HANKEL OPERATORS
ON THE BERGMAN SPACE OF THE UNIT BALL

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ABSTRACT. We characterize the bounded holomorphic functions \( f, g \) in the unit ball of \( \mathbb{C}^n \) for which the operator \( H_\bar{g}^* H_\bar{f} \) is compact. For \( n = 1 \) the result was obtained by Axler and Gorkin in 1988 and by Zheng in 1989.

1. Introduction

Let \( B_n \) be the open unit ball in \( \mathbb{C}^n \) with \( dv \) the normalized volume measure on \( B_n \). When \( n = 1 \) we will denote by \( \mathbb{D} \) the unit disc in \( \mathbb{C} \). The Bergman space \( L^2_a(B_n) \) is the closed subspace of \( L^2(B_n, dv) \) consisting of holomorphic functions. Let \( P \) be the orthogonal projection of \( L^2(B_n, dv) \) onto \( L^2_a(B_n) \). For \( f \in L^2(B_n, dv) \) the Toeplitz operator \( T_f : L^2_a(B_n) \to L^2_a(B_n) \) and the Hankel operator \( H_f : L^2_a(B_n) \to (L^2_a(B))^\perp \) are defined by

\[
T_f(g) = P(fg) \quad \text{and} \quad H_f(g) = fg - P(fg),
\]

respectively. In fact, these operators are densely defined for bounded holomorphic functions and extended to all functions in \( L^2_a(B_n) \).

In 1986 S. Axler ([4], [5]) proved that for \( f \in L^2_a(\mathbb{D}) \) the Hankel operator \( H_f : L^2_a(\mathbb{D}) \to (L^2_a(\mathbb{D}))^\perp \) is bounded if and only if \( f \) is a Bloch function. Moreover, \( H_f \) is compact if and only if \( f \) is in the little Bloch space. Many generalizations of these results have been found since then (see e.g. [3], [7], [8]).

In view of the relation

\[
T_g^* T_f - T_g T_f^* = H_\bar{g}^* H_\bar{f} \quad \text{for} \quad f, g \in L^2_a(B)
\]

the problem of characterization of the functions \( f, g \) for which the operator \( H_\bar{g}^* H_\bar{f} \) is compact seems natural. In the case of the unit disc the problem was proposed by S. Axler [4] and solved by S. Axler and P. Gorkin [6] and independently by D. Zheng [14]. They have proved the following
Theorem A. Let $f, g$ be bounded holomorphic functions on $D$. Then the following statements are equivalent:

(a) $H^*_g H_f$ is compact,

(b) \[ \lim_{|z| \to 1^-} (1 - |z|^2) \min\{|f'(z)|, |g'(z)|\} = 0 , \]

(c) \[ \lim_{|w| \to 1^-} \int_D |f(z) - f(w)||g(z) - g(w)|dv(z) = 0 , \]

(d) either $f$ or $g$ is constant on each Gleason part (except $D$) of the maximal ideal space of $\mathcal{H}^\infty(D)$.

It was also noticed that each of the conditions (b), (c) implies the compactness of the operator $H^*_g H_f$ under the weaker and more natural assumption that $f, g$ are Bloch functions. Moreover, these sufficient conditions for compactness of $H^*_g H_f$ are still valid if $D$ is replaced by the unit ball $B_n, n > 1$.

Here we show that in the case of $D$ (b) ((c)) is also a necessary condition for compactness of $H^*_g H_f$ under the assumption that $f, g$ are Bloch functions. The main result of this paper is a characterization of the functions $f, g$ bounded and holomorphic on $B_n$ for which the operator $H^*_g H_f$ is compact.

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2. Preliminaries

Let $\langle z, w \rangle$ denote the inner product in $\mathbb{C}^n$ given by

\[ \langle z, w \rangle = \sum_{j=1}^n z_j \bar{w}_j , \quad z = (z_1, ..., z_n) , w = (w_1, ..., w_n) . \]

Let $\text{Aut}(B_n)$ be the group of all biholomorphic maps of $B_n$ into $B_n$. It is known that $\text{Aut}(B_n)$ is generated by the unitary operators on $\mathbb{C}^n$ and the involutions $\varphi_a$ of the form

\[ \varphi_a(z) = \frac{a - P_a z - (1 - |a|^2)^{1/2}Q_a z}{1 - \langle z, a \rangle} , \]

where $a \in B_n$, $P_a$ is the orthogonal projection into the space spanned by $a$, i.e.

\[ P_a z = \frac{\langle z, a \rangle a}{|a|^2} , \quad |a|^2 = \langle a, a \rangle , \]

\[ P_0 z = 0 \]

and $Q_a = I - P_a$.

