

## HANKEL OPERATORS ON THE BERGMAN SPACE OF THE UNIT BALL

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ABSTRACT. We characterize the bounded holomorphic functions  $f, g$  in the unit ball of  $\mathbb{C}^n$  for which the operator  $H_g^* H_{\bar{f}}$  is compact. For  $n = 1$  the result was obtained by Axler and Gorkin in 1988 and by Zheng in 1989.

### 1. INTRODUCTION

Let  $B_n$  be the open unit ball in  $\mathbb{C}^n$  with  $dv$  the normalized volume measure on  $B_n$ . When  $n = 1$  we will denote by  $\mathbb{D}$  the unit disc in  $\mathbb{C}$ . The Bergman space  $L_a^2(B_n)$  is the closed subspace of  $L^2(B_n, dv)$  consisting of holomorphic functions. Let  $P$  be the orthogonal projection of  $L^2(B_n, dv)$  onto  $L_a^2(B_n)$ . For  $f \in L^2(B_n, dv)$  the Toeplitz operator  $T_f : L_a^2(B_n) \rightarrow L_a^2(B_n)$  and the Hankel operator  $H_f : L_a^2(B_n) \rightarrow (L_a^2(B_n))^\perp$  are defined by

$$T_f(g) = P(fg) \quad \text{and} \quad H_f(g) = fg - P(fg),$$

respectively. In fact, these operators are densely defined for bounded holomorphic functions and extended to all functions in  $L_a^2(B_n)$ .

In 1986 S. Axler ([4], [5]) proved that for  $f \in L_a^2(\mathbb{D})$  the Hankel operator  $H_{\bar{f}} : L_a^2(\mathbb{D}) \rightarrow (L_a^2(\mathbb{D}))^\perp$  is bounded if and only if  $f$  is a Bloch function. Moreover,  $H_{\bar{f}}$  is compact if and only if  $f$  is in the little Bloch space. Many generalizations of these results have been found since then (see e.g. [3], [7], [8]).

In view of the relation

$$(1) \quad T_f^* T_g - T_g T_f^* = H_{\bar{g}}^* H_{\bar{f}} \quad \text{for } f, g \in L_a^2(B)$$

the problem of characterization of the functions  $f, g$  for which the operator  $H_{\bar{g}}^* H_{\bar{f}}$  is compact seems natural. In the case of the unit disc the problem was proposed by S. Axler [4] and solved by S. Axler and P. Gorkin [6] and independently by D. Zheng [14]. They have proved the following

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**Theorem A.** *Let  $f, g$  be bounded holomorphic functions on  $\mathbb{D}$ . Then the following statements are equivalent:*

- (a)  $H_g^* H_{\bar{f}}$  is compact ,
- (b)  $\lim_{|z| \rightarrow 1^-} (1 - |z|^2) \min\{|f'(z)|, |g'(z)|\} = 0$  ,
- (c)  $\lim_{|w| \rightarrow 1^-} \int_{\mathbb{D}} |f(z) - f(w)| |g(z) - g(w)| dv(z) = 0$  ,
- (d) either  $f$  or  $g$  is constant on each Gleason part (except  $\mathbb{D}$ ) of the maximal ideal space of  $H^\infty(\mathbb{D})$  .

It was also noticed that each of the conditions (b), (c) implies the compactness of the operator  $H_g^* H_{\bar{f}}$  under the weaker and more natural assumption that  $f, g$  are Bloch functions. Moreover, these sufficient conditions for compactness of  $H_g^* H_{\bar{f}}$  are still valid if  $\mathbb{D}$  is replaced by the unit ball  $B_n, n > 1$  .

Here we show that in the case of  $\mathbb{D}$  (b) ((c)) is also a necessary condition for compactness of  $H_g^* H_{\bar{f}}$  under the assumption that  $f, g$  are Bloch functions. The main result of this paper is a characterization of the functions  $f, g$  bounded and holomorphic on  $B_n$  for which the operator  $H_g^* H_{\bar{f}}$  is compact.

