ORDINARY DIFFERENTIAL INEQUALITIES
AND QUASIMONOTONICITY
IN ORDERED TOPOLOGICAL VECTOR SPACES

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Abstract. A well known comparison theorem on ordinary differential inequalities with quasimonotone right-hand side \( f(t, x) \) was carried over by Volkmann (1972) to (pre)ordered topological vector spaces. We prove that the quasimonotonicity of \( f \) is a necessary condition here if \( f \) is continuous. Then it is shown that quasimonotonicity can be verified by considering only a few positive continuous linear functionals in the definition (for instance in \( \ell_\infty \) by taking coordinate functionals).

1. Introduction

Quasimonotonicity has its origins in initial value problems

\[
\begin{align*}
  u(t_0) &= x_0, \\
  u'(t) &= f(t, u(t)) \\
  (t_0 \leq t \leq t_1)
\end{align*}
\]

and corresponding differential inequalities, when certain theorems were carried over from the scalar case to problems in \( \mathbb{R}^n \). First, Müller (1927) [3] proved the existence of a solution to (1) between given lower and upper solutions, and Kamke [1] established extremal solutions. They referred to the componentwise ordering in \( \mathbb{R}^n \), and they assumed that, roughly speaking, each component \( f_i(t, x_1, ..., x_n) \) is monotone increasing in every \( x_j \) with \( j \neq i \). Later on, Walter [12] called such functions quasimonotone increasing in \( x \). In some applications the term cooperative is used.

Finally, Volkmann (1972) [8] gave the general definition of quasimonotonicity in topological vector spaces preordered by a cone, which makes use of the dual cone. He carried over a very useful comparison theorem on ordinary differential inequalities where the right-hand side \( f(t, x) \) is quasimonotone increasing in \( x \). Our purpose is to prove that, conversely, the quasimonotonicity of \( f \) is a necessary condition here in case \( f \) is continuous. Then we will show that quasimonotonicity can be verified by considering only some small subset of the dual cone.

A generalization of Müller’s theorem to preordered Banach spaces is given in [7]. Note that (1) need not have any local solution if \( f \) is only assumed to be continuous and bounded. It should be pointed out that in some ordered Banach spaces the quasimonotonicity of \( f \) implies the existence of solutions to (1), in fact extremal solutions, without any Lipschitz or compactness condition. Some of these results
are based on the fixed point theorem of Lemmert [2]. See the survey [10]. Finally, quasimonotonicity is also significant for systems of parabolic differential equations, cf. [6], [12], and for fixed points of discontinuous functions, cf. [4].

2. Notations

Let $E$ be a Hausdorff topological vector space, and let $K \subset E$ be a cone (i.e. $K$ is closed, convex, nonempty, and satisfies $\lambda x \in K$ for all $\lambda \geq 0$, $x \in K$). By the definition

$$x \leq y \iff y - x \in K \quad (x, y \in E),$$

a preordering (a reflexive transitive relation) on $E$ is given; this preordering is an ordering (also antisymmetric) iff $K$ is strict (i.e. $x, -x \in K \implies x = 0_E$). Now $E$ is said to be preordered or ordered, respectively. We write $x \ll y \iff y \gg x \iff y - x \in \text{Int} K \quad (x, y \in E)$, where $\text{Int} K$ denotes the interior of $K$. The dual cone of $K$ is $K^* = \{ \varphi \in E_R^* : \varphi(x) \geq 0 \text{ for all } x \in K\}$, where $E_R^*$ denotes the dual of $E_R$ ($E$ regarded as a real space).

A function $f : G \to E$ on $G \subset E$ is said to be quasimonotone increasing if for all $x, y \in G$ and all $\varphi \in K^*$ the implication

$$(2) \quad x \leq y, \varphi(x) = \varphi(y) \implies \varphi(f(x)) \leq \varphi(f(y))$$

holds. Finally, (P) will denote the following property, where now $f : D \to E$ has domain $D \subset \mathbb{R} \times E$.

(P) If $v, w : [t_0, t_1] \to E$ ($t_0 < t_1$) are any differentiable functions such that graph $v$, graph $w \subset D$, $v(t_0) \ll w(t_0)$ and

$$(3) \quad v'(t) - f(t, v(t)) \ll w'(t) - f(t, w(t)) \quad (t_0 \leq t \leq t_1),$$

then $v(t) \ll w(t)$ for $t_0 \leq t \leq t_1$.

3. (P) implies quasimonotonicity

Roughly speaking, property (P) and quasimonotonicity are equivalent.

