^{**}) \) (resp. \( Z_2(\mathfrak{A}^{**}) \)) of \( \mathfrak{A}^{**} \) is defined by
\[
Z_1(\mathfrak{A}^{**}) = \{ m \in \mathfrak{A}^{**} : n \mapsto m \Box n \text{ is weak*-weak* continuous} \},
\]
\[
Z_2(\mathfrak{A}^{**}) = \{ m \in \mathfrak{A}^{**} : n \mapsto n \Box m \text{ is weak*-weak* continuous} \}.
\]

It follows easily from the definition of \( Z_1(\mathfrak{A}^{**}) \) (resp. \( Z_2(\mathfrak{A}^{**}) \)) that \( m \in Z_1(\mathfrak{A}^{**}) \) (resp. \( m \in Z_2(\mathfrak{A}^{**}) \)) if and only if \( m \Box n = m \Diamond n \) (resp. \( n \Box m = n \Diamond m \)) for all \( n \in \mathfrak{A}^{**} \). In general \( Z_1(\mathfrak{A}^{**}) \) and \( Z_2(\mathfrak{A}^{**}) \) need not be equal (see [5], Example 2.5 and Remark 6.j, p.1211), but they both contain \( \mathfrak{A} \) (= the image of \( \mathfrak{A} \) in \( \mathfrak{A}^{**} \) under the canonical mapping).

Anthony To-Ming Lau and Ali Ülger have shown in [5] that if \( \mathfrak{A}^{*} \) factors on one side but not on the other, then \( Z_1(\mathfrak{A}^{**}) \neq Z_2(\mathfrak{A}^{**}) \). In connection to this result they have asked:

**Question.** Suppose that \( \mathfrak{A} \) has a bounded approximate identity. Does the equality \( \mathfrak{A}^{*}\mathfrak{A} = \mathfrak{A}\mathfrak{A}^{*} \) imply that \( Z_1(\mathfrak{A}^{**}) = Z_2(\mathfrak{A}^{**}) \)?

We answer this question in the negative. In fact our counter example is a modification of one of the examples in [5].

For a Banach algebra \( \mathfrak{A} \) we let \( \mathfrak{A}^{\sharp} \) be the unitization of \( \mathfrak{A} \). Thus \( \mathfrak{A}^{\sharp} = \mathfrak{A} \oplus \mathbb{C} \), where \( \mathbb{C} \) is the field of complex numbers. For \( a, b \in \mathfrak{A}, \alpha, \beta \in \mathbb{C} \) we have \( \| (a, \alpha) \| = \| a \| + | \alpha | \) and \( (a, \alpha)(b, \beta) = (ab + \alpha b + \beta a, \alpha \beta) \).

1. **Lemma.** For any Banach algebra \( \mathfrak{A} \), \( Z_1(\mathfrak{A}^{\sharp^{**}}) = Z_2(\mathfrak{A}^{\sharp^{**}}) \) and a similar result holds for the right topological centers.

**Proof.** First we note that the two Banach algebras \( (\mathfrak{A}^{\sharp})^{**} \) and \( (\mathfrak{A}^{**})^{\sharp} \) are isomorphic, when they both have the first Arens product. Then the rest is straightforward from the definition of topological centers.

The following answers question (6d) in [5].

2. **Theorem.** There exists a Banach algebra \( \mathfrak{A} \) possessing a bounded approximate identity and satisfying \( \mathfrak{A}^{*}\mathfrak{A} = \mathfrak{A}\mathfrak{A}^{*} = \mathfrak{A}^{*} \), but \( Z_1(\mathfrak{A}^{**}) \neq Z_2(\mathfrak{A}^{**}) \).

**Proof.** It is known that there exists a Banach algebra \( \mathfrak{B} \) having a bounded approximate identity for which \( Z_1(\mathfrak{B}^{**}) \neq Z_2(\mathfrak{B}^{**}) \) (see [5]). Then for \( \mathfrak{A} = \mathfrak{B}^{\sharp} \), by Lemma 1 we have \( Z_1(\mathfrak{A}^{**}) \neq Z_2(\mathfrak{A}^{**}) \). However, \( \mathfrak{A}^{*}\mathfrak{A} = \mathfrak{A}\mathfrak{A}^{*} = \mathfrak{A}^{*} \), since \( \mathfrak{A} \) is unital.

For the origin of the following question see [5, question 6e].