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## 2. PRELIMINARIES

Let  $\langle z, w \rangle$  denote the inner product in  $\mathbb{C}^n$  given by

$$\langle z, w \rangle = \sum_{j=1}^n z_j \bar{w}_j, \quad z = (z_1, \dots, z_n), \quad w = (w_1, \dots, w_n).$$

Let  $\text{Aut}(B_n)$  be the group of all biholomorphic maps of  $B_n$  into  $B_n$  . It is known that  $\text{Aut}(B_n)$  is generated by the unitary operators on  $\mathbb{C}^n$  and the involutions  $\varphi_a$  of the form

$$(2) \quad \varphi_a(z) = \frac{a - P_a z - (1 - |a|^2)^{1/2} Q_a z}{1 - \langle z, a \rangle},$$

where  $a \in B_n$  ,  $P_a$  is the orthogonal projection into the space spanned by  $a$ , i.e.

$$P_a z = \frac{\langle z, a \rangle a}{|a|^2}, \quad |a|^2 = \langle a, a \rangle, \\ P_0 z = 0$$

and  $Q_a = I - P_a$  .

For fixed  $a \in B_n$  and  $r, 0 < r < 1$  , define

$$E_n(a, r) = \varphi_a(rB_n).$$

Since  $\varphi_a$  is an involution,  $z \in E_n(a, r)$  if and only if  $|\varphi_a(z)| < r$  .

As in [11] we say that a function  $f \in C^2(B_n)$  is  $\mathcal{M}$ -harmonic in  $B_n$  if  $\tilde{\Delta} f(z) = 0$  for every  $z \in B_n$  . The operator  $\tilde{\Delta}$  is defined by

$$(\tilde{\Delta} f)(z) = \Delta(f \circ \varphi_z)(0),$$

where  $\Delta$  is the ordinary Laplacian. We will need to use the following recent results on  $\mathcal{M}$ -harmonic functions.

**Theorem B** ([2]). *Suppose that  $f$  and  $g$  are nonconstant holomorphic functions in  $B_n$ , and that  $f\bar{g}$  is  $\mathcal{M}$ -harmonic.*

- (a) *When  $n = 2$ , this cannot happen.*
- (b) *When  $n \geq 3$ , then there exist*
  - (i) *an integer  $m$ ,  $2 \leq m \leq n - 1$ ,*
  - (ii) *entire functions  $\varphi : \mathbb{C}^{m-1} \rightarrow \mathbb{C}$  and  $\psi : \mathbb{C}^{n-m} \rightarrow \mathbb{C}$ , such that*

$$f(Uz) = \varphi\left(\frac{z_2}{1-z_1}, \dots, \frac{z_m}{1-z_1}\right), \quad g(Uz) = \psi\left(\frac{z_{m+1}}{1-z_1}, \dots, \frac{z_n}{1-z_1}\right).$$

Moreover,  $f(B_n) = \varphi(\mathbb{C}^{m-1})$ ,  $g(B_n) = \psi(\mathbb{C}^{n-m})$ , and  $(f\bar{g})(B_n) = \mathbb{C}$  or  $\mathbb{C} \setminus \{0\}$ . (The latter case occurs only when both  $\varphi$  and  $\psi$  omit the value 0.)

**Theorem C** ([1]). *Assume that for  $f \in L^1(B_n, dv)$  the linear operator  $T_0$  is defined by*

$$(T_0f)(z) = \int_{B_n} (f \circ \varphi_z) dv, \quad z \in B_n.$$

- (a) *If  $f \in L^\infty(B_n)$  and  $T_0f = f$  then  $\tilde{\Delta}f = 0$ .*
- (b) *If  $f \in L^1(B_n, dv)$ ,  $n < 12$  and  $T_0f = f$ , then  $\tilde{\Delta}f = 0$ ; this fails for all  $n \geq 12$ .*

