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HANKEL OPERATORS ON THE BERGMAN SPACE OF THE UNIT BALL

MARIA NOWAK

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Abstract. We characterize the bounded holomorphic functions f, g in the unit ball of $\mathbb{C}^n$ for which the operator $H_{\bar{g}}^* H_{\bar{f}}$ is compact. For $n = 1$ the result was obtained by Axler and Gorkin in 1988 and by Zheng in 1989.

1. Introduction

Let B_n be the open unit ball in $\mathbb{C}^n$ with dv the normalized volume measure on B_n. When $n = 1$ we will denote by $\mathbb{D}$ the unit disc in $\mathbb{C}$. The Bergman space $L_a^2(B_n)$ is the closed subspace of $L^2(B_n, dv)$ consisting of holomorphic functions. Let P be the orthogonal projection of $L^2(B_n, dv)$ onto $L_a^2(B_n)$. For $f \in L^2(B_n, dv)$ the Toeplitz operator $T_f : L_a^2(B_n) \to L_a^2(B_n)$ and the Hankel operator $H_f : L_a^2(B_n) \to (L_a^2(B))^\perp$ are defined by

$$T_f(g) = P(fg) \quad \text{and} \quad H_f(g) = fg - P(fg),$$

respectively. In fact, these operators are densely defined for bounded holomorphic functions and extended to all functions in $L_a^2(B_n)$.

In 1986 S.Axler ([4], [5]) proved that for $f \in L_a^2(\mathbb{D})$ the Hankel operator $H_{\bar{f}} : L_a^2(\mathbb{D}) \to (L_a^2(\mathbb{D}))^\perp$ is bounded if and only if f is a Bloch function. Moreover, $H_{\bar{f}}$ is compact if and only if f in the little Bloch space. Many generalizations of these results have been found since then (see e.g. [3], [7], [8]).

In view of the relation

$$T_f^* T_g - T_g T_f^* = H_{\bar{g}}^* H_{\bar{f}} \quad \text{for } f, g \in L_a^2(B) \tag{1}$$

the problem of characterization of the functions f, g for which the operator $H_{\bar{g}}^* H_{\bar{f}}$ is compact seems natural. In the case of the unit disc the problem was proposed by S. Axler [4] and solved by S. Axler and P. Gorkin [6] and independently by D. Zheng [14]. They have proved the following

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Theorem A. *Let* f, g *be bounded holomorphic functions on* $\mathbb{D}$. *Then the following statements are equivalent*:

(a) $H^*_{\bar{g}} H_{\bar{f}}$ *is compact* ,

(b) $\lim_{|z|\to 1^-} (1-|z|^2) \min\{|f'(z)|, |g'(z)|\} = 0$,

(c) $\lim_{|w|\to 1^-} \int_{\mathbb{D}} |f(z)-f(w)||g(z)-g(w)| dv(z) = 0$,

(d) *either* f *or* g *is constant on each Gleason part* (*except* $\mathbb{D}$) *of the maximal ideal space of* $H^\infty(\mathbb{D})$.

It was also noticed that each of the conditions (b), (c) implies the compactness of the operator $H^*_{\bar{g}} H_{\bar{f}}$ under the weaker and more natural assumption that f, g are Bloch functions. Moreover, these sufficient conditions for compactness of $H^*_{\bar{g}} H_{\bar{f}}$ are still valid if $\mathbb{D}$ is replaced by the unit ball B_n , $n > 1$.

Here we show that in the case of $\mathbb{D}$ (b) ((c)) is also a necessary condition for compactness of $H^*_{\bar{g}} H_{\bar{f}}$ under the assumption that f, g are Bloch functions. The main result of this paper is a characterization of the functions f, g bounded and holomorphic on B_n for which the operator $H^*_{\bar{g}} H_{\bar{f}}$ is compact.

