TRANSITIVE AND FULLY TRANSITIVE GROUPS

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Abstract. The notions of transitivity and full transitivity for abelian $p$-groups were introduced by Kaplansky in the 1950s. Important classes of transitive and fully transitive $p$-groups were discovered by Hill, among others. Since a 1976 paper by Corner, it has been known that the two properties are independent of one another. We examine how the formation of direct sums of $p$-groups affects transitivity and full transitivity. In so doing, we uncover a far-reaching class of $p$-groups for which transitivity and full transitivity are equivalent. This result sheds light on the relationship between the two properties for all $p$-groups.

1. Introduction

Throughout this note, we will denote the $p$-height sequence of an element $x$ in a $p$-local abelian group $G$ by $U_G(x)$ or simply $U(x)$. Recall that $G$ is transitive if $x$ can be mapped to $y$ by an automorphism of $G$ whenever $x, y \in G$ satisfy $U(x) = U(y)$; and fully transitive if this can be accomplished by an endomorphism of $G$ whenever $U(x) \leq U(y)$ pointwise. Extensive classes of abelian $p$-groups with both transitivity properties—including separable and totally projective $p$-groups—are set forth in [Co], [Gr], [Hi] and [Ka]. Examples of $p$-groups with neither of the properties are given in [Me] and [Hi].

Can an abelian $p$-group be fully transitive but not transitive, or vice versa? In the earliest account [Ka], Kaplansky proved that transitive $p$-groups are indeed fully transitive provided $p \neq 2$. More than twenty years later, however, Corner [Co] answered the question in the negative by constructing fully transitive $p$-groups which fail to be transitive, and a transitive 2-group which is not fully transitive.

Despite the independence of transitivity and full transitivity for abelian $p$-groups, it has become increasingly clear (see e.g. [CaGo]) that there is, indeed, some basic connection between the two. The most striking of the results in this present note is the fundamental, but apparently unknown

**Corollary 3.** A $p$-group $G$ is fully transitive if and only if its square $G \oplus G$ is transitive.
In Theorem 1, we will set forth an extensive class of $p$-groups for which transitivity and full transitivity are equivalent. Corollary 3 is symptomatic of the fact that this class contains the square of every abelian $p$-group.

Throughout this note all groups are reduced $p$-local abelian groups, and we refer to them simply as groups. Notation follows the standard works of Fuchs [Fu] and Kaplansky [Ka] with the exception that maps are written on the right; all undefined terms may be found in these references.

2. Transitivity and Full Transitivity

It was shown by Megibben [Me, Theorem 2.4] that the direct sum of two fully transitive $p$-groups need not be fully transitive. In order to obtain closure under direct sums, we first extend the notion of full transitivity in a fairly obvious way.

**Definition 1.** If $G_1$ and $G_2$ are groups, then $\{G_1, G_2\}$ is a fully transitive pair if for every $x \in G_i$, $y \in G_j$ ($i, j \in \{1, 2\}$) which satisfy $U_G(x) \leq U_G(y)$, there exists $\alpha \in \text{Hom}(G_i, G_j)$ with $x\alpha = y$.

For example, it is an easy exercise to verify that $\{G_1, G_2\}$ is a fully transitive pair whenever $G_1$ and $G_2$ are direct summands of a fully transitive group. The next result shows that all fully transitive pairs of $p$-groups arise in this way.

**Proposition 1.** Let $\{G_i\}_{i \in I}$ be a collection of $p$-groups such that for each $i, j \in I$, $\{G_i, G_j\}$ is a fully transitive pair. Then the (external) direct sum $\bigoplus_{i \in I} G_i$ is fully transitive.

**Proof.** It suffices to consider the case where $I = \{1, \ldots, n\}$ is finite. Denote $G = G_1 \oplus \cdots \oplus G_n$ and suppose $x, y \in G$ satisfy $U_G(x) \leq U_G(y)$. We will obtain an endomorphism of $G$ mapping $x$ to $y$ by inducting on the order of $y$. First suppose $py = 0$. Write $x = (x_1, \ldots, x_n)$ and $y = (y_1, \ldots, y_n)$. By relabelling, we may assume that the $p$-heights satisfy $ht_G(x) = ht_G(x_1)$. Observe, since $py = 0$, that $U_G(x_1) \leq U_G(y_1) \leq U_G(y_i)$ for all $i$. By assumption, there exist $\alpha_i \in \text{Hom}(G_1, G_i)$ with $x_1\alpha_i = y_i$ for $1 \leq i \leq n$. Clearly, the $n \times n$ matrix with first row $(\alpha_1, \ldots, \alpha_n)$ and other rows zero represents an endomorphism of $G$ mapping $x$ to $y$.