Theorem 1. Let $E$ be a Hausdorff topological vector space preordered by a cone $K$ with $\text{Int} K \neq \emptyset$. Suppose $D \subset \mathbb{R} \times E$ is such that for every $(t, x) \in D$ there exist $\varepsilon > 0$ and a neighborhood $G$ of $x$ satisfying $[t, t + \varepsilon) \times G \subset D$. Assume $f : D \to E$ is continuous. Then (P) holds if and only if for each $t \in \mathbb{R}$ the function $x \mapsto f(t, x)$ is quasimonotone increasing.

Note that the quasimonotonicity of $f$ in $x$ always implies (P), by Volkmann [8], without any assumptions on $D$ and $f$. Simon and Volkmann [5] prove the converse in case $E$ is a Banach space. Here we give a proof for the general case, which is nevertheless much shorter.

Proof. Assume (P). Suppose $(t_0, x), (t_0, y) \in D$ and $\varphi \in K^*$ satisfy

$$x \leq y, \quad \varphi(x) = \varphi(y).$$
Fix any \( p \in \text{Int} \, K \). For each \( t_1 \geq t_0 \), define \( v, w : [t_0, t_1] \to E \) by
\[
  v(t) = x + (t - t_0)(f(t_0, x) - p), \quad w(t) = y + (t_1 - t_0) p + (t - t_0)(f(t_0, y) + p)
\]
\((t_0 \leq t \leq t_1)\). Choosing \( t_1 > t_0 \) sufficiently close to \( t_0 \), we may assume \( \text{graph} \, v, \text{graph} \, w \subset D \), and
\[
  f(t, v(t)) - f(t_0, x) + p \in \text{Int} \, K, \\
  f(t_0, y) + p - f(t, w(t)) \in \text{Int} \, K
\]
for \( t_0 \leq t \leq t_1 \). For the left sides may be regarded as continuous functions of \((t_1, t)\), which assign \( p \) to \((t_0, t_0)\). Thus we have
\[
  v'(t) \ll f(t, v(t)), \quad w'(t) \gg f(t, w(t)) \quad (t_0 \leq t \leq t_1),
\]
hence \((3)\), and clearly \( v(t_0) \ll w(t_0) \). By \((P)\), this implies \( v(t_1) \leq w(t_1) \). We conclude that \( \varphi(v(t_1)) \leq \varphi(w(t_1)) \), hence
\[
  \varphi(f(t_0, x) - p) \leq \varphi(f(t_0, y) + 2p),
\]
and, letting \( p \to 0_E, \varphi(f(t_0, x)) \leq \varphi(f(t_0, y)) \). Thus the quasimonotonicity of \( f \) is verified. \( \square \)

Remark 1. If we replace \((3)\) by \((4)\) in \((P)\) then Theorem 1 still holds, by the preceding proof. Moreover, if the three \( \ll \) signs in \((P)\) are replaced by \( \leq \) then this modified property \((P_\leq)\) also implies the quasimonotonicity of \( f \) in \( x \), under the assumptions of Theorem 1. Note that the converse is valid if, in addition, \( E \) is a normed space and \( f \) satisfies a local Lipschitz condition with respect to \( x \); cf. [9].

4. Another characterization of quasimonotonicity

To verify quasimonotonicity, it suffices to show \((2)\) only for a few \( \varphi \in K^* \) if these \( \varphi \), regarded as supporting functionals of \( K \), provide sufficiently many supporting points.

Theorem 2. Let \( E \) be a Hausdorff topological vector space preordered by a cone \( K \) with \( \text{Int} \, K \neq \emptyset \). Let \( S \subset K^* \) be such that the set
\[
  \{x \in K : \varphi(x) = 0 \text{ for some nontrivial } \varphi \in S\}
\]
is dense in the boundary of \( K \). Suppose \( G \subset E \) is open, and \( f : G \to E \) is continuous. If the implication \((2)\) holds for all \( x, y \in G \) and all \( \varphi \in S \), then \( f \) is quasimonotone increasing.

This result was motivated by Walter [11], [12, Theorem 12 XII] where the ordered Banach space \( E = \ell_\infty(A) \) is underlying; cf. Remark 2 and [8, Beispiel 3].