**Question.** Does the inclusion \( \mathfrak{A}Z_1(\mathfrak{A}^{**}) \subseteq \mathfrak{A} \) imply \( Z_2(\mathfrak{A}^{**})\mathfrak{A} \subseteq \mathfrak{A} \)?
To answer this question, we first recall that if $\mathfrak{A}$ is a Banach algebra then $\mathfrak{A}^{op}$ is the algebra obtained by reversing the order of multiplication in $\mathfrak{A}$; i.e., $\mathfrak{A}^{op}$ has the product $\circ$ given by $a \circ b = ba$, for every $a, b \in \mathfrak{A}$.

3. Theorem. There exists a Banach algebra $\mathfrak{A}$ with a bounded approximate identity for which $\hat{\mathfrak{A}}Z_1(\mathfrak{A}^{**}) \subset \mathfrak{A}$, but $Z_2(\mathfrak{A}^{**})\mathfrak{A}$ is not contained in $\mathfrak{A}$.

Proof. There exists a Banach algebra $\mathfrak{B}$, with a bounded approximate identity, such that $Z_2(\mathfrak{B}^{**}) = \mathfrak{B}$ but $Z_1(\mathfrak{B}^{**})$ is larger than $\mathfrak{B}$ (see [5]). It is well known that switching to the opposite multiplication in a Banach algebra results in interchanging the first and the second Arens products in its second dual space. From this and by Lemma 1, for $\mathfrak{A} = (\mathfrak{B}^{op})^2$ we have

$$Z_1(\mathfrak{A}^{**}) = \mathfrak{B}^{op} \oplus C = \hat{\mathfrak{A}},$$

and

$$Z_2(\mathfrak{A}^{**}) = Z_1(\mathfrak{B}^{**}) \oplus C.$$

Hence

$$\hat{\mathfrak{A}}Z_1(\mathfrak{A}^{**}) = (\mathfrak{B}^{op} \oplus C)(\mathfrak{B}^{op} \oplus C) = \mathfrak{B}^{op} \oplus C = \hat{\mathfrak{A}},$$

and

$$Z_2(\mathfrak{A}^{**})\hat{\mathfrak{A}} = (Z_1(\mathfrak{B}^{**}) \oplus C)(\mathfrak{B}^{op} \oplus C) = Z_1(\mathfrak{B}^{**}) \oplus C.$$

But $Z_1(\mathfrak{B}^{**}) \oplus C$ is larger than $\hat{\mathfrak{A}}$, since $Z_1(\mathfrak{B}^{**})$ is larger than $\hat{\mathfrak{B}}$.

In [5], Lau and Ülger ask (question 6j): if $Z_1(\mathfrak{A}^{**}) = Z_2(\mathfrak{A}^{**}) = \hat{\mathfrak{A}}$, must $\mathfrak{A}$ be weakly sequentially complete?

We give an example of a Banach algebra $\mathfrak{A}$ with identity, and such that $Z_1(\mathfrak{A}^{**}) = Z_2(\mathfrak{A}^{**}) = \hat{\mathfrak{A}}$, but with $\mathfrak{A}$ not weakly sequentially complete. First we describe a general construction: this construction is known and has been used by workers in the area of Automatic Continuity (see [2], p. 647).

Let $A$ be a Banach algebra, and let $M$ be a Banach $\mathfrak{A}$-bimodule. Then $\mathfrak{A} = M \oplus A$, with norm $\| (m, a) \| = \| m \| + \| a \|$, and product $(m, a)(n, b) = (mb + an, ab)$ is a Banach algebra. If $A$ has a bounded approximate identity $(e_j)$ and if $M$ factors $(M = MA = AM)$, then $(0, e_j)$ is a bounded approximate identity for $\mathfrak{A}$. In particular, if $A$ has an identity $e$, and $M$ is unital, then $(0, e)$ is an identity for $\mathfrak{A}$. Note that $\mathfrak{A}^* = M^* \times A^*$ (with norm $\| (s, f) \| = \max \{ \| s \|, \| f \| \}$), and that $\mathfrak{A}^{**} = M^{**} \oplus A^{**}$. It is routine to check that $\mathfrak{A}$ is weakly sequentially complete if and only if each of $A$ and $M$ is weakly sequentially complete. Also, it can be checked that the first Arens product on $\mathfrak{A}^{**}$ satisfies

(1) $$(\mu, \alpha) \otimes (\nu, \beta) = (\mu \otimes (\beta + \alpha) \otimes \nu, \alpha \otimes \beta),$$
for $\mu, \nu \in M^{**}$ and $\alpha, \beta \in A^{**}$.