Now assume $o(y) > p$. Note $U_G(py) \leq U_G(py)$. Since $o(py) < o(y)$, induction yields $\theta \in \text{End}(G)$ with $(py)\theta = py$. Set $x' = x\theta$. Then $y - x' \in G[p]$ and $U_G(x) \leq U_G(y - x')$; hence by the first paragraph there exists $\alpha \in \text{End}(G)$ with $x\alpha = y - x'$. Now $\theta + \alpha$ maps $x$ to $x' + y - x' = y$, as desired. \qed

We will require the following consequence of Proposition 1, indicating how a single fully transitive $p$-group can be used to produce many more.

**Corollary 1.** Let $G$ be a fully transitive $p$-group and $\lambda$ any cardinal. Then all direct summands of the power $\bigoplus_{\lambda} G$ are fully transitive.

**Proof.** Because $G$ is fully transitive, $\{G, G\}$ is a fully transitive pair. Proposition 1 implies $\bigoplus_{\lambda} G$ is fully transitive, and the claim follows since direct summands of fully transitive groups are fully transitive. \qed

In [Co, Proposition 2.2], Corner proves that if $G = G_1 \oplus G_2$ is a fully transitive $p$-group such that the Ulm subgroups $p^\omega G_1, p^\omega G_2$ are nontrivial and $p^\omega G$ is homocyclic, then $G$ is transitive. His example of a non-transitive, fully transitive $p$-group $G$ is such that $p^{\omega + 1} G = 0$, and he notes the “curious consequence” of Proposition 2.2 that $G \oplus G$ must be transitive (note that it is fully transitive by Corollary 1.
Choosing \( G \) of all ordinal numbers

Because \( \sigma \) is a fully transitive group, \( \sigma \) is a fully transitive pair, we have \( \{ G \} \) as above. It was Corner’s result that motivated the theorem we shall soon prove. We need two preparatory lemmas.

**Lemma 1.** Assume \( G = G_1 \oplus G_2 \) is a fully transitive group and \( x_i, y_i \in G_i \) \((i = 1, 2)\). If \( U_{G_1}(x_1) \leq U_{G_2}(y_2 - x_2) \) and \( U_{G_1}(y_1) \leq U_{G_2}(y_1 - x_1) \), then there is an automorphism of \( G \) mapping \((x_1, x_2)\) to \((y_1, y_2)\).

**Proof.** Because \( \{ G_1, G_2 \} \) is a fully transitive pair, there exist \( \alpha \in \text{Hom}(G_1, G_2) \) and \( \beta \in \text{Hom}(G_2, G_1) \) with \( x_1 \alpha = y_2 - x_2 \) and \( y_2 \beta = y_1 - x_1 \). The matrix \( \phi = \begin{pmatrix} 1 + \alpha \beta & \alpha \\ \beta & 1 \end{pmatrix} \) represents an automorphism of \( G_1 \oplus G_2 \), and an easy check verifies that \((x_1, x_2)\phi = (y_1, y_2)\). \( \square \)

If \( G \) is a group and \( \sigma \) an ordinal number, we use \( f_G(\sigma) \) to denote the classical Ulm invariant of \( G \) at \( \sigma \) (see \([Fu]\) or \([Ka]\)).

**Definition 2.** If \( G \) is a reduced group, the Ulm support \( \text{supp}(G) \) of \( G \) is the set of all ordinal numbers \( \sigma \) less than the \( p \)-length of \( G \) for which \( f_G(\sigma) \) is nonzero.

If \( G_1 \) and \( G_2 \) are \( p \)-groups with \( \text{supp}(G_1) \subseteq \text{supp}(G_2) \), it follows that every \( U \)-sequence relative to \( G_1 \) is also a \( U \)-sequence relative to \( G_2 \). In particular, we note that for every \( x \in G_1 \) there is an element \( y \in G_2 \) such that \( U_{G_1}(x) = U_{G_2}(y) \) (see \([Ka, \text{Lemma 24}]\)). We employ this fact in the proof of the following crucial lemma.