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2. Preliminaries

Let $\langle z, w\rangle$ denote the inner product in $\mathbb{C}^n$ given by

$$\langle z, w\rangle = \sum_{j=1}^n z_j \bar{w}_j \ , \qquad z = (z_1, ..., z_n) \ , \ w = (w_1, ..., w_n) \ .$$

Let $\mathrm{Aut}(B_n)$ be the group of all biholomorphic maps of B_n into B_n . It is known that $\mathrm{Aut}(B_n)$ is generated by the unitary operators on $\mathbb{C}^n$ and the involutions φ_a of the form

$$\varphi_a(z) = \frac{a - P_a z - (1-|a|^2)^{1/2} Q_a z}{1 - \langle z, a\rangle} \ , \tag{2}$$

where $a \in B_n$, P_a is the orthogonal projection into the space spanned by a, i.e.

$$P_a z = \frac{\langle z, a\rangle a}{|a|^2} \ , \quad |a|^2 = \langle a, a\rangle \ ,$$
$$P_0 z = 0$$

and $Q_a = I - P_a$.

For fixed $a \in B_n$ and r, $0 < r < 1$, define

$$E_n(a, r) = \varphi_a(rB_n) \ .$$

Since φ_a is an involution, $z \in E_n(a, r)$ if and only if $|\varphi_a(z)| < r$.

As in [11] we say that a function $f \in C^2(B_n)$ is $\mathcal{M}$- harmonic in B_n if $\widetilde{\Delta} f(z) = 0$ for every $z \in B_n$. The operator $\widetilde{\Delta}$ is defined by

$$(\widetilde{\Delta} f)(z) = \Delta(f \circ \varphi_z)(0) \ ,$$

where Δ is the ordinary Laplacian. We will need to use the following recent results on $\mathcal{M}$-harmonic functions.

Theorem B ([2]). *Suppose that f and g are nonconstant holomorphic functions in B_n , and that $f\bar{g}$ is $\mathcal{M}$-harmonic.*

(a) *When $n = 2$, this cannot happen.*

(b) *When $n \geq 3$,then there exist*

(i) *an integer m , $2 \leq m \leq n-1$,*

(ii) *entire functions $\varphi : \mathbb{C}^{m-1} \to \mathbb{C}$ and $\psi : \mathbb{C}^{n-m} \to \mathbb{C}$, such that*

$$f(Uz) = \varphi\left(\frac{z_2}{1-z_1}, ..., \frac{z_m}{1-z_1}\right) \ , \ g(Uz) = \psi\left(\frac{z_{m+1}}{1-z_1}, ..., \frac{z_n}{1-z_1}\right) \ .$$

Moreover, $f(B_n) = \varphi(\mathbb{C}^{m-1})$, $g(B_n) = \psi(\mathbb{C}^{n-m})$, and $(f\bar{g})(B_n) = \mathbb{C}$ or $\mathbb{C} \setminus \{0\}$. (The latter case occurs only when both φ and ψ omit the value 0.)

Theorem C ([1]). *Assume that for $f \in L^1(B_n, dv)$ the linear operator T_0 is defined by*

$$(T_0 f)(z) = \int_{B_n} (f \circ \varphi_z) dv \ , \quad z \in B_n \ .$$

(a) *If $f \in L^\infty(B_n)$ and $T_0 f = f$ then $\widetilde{\Delta} f = 0$.*

(b) *If $f \in L^1(B_n, dv)$,$n < 12$ and $T_0 f = f$, then $\widetilde{\Delta} f = 0$; this fails for all $n \geq 12$.*

We say that a holomorphic function f is a Bloch function if

$$\|f\|_{\mathcal{B}} = \sup\{|\nabla_z f|(1-|z|^2);\ z \in B_n\ \} < \infty \ . \tag{3}$$