We say that a holomorphic function  $f$  is a Bloch function if

$$(3) \quad \|f\|_B = \sup\{|\nabla_z f|(1 - |z|^2); z \in B_n\} < \infty.$$

Bloch functions on bounded homogeneous domains in  $\mathbb{C}^n$  were defined by R. Timoney [12] and studied by many authors.(e.g. [9], [11], [12]). One of the most important properties of the space of Bloch functions on  $B_n$ ,  $n \geq 2$ , is that (3) is equivalent to the following conditions (for some  $K \geq 0$ ):

$$\begin{aligned} |\langle \nabla_z f, \bar{z} \rangle|(1 - |z|^2) &\leq K, \\ |\langle \nabla_z f, \bar{x} \rangle|(1 - |z|^2) &\leq K \text{ for all } x \in \mathbb{C}^n \text{ satisfying } |x| = 1 \text{ and } \langle x, \bar{z} \rangle = 0 \end{aligned}$$

for all  $z \in B_n$ . This means that  $f$  is a Bloch function on  $B_n$  if and only if the radial derivative of  $f$  is  $O(1/(1 - |z|^2))$  and the directional derivatives of  $f$  in directions perpendicular to the radial direction are  $O(1/(1 - |z|^2)^{1/2})$ .

### 3. MAIN RESULTS

We start with the following

**Lemma 1.** *Let  $f, g$  be bounded holomorphic functions on  $B_n$ . If the operator  $H_{\bar{g}}^* H_f$  is compact then*

$$(5) \quad \lim_{|a| \rightarrow 1^-} (1 - |a|^2) \min\{|\nabla_a f|, |\nabla_a g|\} = 0.$$

*Proof.* For  $a \in B_n$  define

$$k_a(z) = \frac{(1 - |a|^2)^{(n+1)/2}}{(1 - \langle z, a \rangle)^{n+1}}, \quad z \in B_n.$$

Because for  $f \in L_a^2(B_n)$

$$\langle f, k_a \rangle = (1 - |a|^2)^{\frac{n+1}{2}} \int_{B_n} \frac{f(z)}{(1 - \langle a, z \rangle)^{n+1}} dv(z) = (1 - |a|^2)^{\frac{n+1}{2}} f(a)$$

and holomorphic bounded functions are dense in  $L_a^2(B_n)$ ,  $k_a$  tends weakly to zero in  $L_a^2$  for  $|a| \rightarrow 1^-$ . Hence the compactness of  $H_{\bar{g}}^* H_{\bar{f}}$  implies

$$\begin{aligned} 0 &= \lim_{|a| \rightarrow 1^-} \langle H_{\bar{g}}^* H_{\bar{f}} k_a, k_a \rangle = \lim_{|a| \rightarrow 1^-} \langle H_{\bar{f}} k_a, H_{\bar{g}} k_a \rangle \\ &= \lim_{|a| \rightarrow 1^-} \int_{B_n} (\bar{f} \circ \varphi_a - \bar{f}(a))(g \circ \varphi_a - g(a)) dv . \end{aligned}$$

Suppose that (5) does not hold. Then there exists a sequence  $\{a_m\}$  in  $B$  such that  $\lim_{m \rightarrow \infty} |a_m| = 1$  and

$$(6) \quad \lim_{m \rightarrow \infty} (1 - |a_m|^2) |\nabla_{a_m} f| = a > 0 \quad \text{and} \quad \lim_{m \rightarrow \infty} (1 - |a_m|^2) |\nabla_{a_m} g| = b > 0 .$$

Since the families of functions

$$\{f \circ \varphi_a - f(a); a \in B_n\} \quad \text{and} \quad \{g \circ \varphi_a - g(a); a \in B_n\}$$

are compact, there exists a subsequence of  $\{a_m\}$  (which will be denoted also by  $\{a_m\}$ ) and holomorphic functions  $F, G$  such that

$$f \circ \varphi_{a_m} - f(a_m) \xrightarrow{m \rightarrow \infty} F \quad \text{and} \quad g \circ \varphi_{a_m} - g(a_m) \xrightarrow{m \rightarrow \infty} G$$

uniformly on compact subsets of  $B_n$ .

Now let  $w \in B_n$  be fixed. Then also

$$f \circ \varphi_{a_m} \circ \varphi_w - f(a_m) \xrightarrow{m \rightarrow \infty} F \circ \varphi_w \quad \text{and} \quad g \circ \varphi_{a_m} \circ \varphi_w - g(a_m) \xrightarrow{m \rightarrow \infty} G \circ \varphi_w$$

uniformly on compacta.