**Lemma 3.** Assume \( G = G_1 \oplus G_2 \) is a fully transitive \( p \)-group and \( \text{supp}(p^\omega G_1) \subseteq \text{supp}(p^\omega G_2) \). If \( x \in p^\omega G \), there is an automorphism of \( G \) mapping \( x \) to an element \((c, d) \in G_1 \oplus G_2 \) with \( U_G(x) = U_{G_2}(d) \).

**Proof.** Write \( x = (a, b) \) and assume for the moment that we have shown that there exists an automorphism \( \phi \) of \( G \) with \( x\phi = (a_1, b_1) \) and \( \text{ht}_{G_i}(p^a_1) \neq \text{ht}_{G_2}(p^b_1) \) whenever \( p^a_1 \neq 0 \). But then, as noted above, our assumption on the Ulm supports means we can choose \( b_2 \in p^\omega G_2 \) such that \( U_{G_1}(a_1) = U_{G_2}(b_2) \). By full transitivity, \( b_2 = a_1 \alpha \) for some homomorphism \( \alpha : G_1 \rightarrow G_2 \). The composite automorphism \( \phi \begin{pmatrix} 1 & 0 \\ \alpha & 1 \end{pmatrix} \) of \( G_1 \oplus G_2 \) maps \( x \) to \((a_1, b_1 + b_2)\). Since \( \text{ht}(p^a_1) \neq \text{ht}(p^b_2) \) when \( p^b_1 \neq 0 \), we compute

\[
U_{G_2}(b_1 + b_2) = U_{G_2}(b_1) \land U_{G_2}(b_2) = U_{G_2}(b_1) \land U_{G_1}(a_1) = U_{G_1}(x).
\]

Choosing \( d = (b_1 + b_2) \) gives the desired result. It remains only to establish the existence of the elements \( a_1, b_1 \) and the automorphism \( \phi \) as above.

We prove this by induction on the maximum \( m \) of the set

\[
S_G(a, b) = \{ i < \omega : \text{ht}_{G_i}(p^i a) = \text{ht}_{G_2}(p^i b) \neq \infty \}.
\]

If \( S = \emptyset \), simply take \( \phi = 1_G \). If \( m = 0 \), we proceed as follows. Clearly \( \text{ht}(pa) > \text{ht}(pb) \) or \( \text{ht}(pa) < \text{ht}(pb) \) by definition of \( S_G(a, b) \), say the former. Then \( \text{ht}_{G_1}(pa) > \text{ht}_{G_1}(a) + 1 \); hence \( pa = pa_1 \) for some \( a_1 \in G_1 \) with \( \text{ht}(a_1) > \text{ht}(a) \). Put \( b_1 = b \). Clearly, \( \text{ht}(p^a_1) = \text{ht}(p^b_1) \) only if \( p^a_1 = 0 \). Since \( U_{G_1}(a_1 - a) = (\text{ht}(a), \infty, \ldots) \geq U_{G_2}(b_1) \) and \( \{ G_1, G_2 \} \) is a fully transitive pair, we have \( a_1 - a = b_1 \alpha \) for some \( \alpha \in \text{Hom}(G_2, G_1) \). The automorphism \( \begin{pmatrix} 1 & 0 \\ \alpha & 1 \end{pmatrix} \) of \( G_1 \oplus G_2 \) maps \((a, b)\) to \((a_1, b_1)\) as desired. If \( \text{ht}(pa) < \text{ht}(pb) \), we proceed as above to obtain a suitable automorphism of the form \( \begin{pmatrix} 1 & 0 \\ \alpha & 1 \end{pmatrix} \), finishing the case \( m = 0 \).
Now assume that $S_G(a,b)$ is nonempty and has maximum $m > 0$. Note that $S_G(pa,pb)$ has maximum $< m$. By induction, there exists $ψ ∈ \text{Aut}(G)$ such that $(pa,pb)ψ = (a_2,b_2)$ and $ht(p'a_2) ≠ ht(p'b_2)$ whenever $p'a_2 ≠ 0$. Set $x' = (a',b') = xψ$. Because $px' = (a_2,b_2)$, it follows that $S_G(a',b')$ is empty or has maximum 0. By the above paragraph, there exists $φ ∈ \text{Aut}(G)$ such that $(a',b')φ = xψφ = (a_1,b_1)$ and $ht(p'a_1) ≠ ht(p'b_1)$ whenever $p'a_1 ≠ 0$. This establishes our claim.