Bloch functions on bounded homogeneous domains in $\mathbb{C}^n$ were defined by R. Timoney [12] and studied by many authors.(e.g. [9], [11], [12]). One of the most important properties of the space of Bloch functions on B_n, $n \geq 2$, is that (3) is equivalent to the following conditions (for some $K \geq 0$):

$$|\langle \nabla_z f, \bar{z}\rangle|(1-|z|^2) \leq K,$$

$$|\langle \nabla_z f, \bar{x}\rangle|(1-|z|^2) \leq K \text{ for all } x \in \mathbb{C}^n \text{ satisfying } |x| = 1 \text{ and } \langle x, \bar{z}\rangle = 0$$

for all $z \in B_n$. This means that f is a Bloch function on B_n if and only if the radial derivative of f is $O(1/(1-|z|^2))$ and the directional derivatives of f in directions pependicular to the radial direction are $O(1/(1-|z|^2)^{1/2})$.

3. Main results

We start with the following

Lemma 1. *Let f, g be bounded holomorphic functions on B_n . If the operator $H_{\bar{g}}^* H_{\bar{f}}$ is compact then*

$$\lim_{|a| \to 1^-} (1-|a|^2) \min\{|\nabla_a f|, |\nabla_a g|\} = 0 \ . \tag{5}$$

Proof. For $a \in B_n$ define

$$k_a(z) = \frac{(1-|a|^2)^{(n+1)/2}}{(1-\langle z, a\rangle)^{n+1}} \ , \quad z \in B_n \ .$$

Because for $f \in L_a^2(B_n)$

$$\langle f, k_a\rangle = (1-|a|^2)^{\frac{n+1}{2}} \int_{B_n} \frac{f(z)}{(1-\langle a, z\rangle)^{n+1}} dv(z) = (1-|a|^2)^{\frac{n+1}{2}} f(a)$$

and holomorphic bounded functions are dense in $L^2_a(B_n)$, k_a tends weakly to zero in L^2_a for $|a| \to 1^-$. Hence the compactness of $H^*_{\bar g} H_{\bar f}$ implies

$$0 = \lim_{|a|\to 1^-} \langle H^*_{\bar g} H_{\bar f} k_a, k_a\rangle = \lim_{|a|\to 1^-} \langle H_{\bar f} k_a, H_{\bar g} k_a\rangle$$
$$= \lim_{|a|\to 1^-} \int_{B_n} (\bar f \circ \varphi_a - \bar f(a))(g \circ \varphi_a - g(a))dv \ .$$

Suppose that (5) does not hold. Then there exists a sequence $\{a_m\}$ in B such that $\lim_{m\to\infty} |a_m| = 1$ and

$$\lim_{m\to\infty} (1-|a_m|^2)|\nabla_{a_m} f| = a > 0 \quad \text{and} \quad \lim_{m\to\infty} (1-|a_m|^2)|\nabla_{a_m} g| = b > 0 \ . \tag{6}$$

Since the families of functions

$$\{f \circ \varphi_a - f(a);\ a \in B_n\ \} \quad \text{and} \quad \{g \circ \varphi_a - g(a);\ a \in B_n\ \}$$

are compact, there exists a subsequence of $\{a_m\}$ (which will be denoted also by $\{a_m\}$) and holomorphic functions F, G such that

$$f \circ \varphi_{a_m} - f(a_m) \xrightarrow[m\to\infty]{} F \quad \text{and} \quad g \circ \varphi_{a_m} - g(a_m) \xrightarrow[m\to\infty]{} G$$

uniformly on compact subsets of B_n .

Now let $w \in B_n$ be fixed. Then also

$$f \circ \varphi_{a_n} \circ \varphi_w - f(a_m) \xrightarrow[m\to\infty]{} F \circ \varphi_w \quad \text{and} \quad g \circ \varphi_{a_n} \circ \varphi_w - g(a_m) \xrightarrow[m\to\infty]{} G \circ \varphi_w$$

uniformly on compacta.