For fixed $a \in B_n$ and $r$, $0 < r < 1$, define

\[ E_n(a, r) = \varphi_a(rB_n) . \]

Since $\varphi_a$ is an involution, $z \in E_n(a, r)$ if and only if $|\varphi_a(z)| < r$.

As in [11] we say that a function $f \in C^2(B_n)$ is $\mathcal{M}$-harmonic in $B_n$ if $\tilde{\Delta} f(z) = 0$ for every $z \in B_n$. The operator $\tilde{\Delta}$ is defined by

\[ (\tilde{\Delta}f)(z) = \Delta(f \circ \varphi_z)(0) , \]

where $\Delta$ is the ordinary Laplacian. We will need to use the following recent results on $\mathcal{M}$-harmonic functions.
Theorem B ([2]). Suppose that $f$ and $g$ are nonconstant holomorphic functions in $B_n$, and that $fg$ is $M$-harmonic.

(a) When $n = 2$, this cannot happen.

(b) When $n \geq 3$, then there exist

(i) an integer $m$, $2 \leq m \leq n - 1$,

(ii) entire functions $\varphi : \mathbb{C}^{m-1} \to \mathbb{C}$ and $\psi : \mathbb{C}^{n-m} \to \mathbb{C}$, such that

$$f(Uz) = \varphi \left( \frac{z_2}{1 - z_1}, \ldots, \frac{z_m}{1 - z_1} \right), \quad g(Uz) = \psi \left( \frac{z_{m+1}}{1 - z_1}, \ldots, \frac{z_n}{1 - z_1} \right).$$

Moreover, $f(B_n) = \varphi(\mathbb{C}^{m-1})$, $g(B_n) = \psi(\mathbb{C}^{n-m})$, and $(fg)(B_n) = \mathbb{C}$ or $\mathbb{C} \setminus \{0\}$. (The latter case occurs only when both $\varphi$ and $\psi$ omit the value 0.)

Theorem C ([1]). Assume that for $f \in L^1(B_n, dv)$ the linear operator $T_0$ is defined by

$$(T_0 f)(z) = \int_{B_n} (f \circ \varphi_x) dv, \quad z \in B_n.$$

(a) If $f \in L^\infty(B_n)$ and $T_0 f = f$ then $\tilde{\Delta} f = 0$.

(b) If $f \in L^1(B_n, dv), n < 12$ and $T_0 f = f$, then $\tilde{\Delta} f = 0$; this fails for all $n \geq 12$.

We say that a holomorphic function $f$ is a Bloch function if

$$\|f\|_B = \sup \{|\nabla\bar{f}|(1 - |z|^2); \quad z \in B_n\} < \infty.$$ 

Bloch functions on bounded homogeneous domains in $\mathbb{C}^n$ were defined by R. Timoney [12] and studied by many authors (e.g. [9], [11], [12]). One of the most important properties of the space of Bloch functions on $B_n$, $n \geq 2$, is that (3) is equivalent to the following conditions (for some $K \geq 0$):

$$|\langle \nabla f, \bar{z} \rangle| (1 - |z|^2) \leq K,$$

or

$$|\langle \nabla f, \bar{x} \rangle| (1 - |z|^2) \leq K$$

for all $z \in B_n$. This means that $f$ is a Bloch function on $B_n$ if and only if the radial derivative of $f$ is $O(1/(1 - |z|^2))$ and the directional derivatives of $f$ in directions perpendicular to the radial direction are $O(1/(1 - |z|^2)^{1/2})$.