Before turning to the main theorem, we observe a consequence of Lemmas 1 and 2 which indicates that a fully transitive $p$-group $G$ with transitive direct summand $H$ is itself transitive provided $p^2G$ and $p^2H$ have the same Ulm supports.

**Proposition 2.** Assume $G = G_1 ∗ G_2$ is a fully transitive $p$-group and $\text{supp}(p^2G_1) ⊆ \text{supp}(p^2G_2)$. If $G_2$ is transitive, then $G$ is transitive.

**Proof.** By [Co, Lemma 2.1], we need only verify that $\text{Aut}(G)$ acts transitively on $p^2G$. Suppose $x,y ∈ p^2G$ have the same Ulm sequence in $G$. By Lemma 2 there exist $φ_1,φ_2 ∈ \text{Aut}(G)$ such that if $xφ_1 = (x_1,x_2)$ and $yφ_2 = (y_1,y_2)$, then $U_{G_1}(x_2) = U_{G_2}(x) = U_{G_2}(y) = U_{G_2}(y_2)$. Note that $U_{G_2}(x_2) ≤ U_{G_1}(y_1-x_1)$, since $U_{G_2}(x_1-y_1,x_2-y_2) ≥ U_G(xφ_1)$. Since $G_2$ is transitive and $G$ is fully transitive, there are $β ∈ \text{Aut}(G_2)$ and $α ∈ \text{Hom}(G_2,G_1)$ with $x_2β = y_2$ and $x_2α = y_1-x_1$. Put $ψ = \begin{pmatrix} 1 & 0 \\ α & β \end{pmatrix}$, an automorphism of $G$. An easy check verifies that $xφ_1ψφ_2^{-1} = y$, as required.

The following result is complementary to Corollary 1.

**Corollary 2.** Assume the $p$-group $G$ is transitive and fully transitive. If $\{H_i\}$ is a collection of direct summands of any power of $G$, then the external direct sum $G ∗ (∐ H_i)$ is transitive and fully transitive.

**Proof.** Proposition 1 implies $G ∗ (∐ H_i)$ is fully transitive. Since $\text{supp}(p^2G)$ contains $\text{supp}(p^2H_i)$, the direct sum is also transitive by Proposition 2.

We now give a result indicating when transitivity and full transitivity are equivalent.

**Theorem 1.** Assume $G$ is a $p$-group which has a decomposition $G = G_1 ∗ G_2$ such that $p^2G_1$ and $p^2G_2$ have the same Ulm supports. Then $G$ is fully transitive if and only if $G$ is transitive.

**Proof.** Suppose $G$ is fully transitive and that $x,y ∈ p^2G$ satisfy $U_G(x) = U_G(y)$. By Lemma 2, there are $φ_1,φ_2 ∈ \text{Aut}(G)$ such that $xφ_1 = (x_1,x_2)$ and $yφ_2 = (y_1,y_2) ∈ G_1 ∗ G_2$ satisfy $U_G(x) = U_{G_1}(x_1)$ and $U_G(y) = U_{G_2}(y_2)$. Because $U_G(x) = U_G(y)$ we have $U_{G_1}(x_1) ≤ U_{G_2}(x_2),U_{G_2}(y_2)$; hence $U_{G_1}(x_1) ≤ U_{G_2}(y_2-x_2)$. Similarly, $U_{G_2}(y_2) ≤ U_{G_1}(y_1-x_1)$. The conditions of Lemma 1 are fulfilled, hence there exists $ψ ∈ \text{Aut}(G)$ with $(x_1,x_2)ψ = (y_1,y_2)$. Now $xφ_1ψφ_2^{-1} = y$, and we see that $\text{Aut}(G)$ acts transitively on $p^2G$. By [Co, Lemma 2.1], $G$ is transitive.

Conversely, assume $G$ is transitive. Let $B$ denote the square of the standard basic $p$-group. Then $H = G ∗ B$ is transitive since $B$ is separable ([CaGo, Proposition 2.6]). The structure of the groups $B$ and $p^2H = p^2G_1 ∗ p^2G_2$ implies that $H$ has no Ulm invariants equal to one. Therefore $H$ is fully transitive by [Ka, Theorem 26(b)], whence $G$ is fully transitive.
Theorem 1 has many corollaries. Corollary 3 is merely a noteworthy special case of Corollary 4.