Let $a'_m = \varphi_{a_m}(w)$, $m \in \mathbb{N}$. There exists a unique unitary transformation U_m for which $\varphi_{a_m} \circ \varphi_w = \varphi_{a'_m} \circ U_m$ [12, p.29].

It follows from the relation [12, p.26]

$$1 - |\varphi_a(w)|^2 = \frac{(1-|a|^2)(1-|w|^2)}{|1-\langle w, a\rangle|^2} \tag{7}$$

that $|a'_m| \to 1^-$ if $|a_m| \to 1^-$. Thus by the comapctness of $H^*_{\bar g} H_{\bar f}$

$$0 = \lim_{m\to\infty} \int_{B_n} (\bar f \circ \varphi_{a'_m} - \bar f(a'_m))(g \circ \varphi_{a'_m} - g(a'_m))dv$$
$$= \lim_{m\to\infty} \int_{B_n} (\bar f \circ \varphi_{a'_m} \circ U - \bar f(a'_m))(g \circ \varphi_{a'_m} \circ U - g(a'_m))dv$$
$$= \lim_{m\to\infty} \int_{B_n} (\bar f \circ \varphi_{a_m} \circ \varphi_w - \bar f(\varphi_{a_m} \circ \varphi_w)(0))(g \circ \varphi_{a_m} \circ \varphi_w - g(\varphi_{a_m} \circ \varphi_w)(0))dv$$

In view of boundedness of f, g and uniform convergence on compacta we get

$$\begin{aligned} &\lim_{m\to\infty} \int_{B_n} [(\bar f \circ \varphi_{a_m} \circ \varphi_w(z) - \bar f(a_m)) - (\bar f(\varphi_{a_m} \circ \varphi_w)(0) - \bar f(a_m))] \\ &\qquad \times [(g \circ \varphi_{a_m} \circ \varphi_w(z) - g(a_m)) - (g(\varphi_{a_m} \circ \varphi_w)(0) - g(a_m))]dv(z) \\ &= \int_{B_n} (\bar F \circ \varphi_w(z) - \bar F \circ \varphi_w(0))(G \circ \varphi_w(z) - G \circ \varphi_w(0))dv(z) = 0. \end{aligned} \tag{8}$$

The last equality can be rewritten in the form

$$\int_{B_n} \bar F \circ \varphi_w(z) G \circ \varphi_w(z) dv(z) = \bar F(w) G(w) \ .$$

Now Theorem C implies that $\bar{F}G$ is $\mathcal{M}$-harmonic. Because F, G are bounded the Ahern-Rudin theorem (Theorem B) and Liouville's theorem imply that either F or G is constant on B_n . Suppose that F is constant. Then by the chain rule and the symmetry of the matrix $\varphi_a'(0) = (1-|a|^2)P_a - (1-|a|^2)^{1/2}Q_a$ we get

$$\begin{aligned}
0 = |\nabla_0 F|^2 &= \lim_{m\to\infty} |\nabla_0(f\circ\varphi_{a_m})|^2 = \lim_{m\to\infty} \langle \nabla_{a_m} f\varphi_{a_m}'(0), \nabla_{a_m} f\varphi_{a_m}'(0)\rangle \\
&= \lim_{m\to\infty} \langle \varphi_{\bar{a}_m}'(0)\nabla_{a_m} f, \varphi_{\bar{a}_m}'(0)\nabla_{a_m} f\rangle \\
&= \lim_{m\to\infty} \left((1-|a_m|^2)^2|P_{\bar{a}_m}\nabla_{a_m} f|^2 + (1-|a_m|^2)|Q_{\bar{a}_m}\nabla_{a_m} f|^2\right) \\
&\geq \lim_{m\to\infty} (1-|a_m|^2)^2|\nabla_{a_m} f|^2.
\end{aligned}$$