Let  $a'_m = \varphi_{a_m}(w)$ ,  $m \in \mathbb{N}$ . There exists a unique unitary transformation  $U_m$  for which  $\varphi_{a_m} \circ \varphi_w = \varphi_{a'_m} \circ U_m$  [12, p.29].

It follows from the relation [12, p.26]

$$(7) \quad 1 - |\varphi_a(w)|^2 = \frac{(1 - |a|^2)(1 - |w|^2)}{|1 - \langle w, a \rangle|^2}$$

that  $|a'_m| \rightarrow 1^-$  if  $|a_m| \rightarrow 1^-$ . Thus by the compactness of  $H_{\bar{g}}^* H_{\bar{f}}$

$$\begin{aligned} 0 &= \lim_{m \rightarrow \infty} \int_{B_n} (\bar{f} \circ \varphi_{a'_m} - \bar{f}(a'_m))(g \circ \varphi_{a'_m} - g(a'_m)) dv \\ &= \lim_{m \rightarrow \infty} \int_{B_n} (\bar{f} \circ \varphi_{a'_m} \circ U - \bar{f}(a'_m))(g \circ \varphi_{a'_m} \circ U - g(a'_m)) dv \\ &= \lim_{m \rightarrow \infty} \int_{B_n} (\bar{f} \circ \varphi_{a_m} \circ \varphi_w - \bar{f}(\varphi_{a_m} \circ \varphi_w(0)))(g \circ \varphi_{a_m} \circ \varphi_w - g(\varphi_{a_m} \circ \varphi_w(0))) dv \end{aligned}$$

In view of boundedness of  $f, g$  and uniform convergence on compacta we get

$$\begin{aligned} (8) \quad & \lim_{m \rightarrow \infty} \int_{B_n} [(\bar{f} \circ \varphi_{a_m} \circ \varphi_w(z) - \bar{f}(a_m)) - (\bar{f}(\varphi_{a_m} \circ \varphi_w(0)) - \bar{f}(a_m))] \\ & \times [(g \circ \varphi_{a_m} \circ \varphi_w(z) - g(a_m)) - (g(\varphi_{a_m} \circ \varphi_w(0)) - g(a_m))] dv(z) \\ & = \int_{B_n} (\bar{F} \circ \varphi_w(z) - \bar{F} \circ \varphi_w(0))(G \circ \varphi_w(z) - G \circ \varphi_w(0)) dv(z) = 0 . \end{aligned}$$

The last equality can be rewritten in the form

$$\int_{B_n} \bar{F} \circ \varphi_w(z) G \circ \varphi_w(z) dv(z) = \bar{F}(w) G(w) .$$

Now Theorem C implies that  $\bar{F}G$  is  $\mathcal{M}$ -harmonic. Because  $F, G$  are bounded the Ahern-Rudin theorem (Theorem B) and Liouville's theorem imply that either  $F$  or  $G$  is constant on  $B_n$ . Suppose that  $F$  is constant. Then by the chain rule and the symmetry of the matrix  $\varphi'_a(0) = (1 - |a|^2)P_a - (1 - |a|^2)^{1/2}Q_a$  we get

$$\begin{aligned} 0 &= |\nabla_0 F|^2 = \lim_{m \rightarrow \infty} |\nabla_0(f \circ \varphi_{a_m})|^2 = \lim_{m \rightarrow \infty} \langle \nabla_{a_m} f \varphi'_{a_m}(0), \nabla_{a_m} f \varphi'_{a_m}(0) \rangle \\ &= \lim_{m \rightarrow \infty} \langle \varphi'_{a_m}(0) \nabla_{a_m} f, \varphi'_{a_m}(0) \nabla_{a_m} f \rangle \\ &= \lim_{m \rightarrow \infty} ((1 - |a_m|^2)^2 |P_{\bar{a}_m} \nabla_{a_m} f|^2 + (1 - |a_m|^2) |Q_{\bar{a}_m} \nabla_{a_m} f|^2) \\ &\geq \lim_{m \rightarrow \infty} (1 - |a_m|^2)^2 |\nabla_{a_m} f|^2. \end{aligned}$$