3. Main results

We start with the following

Lemma 1. Let $f, g$ be bounded holomorphic functions on $B_n$. If the operator $H^*_g H_f$ is compact then

$$(5) \quad \lim_{|a| \to 1^-} (1 - |a|^2) \min \{|\nabla a f|, |\nabla a g|\} = 0.$$

Proof. For $a \in B_n$ define

$$k_a(z) = \frac{(1 - |a|^2)^{n+1/2}}{(1 - \langle a, z \rangle)^{n+1}}, \quad z \in B_n.$$

Because for $f \in L^2(B_n)$

$$\langle f, k_a \rangle = (1 - |a|^2)^{n+1} \int_{B_n} \frac{f(z)}{(1 - \langle a, z \rangle)^{n+1}} dv(z) = (1 - |a|^2)^{n+1} f(a)$$
and holomorphic bounded functions are dense in \(L^2_\alpha(B_n)\), \(k_\alpha\) tends weakly to zero in \(L^2_\alpha\) for \(|a| \to 1^−\). Hence, the compactness of \(H^*_\beta H_f\) implies
\[
0 = \lim_{|a| \to 1^-} \langle H^*_\beta H_f k_\alpha, k_\alpha \rangle = \lim_{|a| \to 1^-} \langle H_f k_\alpha, H^*_\beta k_\alpha \rangle
\]
\[
= \lim_{|a| \to 1^-} \int_{B_n} (\bar{f} \circ \varphi_a - \bar{f}(a))(g \circ \varphi_a - g(a))dv.
\]
Suppose that (5) does not hold. Then there exists a sequence \(\{a_m\}\) in \(B\) such that \(\lim_{m \to \infty} |a_m| = 1\) and
\[
\lim_{m \to \infty} (1 - |a_m|^2)|\nabla_{a_m} f| = a > 0 \quad \text{and} \quad \lim_{m \to \infty} (1 - |a_m|^2)|\nabla_{a_m} g| = b > 0 .
\]
Since the families of functions
\[
\{f \circ \varphi_a - f(a); \ a \in B_n\} \quad \text{and} \quad \{g \circ \varphi_a - g(a); \ a \in B_n\}
\]
are compact, there exists a subsequence of \(\{a_m\}\) (which will be denoted also by \(\{a_m\}\)) and holomorphic functions \(F, G\) such that
\[
f \circ \varphi_{a_m} - f(a_m) \xrightarrow{m \to \infty} F \quad \text{and} \quad g \circ \varphi_{a_m} - g(a_m) \xrightarrow{m \to \infty} G
\]
uniformly on compact subsets of \(B_n\).

Now let \(w \in B_n\) be fixed. Then also
\[
f \circ \varphi_{a_m} \circ \varphi_w - f(a_m) \xrightarrow{m \to \infty} F \circ \varphi_w \quad \text{and} \quad g \circ \varphi_{a_m} \circ \varphi_w - g(a_m) \xrightarrow{m \to \infty} G \circ \varphi_w
\]
uniformly on compacta.

Let \(a'_m = \varphi_{a_m}(w), \ m \in \mathbb{N}\). There exists a unique unitary transformation \(U_m\)
for which \(\varphi_{a_m} \circ \varphi_w = \varphi_{a'_m} \circ U_m\) [12, p.29].