Because the sequence $(1-|a_m|^2)|\nabla_{a_m} g|$ is bounded for bounded and holomorphic g , the last inequality implies that

$$\lim_{m\to\infty} (1-|a_m|^2)^2|\nabla_{a_m} g||\nabla_{a_m} f| = 0$$

which contradicts (6). □

Note that for the unit disc the same proof still goes under the assumption that f, g are Bloch functions. In this case the equality (8) follows from the Lebesgue's dominated theorem. In fact, if f is a Bloch function on $\mathbb{D}$ then [4, p.320]

$$\int_{\mathbb{D}} |f\circ\varphi_{a_m}(z) - f(a_m)|dv(z) \leq \|f\|_{\mathcal{B}} \int_{\mathbb{D}} |\ln(1-|z|)|dv(z) \leq c\|f\|_{\mathcal{B}} \ .$$

Thus we have

Lemma 2. *If f, g are Bloch functions on $\mathbb{D}$ such that the operator $H_{\bar{g}}^* H_{\bar{f}}$ is compact then*

$$\lim_{|a|\to 1^-} (1-|a|^2)\min\{|f'(a)|, |g'(z)|\} = 0 \ .$$

Analyzing the proof of Lemma 1 one can easily get

Lemma 3. *If f, g are bounded holomorphic functions on B_n and the operator $H_{\bar{g}}^* H_{\bar{f}}$ is compact then*

$$\lim_{|a|\to 1^-} (1-|a|^2)\min\{|P_{\bar{a}}\nabla_a f|, |P_{\bar{a}}\nabla_a g|\} = 0 \tag{9}$$

and

$$\lim_{|a|\to 1^-} (1-|a|^2)^{1/2}\min\{|Q_{\bar{a}}\nabla_a f|, |Q_{\bar{a}}\nabla_a g|\} = 0 \ . \tag{9'}$$

Our main result is the following

Theorem. *Let* f, g *be bounded holomorphic functions on* B_n *and* $0 < r < 1$. *Then the following statements are equivalent*

(a) $$H^*_{\bar g}\, H_{\bar f} \quad \textit{is compact ;}$$

(b) $$\lim_{|a|\to 1}(1-|a|^2)\min\{|P_{\bar a}\nabla_a f|, |P_{\bar a}\nabla_a g|\}$$
$$= \lim_{|a|\to 1}(1-|a|^2)^{1/2}\min\{|Q_{\bar a}\nabla_a f|, |Q_{\bar a}\nabla_a g|\} = 0 \ ;$$

(c) $$\lim_{|a|\to 1}\int_{rB_n}|f\circ\varphi_a - f(a)||g\circ\varphi_a - g(a)|dv = 0 \ ;$$

(d) $$\lim_{|a|\to 1}\frac{1}{|E_n(a,r)|}\int_{E_n(a,r)}|f-f(a)||g-g(a)|dv = 0 \ ;$$

(e) $$\lim_{|a|\to 1}\int_{B_n}|f\circ\varphi_a - f(a)||g\circ\varphi_a - g(a)|dv = 0 \ .$$

Proof. (a)$\Longrightarrow$(b). By Lemma 3.

(b)$\Longrightarrow$(c). Suppose f and g are bounded holomorphic functions on B_n for which statement (b) holds. Let $\{a_m\}$ be a sequence of points in B_n such that $\lim_{m\to\infty}|a_m| = 1$ and

$$F = \lim_{m\to\infty}(f\circ\varphi_{a_m} - f(a_m)) \quad \text{and} \quad G = \lim_{m\to\infty}(g\circ\varphi_{a_m} - g(a_m)) \ ,$$

where the convergence is uniform on compact subsets of B_n and F, G are holomorphic functions on B_n . We will show that

$$\int_{rB_n}|\nabla_z F||\nabla_z G|dv(z) = 0 \ . \tag{10}$$

Applying the definitions of F, G and changing the variables $w = \varphi_{a_m}(z)$ we obtain