Here, in all cases, $\otimes$ has been used to denote that extension of a bilinear operation to second duals, which is weak*-weak* continuous in the left-hand argument; for instance, $(\mu, \beta) \mapsto \mu \otimes \beta$ denotes the extension to $M^{**} \times A^{**}$ of the right module action $(m, b) \mapsto mb$ of $A$ on $M$ which is weak*-weak* continuous in $\mu$.

Now we identify $Z_1(\mathfrak{A}^{**})$. Since a net $(\nu_j, \beta_j)$ in $\mathfrak{A}^{**}$ converges weak* to $(\nu, \beta)$ if and only if $\nu_j \to \nu$ weak* in $M^{**}$ and $\beta_j \to \beta$ weak* in $A^{**}$, it is straightforward to check that $(\mu, \alpha) \in Z_1(\mathfrak{A}^{**})$ if and only if

(2) $\alpha \in Z_1(\mathfrak{A}^{**})$;

(3) $\nu \mapsto \alpha \otimes \nu$ is weak*-weak* continuous on $M^{**}$; and

(4) $\beta \mapsto \mu \otimes \beta$ is weak*-weak* continuous $A^{**} \to M^{**}$. 

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Now suppose $Z_1(A^{**}) = \hat{A}$. Then (2) implies $\alpha \in \hat{A}$, and it is then straightforward to verify that (3) holds. So in this case, $(\mu, \alpha) \in Z_1(\mathfrak{A}^{**})$ if and only if $\alpha \in \hat{A}$ and (4) holds.

Finally, we consider the particular case $M = A^*$. Recall that we are assuming $A$ has a bounded approximate identity. To ensure that $\mathfrak{A}^{**}$ has a bounded approximate identity, we suppose that $A^*$ factors, in the sense of [5]. Note that the case that $A$ has an identity is included.

Now we have $M^* = A^{**}$ and $M^{**} = A^{***}$, and the meanings of the various bilinear operations have to be carefully distinguished; however, it can be shown that, for $\mu \in M^{**}$, $\beta \in A^{**}$, and $\alpha \in M^* = A^*$,

$$\langle \mu \square \beta, \alpha \rangle = \langle \mu, \beta \square \alpha \rangle;$$

that is, the bilinear map $(\mu, \beta) \mapsto \mu \square \beta$ on $M^{**} \times A^{**}$ to $M^{**}$ is the right dual module action of $A^{**}$ on $A^{***} = M^{**}$ corresponding to the first Arens product on $A^{**}$. Suppose (4) holds, and take $\alpha = E$, a right identity for the first Arens product in $A^{**} = M^*$. If $\beta_j \to \beta$ weak in $A^{**}$, then by (4),

$$\langle \mu, \beta_j \rangle = \langle \mu, \beta \rangle E \to \langle \mu \square \beta, E \rangle = \langle \mu, \beta \rangle.$$ 

That is, (4) implies that $\mu$ is a weak*- continuous linear functional on $A^{**}$, so that $\mu \in (A^*)^* = M$. The converse is easy, so we have proved the conclusion $Z_1(\mathfrak{A}^{**}) = \hat{M} \oplus \hat{A} = \hat{\mathfrak{A}}$ in the following proposition; the arguments for $Z_2(\mathfrak{A}^{**})$ are quite similar.

6. Proposition. Let $A$ be a Banach algebra with a bounded approximate identity.

1. Assume $A^*$ factors (in the sense of [5]). Then $\mathfrak{A} = A^* \oplus A$ has a bounded approximate identity.

2. Assume $Z_1(A^{**}) = Z_2(A^{**}) = \hat{A}$. Then $Z_1(\mathfrak{A}^{**}) = Z_2(\mathfrak{A}^{**}) = \hat{\mathfrak{A}}$.

3. $\mathfrak{A}$ is weakly sequentially complete if and only if each of $A$ and $A^*$ is weakly sequentially complete.

If we take $A = l^1(\mathbb{Z})$ (the group algebra of the discrete group of integers), then $Z_1(A^{**}) = Z_2(A^{**}) = A$ (see [4]), and all the assumptions above are satisfied; in particular, $A$ has an identity. Also, $A^* = l^\infty(\mathbb{Z})$ is not weakly sequentially complete. So $\mathfrak{A} = A^* \oplus A$ has an identity, satisfies $Z_1(\mathfrak{A}^{**}) = Z_2(\mathfrak{A}^{**}) = \hat{\mathfrak{A}}$, and $\mathfrak{A}$ is not weakly sequentially complete. That answers (negatively) question 6j of [5].

References


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