It follows from the relation [12, p.26]
\[
1 - |\varphi_a(w)|^2 = \frac{(1 - |a|^2)(1 - |w|^2)}{|1 - \langle w, a \rangle|^2}
\]
that \(|a'_m| \to 1^-\) if \(|a_m| \to 1^-\). Thus by the compactness of \(H^*_\beta H_f\)
\[
0 = \lim_{m \to \infty} \int_{B_n} (\bar{f} \circ \varphi_{a'_m} - \bar{f}(a'_m))(g \circ \varphi_{a'_m} - g(a'_m))dv
\]
\[
= \lim_{m \to \infty} \int_{B_n} (\bar{f} \circ \varphi_{a'_m} \circ U - \bar{f}(a'_m))(g \circ \varphi_{a'_m} \circ U - g(a'_m))dv
\]
\[
= \lim_{m \to \infty} \int_{B_n} (\bar{f} \circ \varphi_{a_m} \circ \varphi_w - \bar{f}(\varphi_{a_m} \circ \varphi_w)(0))(g \circ \varphi_{a_m} \circ \varphi_w - g(\varphi_{a_m} \circ \varphi_w)(0))dv
\]
In view of boundedness of \(f, g\) and uniform convergence on compacta we get
\[
\lim_{m \to \infty} \int_{B_n} [\bar{f}(\varphi_{a_m} \circ \varphi_w(z) - \bar{f}(a_m)) - (\bar{f}(\varphi_{a_m} \circ \varphi_w)(0) - \bar{f}(a_m))] \times [g(\varphi_{a_m} \circ \varphi_w(z) - g(a_m)) - (g(\varphi_{a_m} \circ \varphi_w)(0) - g(a_m))]dv(z)
\]
\[
= \int_{B_n} (\bar{F} \circ \varphi_w(z) - \bar{F}(\varphi_w(0))(G \circ \varphi_w(z) - G \circ \varphi_w(0))dv(z) = 0.
\]
The last equality can be rewritten in the form
\[
\int_{B_n} \bar{F} \circ \varphi_w(z) G \circ \varphi_w(z)dv(z) = \bar{F}(w)G(w) .
\]
Now Theorem C implies that $\tilde{F}G$ is $\mathcal{M}$-harmonic. Because $F, G$ are bounded the Ahern-Rudin theorem (Theorem B) and Liouville’s theorem imply that either $F$ or $G$ is constant on $B_n$. Suppose that $F$ is constant. Then by the chain rule and the symmetry of the matrix $\varphi'_a(0) = (1 - |a|^2)P_a - (1 - |a|^2)^{1/2}Q_a$ we get

$$0 = |\nabla_0 F|^2 = \lim_{m \to \infty} |\nabla_0 (f \circ \varphi_{a_m})|^2 = \lim_{m \to \infty} \langle \nabla_{a_m} f \varphi'_{a_m}(0), \nabla_{a_m} f \varphi'_{a_m}(0) \rangle$$

$$= \lim_{m \to \infty} \langle \varphi'_{a_m}(0) \nabla_{a_m} f, \varphi'_{a_m}(0) \nabla_{a_m} f \rangle$$

$$= \lim_{m \to \infty} \left( (1 - |a_m|^2) |\nabla_{a_m} f|^2 + (1 - |a_m|^2) |Q_{a_m} \nabla_{a_m} f|^2 \right)$$

$$\geq \lim_{m \to \infty} (1 - |a_m|^2) |\nabla_{a_m} f|^2.$$

Because the sequence $(1 - |a_m|^2) |\nabla_{a_m} g|$ is bounded for bounded and holomorphic $g$, the last inequality implies that

$$\lim_{m \to \infty} (1 - |a_m|^2) |\nabla_{a_m} g| |\nabla_{a_m} f| = 0$$

which contradicts (6).

Note that for the unit disc the same proof still goes under the assumption that $f, g$ are Bloch functions. In this case the equality (8) follows from the Lebesgue’s dominated theorem. In fact, if $f$ is a Bloch function on $D$ then [4, p.320]

$$\int_D |f \circ \varphi_{a_m}(z) - f(a_m)| dv(z) \leq \|f\|_B \int_D |ln(1 - |z|)| dv(z) \leq c \|f\|_B.$$ 

Thus we have

**Lemma 2.** If $f, g$ are Bloch functions on $D$ such that the operator $H^*_g H_f$ is compact then

$$\lim_{|a| \to 1^-} (1 - |a|^2) \min\{|f'(a)|, |g'(z)|\} = 0.$$ 

Analyzing the proof of Lemma 1 one can easily get

**Lemma 3.** If $f, g$ are bounded holomorphic functions on $B_n$ and the operator $H^*_g H_f$ is compact then

(9) $$\lim_{|a| \to 1^-} (1 - |a|^2) \min\{|P_{\bar{a}} \nabla_{a} f|, |P_{\bar{a}} \nabla_{a} g|\} = 0$$

and

(9') $$\lim_{|a| \to 1^-} (1 - |a|^2)^{1/2} \min\{|Q_{\bar{a}} \nabla_{a} f|, |Q_{\bar{a}} \nabla_{a} g|\} = 0.$$ 

Our main result is the following
Theorem. Let \( f, g \) be bounded holomorphic functions on \( B_n \) and \( 0 < r < 1 \). Then the following statements are equivalent:

(a) \( H^2 \) is compact;
(b) \[
\lim_{|a| \to 1} (1 - |a|^2) \min\{|P_a \nabla_a f|, |P_a \nabla_a g|\} = 0;
\]
(c) \[
\lim_{|a| \to 1} \int_{rB_n} |f \circ \varphi_a - f(a)||g \circ \varphi_a - g(a)|dv = 0;
\]
(d) \[
\lim_{|a| \to 1} \frac{1}{|E_n(a, r)|} \int_{E_n(a, r)} |f - f(a)||g - g(a)|dv = 0;
\]
(e) \[
\lim_{|a| \to 1} \int_{B_n} |f \circ \varphi_a - f(a)||g \circ \varphi_a - g(a)|dv = 0.
\]

Proof. (a)\(\implies\)(b). By Lemma 3.

(b)\(\implies\)(c). Suppose \( f \) and \( g \) are bounded holomorphic functions on \( B_n \) for which statement (b) holds. Let \( \{a_m\} \) be a sequence of points in \( B_n \) such that \( \lim_{m \to \infty} |a_m| = 1 \) and

\[
F = \lim_{m \to \infty} (f \circ \varphi_{a_m} - f(a_m)); \quad G = \lim_{m \to \infty} (g \circ \varphi_{a_m} - g(a_m)),
\]

where the convergence is uniform on compact subsets of \( B_n \) and \( F, G \) are holomorphic functions on \( B_n \). We will show that

\[
\int_{rB_n} |\nabla z F||\nabla z G|dv(z) = 0.
\]

Applying the definitions of \( F, G \) and changing the variables \( w = \varphi_{a_m}(z) \) we obtain

\[
\int_{rB_n} |\nabla z F||\nabla z G|dv(z) = \lim_{m \to \infty} \int_{rB_n} |\nabla \varphi_{a_m}(z) F \varphi'_{a_m}(z)||\nabla \varphi_{a_m}(z) G \varphi'_{a_m}(z)|dv(z)
\]

\[
= \lim_{m \to \infty} \int_{E_n(a_m, r)} |\nabla w F \varphi'_{a_m}(w)||\nabla w G \varphi'_{a_m}(w)|\frac{(1 - |a_m|^2)^{n+1}}{|1 - (w, a_m)|^{2n+2}}dv(w).
\]

Let \( \varphi_{a_m}(w) = \zeta_m \). To calculate \( \varphi'_{a_m}(\zeta_m) \) notice that for fixed \( w \) and \( a_m \) the biholomorphic map \( \varphi_w \circ \varphi_{a_m} \circ \varphi_{\zeta_m} \) is a unitary operator, say \( U_m \) [12, p.29]. Hence

\[
\varphi'_{a_m}(\zeta_m) = \varphi'_w(0)U_m \varphi'_{\zeta_m}(\zeta_m).
\]

Let \( \|A\| \) denote the standard norm of the linear operator \( A : \mathbb{C}^n \to \mathbb{C}^n \). Because for every \( z \in B_n \)

\[
\varphi'_z(z) = -(1 - |z|^2)^{-1}P_z - (1 - |z|^2)^{-1/2}Q_z,
\]

we have the following estimate:

\[
\|\varphi'_z(\zeta_m)\| \leq \frac{2}{1 - |\zeta_m|^2}.
\]

Moreover, in view of (7) we get

\[
\|\varphi'_{\zeta_m}(\zeta_m)\| \leq \frac{2|1 - \langle w, a_m \rangle|^2}{(1 - |a_m|^2)(1 - |w|^2)}.
\]
\[ \int_{rB_n} |\nabla F||\nabla G|dv(z) \]
\[ \leq 4 \lim_{m \to \infty} \int_{E_n(a_m,r)} (1 - |w|^2)^{-2} |\varphi'(0)\nabla w f||\varphi'(0)\nabla g| \frac{(1 - |a_m|^2)^n}{1 - (w, a_m)} dv(w) \]
\[ \leq 4 \lim_{m \to \infty} \int_{E_n(a_m,r)} (1 - |w|^2)^{-2} \left| (1 - |w|^2)P_w \nabla w f + (1 - |w|^2)^{1/2}Q_w \nabla w f \right| \]
\[ \times \left| (1 - |w|^2)P_w \nabla w g + (1 - |w|^2)^{1/2}Q_w \nabla w g \right| \frac{(1 - |a_m|^2)^n}{1 - (w, a_m)} dv(w) \]
\[ \leq 2^{n+1} \lim_{m \to \infty} \int_{E_n(a_m,r)} |P_w \nabla w f||P_w \nabla w g|(1 - |w|^2)^{-n+1} dv(w) \]
\[ + 2^{n+1} \lim_{m \to \infty} \int_{E_n(a_m,r)} |P_w \nabla w f||Q_w \nabla w g|(1 - |w|^2)^{-n+1/2} dv(w) \]
\[ + 2^{n+1} \lim_{m \to \infty} \int_{E_n(a_m,r)} |Q_w \nabla w f||P_w \nabla w g|(1 - |w|^2)^{-n+1/2} dv(w) \]
\[ + 2^{n+1} \lim_{m \to \infty} \int_{E_n(a_m,r)} |Q_w \nabla w f||Q_w \nabla w g|(1 - |w|^2)^{-n} dv(w) . \]