$$\int_{rB_n}|\nabla_z F||\nabla_z G|dv(z) = \lim_{m\to\infty}\int_{rB_n}|\nabla_{\varphi_{a_m}(z)} f\varphi'_{a_m}(z)||\nabla_{\varphi_{a_m}(z)} g\varphi'_{a_m}(z)|dv(z)$$
$$= \lim_{m\to\infty}\int_{E_n(a_m,r)}|\nabla_w f\varphi'_{a_m}(\varphi_{a_m}(w))||\nabla_w g\varphi'_{a_m}(\varphi_{a_m}(w))|\frac{(1-|a_m|^2)^{n+1}}{|1-\langle w, a_m\rangle|^{2n+2}}dv(w) \ .$$

Let $\varphi_{a_m}(w) = \zeta_m$. To calculate $\varphi'_{a_m}(\zeta_m)$ notice that for fixed w and a_m the biholmorphic maping $\varphi_w\circ\varphi_{a_m}\circ\varphi_{\zeta_m}$ is a unitary operator , say U_m [12, p.29]. Hence

$$\varphi'_{a_m}(\zeta_m) = \varphi'_w(0)U_m\varphi'_{\zeta_m}(\zeta_m) \ .$$

Let $\|A\|$ denote the standard norm of the linear operator $A:\ \mathbb{C}^n\to\mathbb{C}^n$. Because for every $z\in B_n$

$$\varphi'_z(z) = -(1-|z|^2)^{-1}P_z - (1-|z|^2)^{-1/2}Q_z \ ,$$

we have the following estimate:

$$\|\varphi'_{\zeta_m}(\zeta_m)\| \le \frac{2}{1-|\zeta_m|^2} \ .$$

Moreover, in view of (7) we get

$$\|\varphi'_{\zeta_m}(\zeta_m)\| \le \frac{2|1-\langle w, a_m\rangle|^2}{(1-|a_m|^2)(1-|w|^2)} \ .$$

Hence

$$\begin{aligned}
&\int_{rB_n} |\nabla_z F||\nabla_z G| dv(z) \\
&\le 4 \lim_{m\to\infty} \int_{E_n(a_m,r)} (1-|w|^2)^{-2} |\varphi'_{\bar w}(0)\nabla_w f||\varphi'_{\bar w}(0)\nabla_w g| \frac{(1-|a_m|^2)^{n-1}}{|1-\langle w,a_m\rangle|^{2n-2}} dv(w) \\
&\le 4 \lim_{m\to\infty} \int_{E_n(a_m,r)} (1-|w|^2)^{-2} \left|(1-|w|^2)P_{\bar w}\nabla_w f + (1-|w|^2)^{1/2} Q_{\bar w}\nabla_w f\right| \\
&\qquad \times \left|(1-|w|^2)P_{\bar w}\nabla_w g + (1-|w|^2)^{1/2} Q_{\bar w}\nabla_w g\right| \frac{(1-|a_m|^2)^{n-1}}{|1-\langle w,a_m\rangle|^{2n-2}} dv(w) \\
&\le 2^{n+1} \lim_{m\to\infty} \int_{E_n(a_m,r)} |P_{\bar w}\nabla_w f||P_{\bar w}\nabla_w g|(1-|w|^2)^{-n+1} dv(w) \\
&+ 2^{n+1} \lim_{m\to\infty} \int_{E_n(a_m,r)} |P_{\bar w}\nabla_w f||Q_{\bar w}\nabla_w g|(1-|w|^2)^{-n+1/2} dv(w) \\
&+ 2^{n+1} \lim_{m\to\infty} \int_{E_n(a_m,r)} |Q_{\bar w}\nabla_w f||P_{\bar w}\nabla_w g|(1-|w|^2)^{-n+1/2} dv(w) \\
&+ 2^{n+1} \lim_{m\to\infty} \int_{E_n(a_m,r)} |Q_{\bar w}\nabla_w f||Q_{\bar w}\nabla_w g|(1-|w|^2)^{-n} dv(w) \ .
\end{aligned}$$