**Corollary 3.** A $p$-group $G$ is fully transitive if and only if $G \oplus G$ is transitive.

**Corollary 4.** The following conditions are equivalent for a $p$-group $G$.

(i) For all cardinals $\lambda$, $\bigoplus_{\lambda} G$ is fully transitive.

(ii) For some $\lambda > 0$, $\bigoplus_{\lambda} G$ is fully transitive.

(iii) For all $\lambda > 1$, $\bigoplus_{\lambda} G$ is transitive.

(iv) For some $\lambda > 1$, $\bigoplus_{\lambda} G$ is transitive.

Proof. The implications (i) $\Rightarrow$ (ii) and (iii) $\Rightarrow$ (iv) are trivial. Assume (ii) holds, and $\lambda > 1$ is a fixed cardinal. Note that $G$ is a summand of a fully transitive group, hence is fully transitive. By Corollary 1, $\bigoplus_{\lambda} G$ is fully transitive. Because $\lambda > 1$ we can obviously decompose $\bigoplus_{\lambda} G = G_1 \oplus G_2$ in such a way that

$$\text{supp}(p^\omega G_1) = \text{supp}(p^\omega G_2) = \text{supp}(p^\omega G).$$

Hence $\bigoplus_{\lambda} G$ is transitive by Theorem 1. Therefore (iii) holds.

Finally, assume (iv) holds. Writing $\bigoplus_{\lambda} G = G_1 \oplus G_2$ as above, it follows from Theorem 1 that $\bigoplus_{\lambda} G$ is fully transitive since it is transitive. Therefore $G$ is fully transitive, and Corollary 1 yields condition (i).

Since transitive $p$-groups are fully transitive if $p \neq 2$, and squares of fully transitive $p$-groups are necessarily transitive, we obtain

**Corollary 5.** For $p \neq 2$, the class of fully transitive $p$-groups is precisely the class of direct summands of transitive $p$-groups.

Corner [Co] has given an example of a 2-group $G$ which is transitive but not fully transitive. It follows from Corollary 4 that for all $\lambda > 1$, the power $\bigoplus_{\lambda} G$ is neither transitive nor fully transitive. In particular, the square of a transitive 2-group need not be transitive. For $p \neq 2$, Corollary 2 implies that all powers of a transitive $p$-group are both transitive and fully transitive, simply because the group itself is also fully transitive in this case.

The final corollary extends Theorem 1 and Proposition 1 by exploiting Hill’s powerful criteria for transitivity and full transitivity.

**Corollary 6.** Let $\{G_i\}_{i \in I}$ be a collection of $p$-groups. Assume there exists an ordinal $\sigma$ such that $G_i/p^\sigma G_i$ is totally projective and $\{p^\sigma G_i, p^\sigma G_j\}$ is a fully transitive pair for each $i, j \in I$. Then $\bigoplus_{i \in I} G_i$ is fully transitive. If there exists a partition $I = J \cup K$ such that the groups $\bigoplus_{i \in J} p^{\sigma+\omega} G_i$ and $\bigoplus_{i \in K} p^{\sigma+\omega} G_i$ have equal Ulm supports, then $\bigoplus_{i \in I} G_i$ is also transitive.

Proof. Let $G = \bigoplus_{i \in I} G_i$. Proposition 1 shows that $p^\sigma G = \bigoplus_{i \in I} p^\sigma G_i$ is fully transitive. Since $G_i/p^\sigma G_i \cong \bigoplus_{i \in I} G_i/p^\sigma G_i$ is totally projective, $G$ is fully transitive by [Hi, Theorem 4]. If the second condition in the corollary is also met, then $p^\sigma G$ is transitive by Theorem 1, and it follows again from [Hi] that $G$ is transitive.

Observe that the total projectivity of the quotients $G_i/p^\sigma G_i$ in Corollary 6 is automatic if $\sigma$ is a finite ordinal. Megibben’s result [Me, Theorem 2.4] and Hill’s result [Hi, Theorem 6] demonstrate that Corollary 6 can fail if one of these quotients is not totally projective.
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