Because the sequence  $(1 - |a_m|^2) |\nabla_{a_m} g|$  is bounded for bounded and holomorphic  $g$ , the last inequality implies that

$$\lim_{m \rightarrow \infty} (1 - |a_m|^2)^2 |\nabla_{a_m} g| |\nabla_{a_m} f| = 0$$

which contradicts (6). □

Note that for the unit disc the same proof still goes under the assumption that  $f, g$  are Bloch functions. In this case the equality (8) follows from the Lebesgue's dominated theorem. In fact, if  $f$  is a Bloch function on  $\mathbb{D}$  then [4, p.320]

$$\int_{\mathbb{D}} |f \circ \varphi_{a_m}(z) - f(a_m)| dv(z) \leq \|f\|_{\mathcal{B}} \int_{\mathbb{D}} |\ln(1 - |z|)| dv(z) \leq c \|f\|_{\mathcal{B}} .$$

Thus we have

**Lemma 2.** *If  $f, g$  are Bloch functions on  $\mathbb{D}$  such that the operator  $H_g^* H_{\bar{f}}$  is compact then*

$$\lim_{|a| \rightarrow 1^-} (1 - |a|^2) \min\{|f'(a)|, |g'(z)|\} = 0 .$$

Analyzing the proof of Lemma 1 one can easily get

**Lemma 3.** *If  $f, g$  are bounded holomorphic functions on  $B_n$  and the operator  $H_g^* H_{\bar{f}}$  is compact then*

$$(9) \quad \lim_{|a| \rightarrow 1^-} (1 - |a|^2) \min\{|P_{\bar{a}} \nabla_a f|, |P_{\bar{a}} \nabla_a g|\} = 0$$

and

$$(9') \quad \lim_{|a| \rightarrow 1^-} (1 - |a|^2)^{1/2} \min\{|Q_{\bar{a}} \nabla_a f|, |Q_{\bar{a}} \nabla_a g|\} = 0 .$$

Our main result is the following

**Theorem.** Let  $f, g$  be bounded holomorphic functions on  $B_n$  and  $0 < r < 1$ . Then the following statements are equivalent

- (a)  $H_{\bar{g}}^* H_{\bar{f}}$  is compact ;
- (b)  $\lim_{|a| \rightarrow 1} (1 - |a|^2) \min\{|P_{\bar{a}} \nabla_a f|, |P_{\bar{a}} \nabla_a g|\}$   
 $= \lim_{|a| \rightarrow 1} (1 - |a|^2)^{1/2} \min\{|Q_{\bar{a}} \nabla_a f|, |Q_{\bar{a}} \nabla_a g|\} = 0$  ;
- (c)  $\lim_{|a| \rightarrow 1} \int_{rB_n} |f \circ \varphi_a - f(a)| |g \circ \varphi_a - g(a)| dv = 0$  ;
- (d)  $\lim_{|a| \rightarrow 1} \frac{1}{|E_n(a, r)|} \int_{E_n(a, r)} |f - f(a)| |g - g(a)| dv = 0$  ;
- (e)  $\lim_{|a| \rightarrow 1} \int_{B_n} |f \circ \varphi_a - f(a)| |g \circ \varphi_a - g(a)| dv = 0$  .

*Proof.* (a) $\implies$ (b). By Lemma 3.

(b) $\implies$ (c). Suppose  $f$  and  $g$  are bounded holomorphic functions on  $B_n$  for which statement (b) holds. Let  $\{a_m\}$  be a sequence of points in  $B_n$  such that  $\lim_{m \rightarrow \infty} |a_m| = 1$  and

$$F = \lim_{m \rightarrow \infty} (f \circ \varphi_{a_m} - f(a_m)) \quad \text{and} \quad G = \lim_{m \rightarrow \infty} (g \circ \varphi_{a_m} - g(a_m)) ,$$

where the convergence is uniform on compact subsets of  $B_n$  and  $F, G$  are holomorphic functions on  $B_n$ . We will show that

$$(10) \quad \int_{rB_n} |\nabla_z F| |\nabla_z G| dv(z) = 0 .$$

Applying the definitions of  $F, G$  and changing the variables  $w = \varphi_{a_m}(z)$  we obtain