To finish the proof of (10) it is enough to show that each of the four limits on the right-hand side of the last inequality is zero. Indeed, we have

$$\begin{aligned}
&\int_{E_n(a_m,r)} |P_{\bar w}\nabla_w f||P_{\bar w}\nabla_w g|(1-|w|^2)^{-n+1} dv(w) \\
&\le \sup_{w\in E_n(a_m,r)} (1-|w|^2)^2 |P_{\bar w}\nabla_w f||P_{\bar w}\nabla_w f| \int_{E_n(a_m,r)} (1-|w|^2)^{-n-1} dv(w) \\
&= \frac{r^{2n}}{(1-r^2)^{n+1}} \sup_{w\in E_n(a_m,r)} (1-|w|^2)^2 |P_{\bar w}\nabla_w f||P_{\bar w}\nabla_w f| \ .
\end{aligned}$$

Let $\{\zeta_m\}$ be the sequence in B_n such that: $\zeta_m \in E(a_m,r)$ and

$$\sup_{w\in E(a_m,r)} (1-|w|^2)^2 |P_{\bar w}\nabla_w f||P_{\bar w}\nabla_w f| = (1-|\zeta_m^2)^2 |P_{\bar\zeta_m}\nabla_{\zeta_m} f||P_{\bar\zeta_m}\nabla_{\zeta_m} f| \ .$$

Then $\lim_{m\to\infty}|\zeta_m| = 1$ and (b) implies that the last expression tends to 0 . The same reasoning applies to the remaining cases.

Now (10) implies that at least one of the functions F, G must be identically 0 on rB_n . Hence

$$\lim_{m\to\infty} \int_{rB_n} |f\circ\varphi_{a_m} - f(a_m)||g\circ\varphi_{a_m} - g(a_m)| dv = \int_{rB_n} |FG| dv = 0 \ .$$

(c)$\Longleftrightarrow$(d) A change-of-variables yields

$$\begin{aligned}
&\int_{E_n(a,r)} |f-f(a)||g-g(a)| dv \\
&\qquad = \int_{rB_n} |f\circ\varphi_a(z) - f(a)||g\circ\varphi_a(z) - g(a)| \frac{(1-|a|^2)^{n+1}}{|1-\langle z,a\rangle|^{2(n+1)}} dv.
\end{aligned}$$

Because

$$|E_n(a,r)| = \left(\frac{1-|a|^2}{1-r^2|a|^2}\right)^{n+1}$$

the desired equivalence follows from the inequalities

$$\frac{1}{r^{2n}(1+r^2)^{2(n+1)}} \le \frac{1}{|E_n(a,r)|}\frac{(1-|a|^2)^{(n+1)}}{|1-\langle z,a\rangle|^{2(n+1)}} \le \frac{1}{r^{2n}(1-r^2)^{n+1}}\ , \quad z \in rB_n\ .$$

(c)$\Longrightarrow$(e) Let $\{a_m\}$ be a sequence such that $\lim_{m\to\infty}|a_m| = 1$ and F, G be as in the proof of the implication (b)$\Longrightarrow$(c). Then (c) implies that

$$\int_{rB_n} |F||G|dv = 0\ .$$

This in turn implies that either F or G is identically zero on rB_n , and hence on B_n .

(e)$\Longrightarrow$(a) It is enough to proceed analogously to the proof of Theorem 2 of [14]. □

Remark. Notice that the functions f, g defined in assertion (ii) of Theorem B cannot be Bloch functions. Hence in view of Theorem C, when $n < 12$ our theorem holds for Bloch functions instead of bounded holomorphic functions.

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Instytut Matematyki UMCS , pl. Marii Curie-Skłodowskiej 1,20-031 Lublin, Poland

E-mail address: nowakm@golem.umcs.lublin.pl