Proof. By Theorem 1, it suffices to verify \((P)\) rewritten for autonomous \( f \). Suppose \( v, w : [t_0, t_1] \to G (t_0 < t_1) \) are differentiable functions which satisfy \( v(t_0) \ll w(t_0) \) and
\[
  v'(t) - f(v(t)) \ll w'(t) - f(w(t)) \quad (t_0 \leq t \leq t_1).
\]
Put
\[
  u(t) = w(t) - v(t) \quad (t_0 \leq t \leq t_1).
\]
To prove \( u(t) \in \text{Int } K \) for \( t_0 \leq t \leq t_1 \), assume the contrary. Since \( u(t_0) \in \text{Int } K \), there exists \( t_2 \in (t_0, t_1] \) such that

\[
(7) \quad u(t) \in \text{Int } K \quad (t_0 \leq t < t_2)
\]

and such that \( u(t_2) \) lies on the boundary of \( K \).

Choosing some \( t_3 \in (t_0, t_2) \) sufficiently close to \( t_2 \), we may write

\[
f(w(t_2)) - f(v(t_2)) \ll \frac{u(t_2) - u(t_3)}{t_2 - t_3},
\]

by (6). Since \( G \) is open and \( f \) is continuous, there is a neighborhood \( \Delta \) of \( 0_E \) such that \( w(t_2) + d \in G \) and

\[
(8) \quad f(w(t_2) + d) - f(v(t_2)) \leq \frac{u(t_2) + d - u(t_3)}{t_2 - t_3}
\]

for all \( d \in \Delta \). We now choose \( d \in \Delta \) such that \( u(t_2) + d \) is a member of the set (5). Consequently, \( u(t_2) + d \in K \), and \( \varphi(u(t_2) + d) = 0 \) for some nontrivial \( \varphi \in S \). Substituting \( u(t_2) + d = w(t_2) + d - v(t_2) \), we conclude that

\[
\varphi(f(v(t_2))) \leq \varphi(f(w(t_2)) + d),
\]

by the restricted hypothesis (2). Thus the value of \( \varphi \) at the left side of (8) is nonnegative, hence \( 0 \leq \varphi(u(t_2) + d) - \varphi(u(t_3)) \), and therefore \( \varphi(u(t_3)) \leq 0 \). But this contradicts (7), and (P) is verified. \( \square \)

**Remark 2.** Suppose \( E \) is the real Banach space \( \ell_\infty \) ordered by its natural strict cone

\[
K = \{ x \in \ell_\infty : x_n \geq 0 \text{ for all } n \in \mathbb{N} \},
\]

where \( x = (x_n)_{n=1}^\infty \). Then \( \text{Int } K \neq \emptyset \), and the set \( S = \{ \delta_n : n \in \mathbb{N} \} \) of all coordinate functionals \( \delta_n : E \to \mathbb{R}, x \mapsto x_n, \) satisfies the hypothesis of Theorem 2.

We mention some other ordered Banach spaces consisting of functions \( A \to \mathbb{R} \), such that their natural cone has interior points and such that the corresponding set of all evaluation functionals \( x \mapsto x(\alpha) \) \((\alpha \in A) \) may be taken in Theorem 2, similarly: \( \ell_\infty(A) \) \((A \text{ any set})\), \( C^k(A) \) or \( BV(A) \) \((A \subset \mathbb{R} \text{ any compact interval having interior points, } k \in \mathbb{N}_0)\), or \( BC(A) \) \((A \text{ any topological space})\).

**Remark 3.** The following example shows that we cannot omit the continuity condition on \( f \) in Theorem 2. Put \( E = \ell_\infty \) and \( K, S \) as in Remark 2. Define \( f : E \to E \) by \( f(x) = (f_n(x))_{n=1}^\infty \) where

\[
f_n(x) = 1 \text{ if } x_n = 0 \text{ and } x \in K, \quad f_n(x) = -2 \text{ otherwise}
\]

\((n \in \mathbb{N}, x \in E)\). Then (2) holds for all \( x, y \in E \) and all \( \varphi \in S \). On the other hand, the functions \( v, w : [0, 2] \to E \),

\[
v(t) = 0_E, \quad w(t) = (1 + 1/n - t)_{n=1}^\infty \quad (0 \leq t \leq 2),
\]

satisfy \( v(0) \ll w(0) \) and \( v'(t) \ll f(v(t)) \) as well as \( w'(t) \gg f(w(t)) \) for \( 0 \leq t \leq 2 \). But even \( v(t) \leq w(t) \) is false if \( 1 < t < 2 \). Thus \( f \) cannot be quasimonotone increasing. This can also be verified by taking \( x = 0_E, \ y = (1/n)_{n=1}^\infty \) and \( \varphi \) as Banach limit in (2).
REFERENCES


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