$$\begin{aligned} \int_{rB_n} |\nabla_z F| |\nabla_z G| dv(z) &= \lim_{m \rightarrow \infty} \int_{rB_n} |\nabla_{\varphi_{a_m}(z)} f \varphi'_{a_m}(z)| |\nabla_{\varphi_{a_m}(z)} g \varphi'_{a_m}(z)| dv(z) \\ &= \lim_{m \rightarrow \infty} \int_{E_n(a_m, r)} |\nabla_w f \varphi'_{a_m}(\varphi_{a_m}(w))| |\nabla_w g \varphi'_{a_m}(\varphi_{a_m}(w))| \frac{(1 - |a_m|^2)^{n+1}}{|1 - \langle w, a_m \rangle|^{2n+2}} dv(w) . \end{aligned}$$

Let  $\varphi_{a_m}(w) = \zeta_m$ . To calculate  $\varphi'_{a_m}(\zeta_m)$  notice that for fixed  $w$  and  $a_m$  the biholomorphic mapping  $\varphi_w \circ \varphi_{a_m} \circ \varphi_{\zeta_m}$  is a unitary operator, say  $U_m$  [12, p.29]. Hence

$$\varphi'_{a_m}(\zeta_m) = \varphi'_w(0) U_m \varphi'_{\zeta_m}(\zeta_m) .$$

Let  $\|A\|$  denote the standard norm of the linear operator  $A : \mathbb{C}^n \rightarrow \mathbb{C}^n$ . Because for every  $z \in B_n$

$$\varphi'_z(z) = -(1 - |z|^2)^{-1} P_z - (1 - |z|^2)^{-1/2} Q_z ,$$

we have the following estimate:

$$\|\varphi'_{\zeta_m}(\zeta_m)\| \leq \frac{2}{1 - |\zeta_m|^2} .$$

Moreover, in view of (7) we get

$$\|\varphi'_{\zeta_m}(\zeta_m)\| \leq \frac{2|1 - \langle w, a_m \rangle|^2}{(1 - |a_m|^2)(1 - |w|^2)} .$$

Hence

$$\begin{aligned}
 & \int_{rB_n} |\nabla_z F| |\nabla_z G| dv(z) \\
 & \leq 4 \lim_{m \rightarrow \infty} \int_{E_n(a_m, r)} (1 - |w|^2)^{-2} |\varphi'_{\bar{w}}(0) \nabla_w f| |\varphi'_{\bar{w}}(0) \nabla_w g| \frac{(1 - |a_m|^2)^{n-1}}{|1 - \langle w, a_m \rangle|^{2n-2}} dv(w) \\
 & \leq 4 \lim_{m \rightarrow \infty} \int_{E_n(a_m, r)} (1 - |w|^2)^{-2} \left| (1 - |w|^2) P_{\bar{w}} \nabla_w f + (1 - |w|^2)^{1/2} Q_{\bar{w}} \nabla_w f \right| \\
 & \quad \times \left| (1 - |w|^2) P_{\bar{w}} \nabla_w g + (1 - |w|^2)^{1/2} Q_{\bar{w}} \nabla_w g \right| \frac{(1 - |a_m|^2)^{n-1}}{|1 - \langle w, a_m \rangle|^{2n-2}} dv(w) \\
 & \leq 2^{n+1} \lim_{m \rightarrow \infty} \int_{E_n(a_m, r)} |P_{\bar{w}} \nabla_w f| |P_{\bar{w}} \nabla_w g| (1 - |w|^2)^{-n+1} dv(w) \\
 & + 2^{n+1} \lim_{m \rightarrow \infty} \int_{E_n(a_m, r)} |P_{\bar{w}} \nabla_w f| |Q_{\bar{w}} \nabla_w g| (1 - |w|^2)^{-n+1/2} dv(w) \\
 & + 2^{n+1} \lim_{m \rightarrow \infty} \int_{E_n(a_m, r)} |Q_{\bar{w}} \nabla_w f| |P_{\bar{w}} \nabla_w g| (1 - |w|^2)^{-n+1/2} dv(w) \\
 & + 2^{n+1} \lim_{m \rightarrow \infty} \int_{E_n(a_m, r)} |Q_{\bar{w}} \nabla_w f| |Q_{\bar{w}} \nabla_w g| (1 - |w|^2)^{-n} dv(w) .
 \end{aligned}$$