To finish the proof of (10) it is enough to show that each of the four limits on the right-hand side of the last inequality is zero. Indeed, we have
\[ \int_{E_n(a_m,r)} |P_w \nabla w f||P_w \nabla w g|(1 - |w|^2)^{-n+1} dv(w) \]
\[ \leq \sup_{w \in E_n(a_m,r)} (1 - |w|^2)^2 |P_w \nabla w f||P_w \nabla w f| \int_{E_n(a_m,r)} (1 - |w|^2)^{-n-1} dv(w) \]
\[ = 2n \sup_{w \in E_n(a_m,r)} (1 - |w|^2)^2 |P_w \nabla w f||P_w \nabla w f| . \]

Let \( \{\zeta_m\} \) be the sequence in \( B_n \) such that: \( \zeta_m \in E(a_m, r) \) and
\[ \sup_{w \in E(a_m,r)} (1 - |w|^2)^2 |P_w \nabla w f||P_w \nabla w f| = (1 - |\zeta_m|^2)^2 |P_{\zeta_m} \nabla \zeta_m f||P_{\zeta_m} \nabla \zeta_m f| . \]

Then \( \lim_{m \to \infty} |\zeta_m| = 1 \) and (b) implies that the last expression tends to 0. The same reasoning applies to the remaining cases.

Now (10) implies that at least one of the functions \( F, G \) must be identically 0 on \( rB_n \). Hence
\[ \lim_{m \to \infty} \int_{rB_n} |f \circ \varphi_{a_m} - f(a_m)||g \circ \varphi_{a_m} - g(a_m)| dv = \int_{rB_n} |FG| dv = 0 . \]

(c) \( \iff \) (d) A change-of-variables yields
\[ \int_{E_n(a,r)} |f - f(a)||g - g(a)| dv \]
\[ = \int_{rB_n} |f \circ \varphi_a(z) - f(a)||g \circ \varphi_a(z) - g(a)| \frac{(1 - |a|^2)^{n+1}}{1 - (z, a)^2(n+1)} dv. \]
Because

\[ |E_n(a,r)| = \left( \frac{1 - |a|^2}{1 - r^2|a|^2} \right)^{n+1} \]

the desired equivalence follows from the inequalities

\[ \frac{1}{r^{2n}(1 + r^2)^{2(n+1)}} \leq \frac{1}{|E_n(a,r)|} \frac{(1 - |a|^2)^{(n+1)}}{|1 - \langle z, a \rangle|} \leq \frac{1}{r^{2n}(1 - r^2)^{n+1}}, \quad z \in rB_n. \]

(c)⇒(e) Let \( \{a_m\} \) be a sequence such that \( \lim_{m \to \infty} |a_m| = 1 \) and \( F, G \) be as in the proof of the implication (b)⇒(c). Then (c) implies that

\[ \int_{rB_n} |F||G|dv = 0. \]

This in turn implies that either \( F \) or \( G \) is identically zero on \( rB_n \), and hence on \( B_n \).

(c)⇒(a) It is enough to proceed analogously to the proof of Theorem 2 of [14].

Remark. Notice that the functions \( f, g \) defined in assertion (ii) of Theorem B cannot be Bloch functions. Hence in view of Theorem C, when \( n < 12 \) our theorem holds for Bloch functions instead of bounded holomorphic functions.

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