To finish the proof of (10) it is enough to show that each of the four limits on the right-hand side of the last inequality is zero. Indeed, we have

$$\begin{aligned}
 & \int_{E_n(a_m, r)} |P_{\bar{w}} \nabla_w f| |P_{\bar{w}} \nabla_w g| (1 - |w|^2)^{-n+1} dv(w) \\
 & \leq \sup_{w \in E_n(a_m, r)} (1 - |w|^2)^2 |P_{\bar{w}} \nabla_w f| |P_{\bar{w}} \nabla_w f| \int_{E_n(a_m, r)} (1 - |w|^2)^{-n-1} dv(w) \\
 & = \frac{r^{2n}}{(1 - r^2)^{n+1}} \sup_{w \in E_n(a_m, r)} (1 - |w|^2)^2 |P_{\bar{w}} \nabla_w f| |P_{\bar{w}} \nabla_w f| .
 \end{aligned}$$

Let  $\{\zeta_m\}$  be the sequence in  $B_n$  such that:  $\zeta_m \in E(a_m, r)$  and

$$\sup_{w \in E(a_m, r)} (1 - |w|^2)^2 |P_{\bar{w}} \nabla_w f| |P_{\bar{w}} \nabla_w f| = (1 - |\zeta_m|^2)^2 |P_{\bar{\zeta}_m} \nabla_{\zeta_m} f| |P_{\bar{\zeta}_m} \nabla_{\zeta_m} f| .$$

Then  $\lim_{m \rightarrow \infty} |\zeta_m| = 1$  and (b) implies that the last expression tends to 0 . The same reasoning applies to the remaining cases.

Now (10) implies that at least one of the functions  $F, G$  must be identically 0 on  $rB_n$  . Hence

$$\lim_{m \rightarrow \infty} \int_{rB_n} |f \circ \varphi_{a_m} - f(a_m)| |g \circ \varphi_{a_m} - g(a_m)| dv = \int_{rB_n} |FG| dv = 0 .$$

(c)  $\iff$  (d) A change-of-variables yields

$$\begin{aligned}
 & \int_{E_n(a, r)} |f - f(a)| |g - g(a)| dv \\
 & = \int_{rB_n} |f \circ \varphi_a(z) - f(a)| |g \circ \varphi_a(z) - g(a)| \frac{(1 - |a|^2)^{n+1}}{|1 - \langle z, a \rangle|^{2(n+1)}} dv .
 \end{aligned}$$

Because

$$|E_n(a, r)| = \left( \frac{1 - |a|^2}{1 - r^2|a|^2} \right)^{n+1}$$

the desired equivalence follows from the inequalities

$$\frac{1}{r^{2n}(1+r^2)^{2(n+1)}} \leq \frac{1}{|E_n(a, r)|} \frac{(1 - |a|^2)^{(n+1)}}{|1 - \langle z, a \rangle|^{2(n+1)}} \leq \frac{1}{r^{2n}(1-r^2)^{n+1}}, \quad z \in rB_n.$$

(c) $\implies$ (e) Let  $\{a_m\}$  be a sequence such that  $\lim_{m \rightarrow \infty} |a_m| = 1$  and  $F, G$  be as in the proof of the implication (b) $\implies$ (c). Then (c) implies that

$$\int_{rB_n} |F||G|dv = 0.$$

This in turn implies that either  $F$  or  $G$  is identically zero on  $rB_n$ , and hence on  $B_n$ .

(e) $\implies$ (a) It is enough to proceed analogously to the proof of Theorem 2 of [14].  $\square$

*Remark.* Notice that the functions  $f, g$  defined in assertion (ii) of Theorem B cannot be Bloch functions. Hence in view of Theorem C, when  $n < 12$  our theorem holds for Bloch functions instead of bounded holomorphic functions.

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