ENUMERATIONS, COUNTABLE STRUCTURES
AND TURING DEGREES

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Abstract. It is proven that there is a family of sets of natural numbers which
has enumerations in every Turing degree except for the recursive degree. This
implies that there is a countable structure which has representations in all but
the recursive degree. Moreover, it is shown that there is such a structure which
has a recursively represented elementary extension.

1. Introduction

In the following we are concerned with countable structures in a recursive lan-
guage. Researchers have investigated how one could measure the intuitive idea of
information content of such structures and tried to relate each one of them to a
Turing degree [2], [3], [1]. The natural starting point is to look at the collection of
representations. Let $A$ be a structure. If $B$ is an isomorphic structure with universe
$\omega$, then $B$ is called a representation of $A$ (written $B \simeq A$). $D(A)$, its open diagram,
can be regarded as a subset of $\omega$ so that it has a Turing degree, and one can look
at the collection of degrees $\{\deg(D(B)) : B \simeq A\}$. A first guess for capturing the
complexity of $A$ would be to let its degree be the least element of this collection,
especially in the light of the following theorem [2, Theorem 4.1]:

Theorem 1.1 (Knight). Let $A$ be a structure in a relational language. Then ex-
actly one of the following holds: (1) For any $d > \deg(D(A))$, there is a representa-
tion $B$ of $A$ such that $\deg(D(B)) = d$. (2) There is a finite subset $S$ of the universe
of $A$ such that all permutations of the universe which fix $S$ are automorphisms of
$A$.

But this idea fails. For example, Richter [3, Theorem 3.3] shows that for any
countable order $\mathcal{C}$ which has no recursive representation the collection $\{\deg(D(B)) : B \simeq \mathcal{C}\}$ has no least element. Therefore more involved concepts have been tried to
assign degrees to structures [1].

Now for the particular problems addressed in this paper. Steffen Lempp asked
(unpublished): Does a structure with representations in all non-recursive degrees
have a recursive representation? Julia Knight asked some related questions: With
a binary relation $R \subseteq \omega^2$ associate a family of subsets of $\omega$ given by
$\mathcal{F}_R := \{R_n : n \in \omega\}$, where $R_n := \{x : (n, x) \in R\}$; say that $R$ is an enumeration of $\mathcal{F}_R$. She

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asked (also unpublished): If a family $F$ has the feature that for every non-recursive set $X$, $F$ has an enumeration recursive in $X$, does $F$ have a recursive enumeration? Similarly, if for every non-recursive set $X$, $F$ has an enumeration r.e. in $X$, does $F$ have an r.e. enumeration?

In the next section we give some positive results on Knight’s questions under extra hypotheses. In Section 3 we prove that the answer to Knight’s questions is negative, by constructing a single suitable family. This implies that Lempp’s question also has a negative answer, as is shown in the last section. The same finding is obtained in [4], but by another approach. We close by discussing the difference.

The notation is quite standard and follows [5]. All sets considered are subsets of $\omega$, the set of natural numbers. We call countable collections of subsets of $\omega$ families. Let $\varphi$ be a G"odel-numbering of the partial recursive functions (of varying arity) and $W_i := \text{Rng}(\varphi_i)$ be an enumeration of the recursively enumerable sets as usual. $W_{i,s}$ is the set of numbers enumerated in $W_i$ by stage $s$. We use recursive bijections $\langle \cdot, \cdot \rangle$, $\langle \cdot, \cdot, \cdot \rangle$ between $\omega$ and $\omega^2$, $\omega$ and $\omega^3$, respectively. We also use projections $\langle \cdot \rangle_1$ and $\langle \cdot \rangle_2$ so that, for example, $\langle (a, b) \rangle_1 = a$. Let $\langle x, A \rangle$ be the set $\{ \langle x, a \rangle : a \in A \}$. Similarly, $A_2 := \{ (a)_2 : a \in A \}$. We let $A + x = \{ a + x : a \in A \}$ and $A - x = \{ b : b + x \in A \}$.

Fix an effective listing $\Omega$ of recursively enumerable enumerations of all families with r.e. enumerations:

\[ \Omega^{(e)} := \{ (i, x) : (i, x) \in W_e \} \]

and write $\Omega_i^{(e)}$ for $\{ x : (i, x) \in W_i \}$, the $i$-th set of the enumeration $\Omega^{(e)}$. Define $C^{(e)} := \{ \Omega_i^{(e)} : i \in \omega \}$ to be the family enumerated by $\Omega^{(e)}$.

$D : \omega \to 2^\omega$ denotes the canonical enumeration of the family of finite sets; write $D_n$ for the $n$-th finite set. Then the binary predicates $x \in D_n$ and $x = |D_n|$ are recursive.

2. Positive results

In this section we give conditions on a family which ensure that the implications of Knight’s two questions hold. These are given in Theorems 2.3 and 2.4 and are due to Julia Knight. Jockusch gave a proof of Theorem 2.4 which also showed the following: There is a property which families with a recursive (r.e.) enumeration share with families that have, for all non-recursive degrees $d$, an enumeration recursive (r.e.) in $d$. These seem to be the only positive statements possible about such families.

What is this property? By a rather straightforward forcing construction it follows that if a set of natural numbers is recursive (r.e.) in all non-recursive degrees, then it is recursive (r.e.). Therefore, the members of a family are recursive (recursively enumerable) if the family has, for all non-recursive degrees $d$, an enumeration recursive (r.e.) in $d$. Hence such families $F$ are fully described by the index set

\[ I_r(F) := \{ i : (\exists A \in F)(\varphi_i = \chi_A) \} \]

or, respectively,

\[ I_{re}(F) := \{ i : (\exists A \in F)(W_i = A) \} \].
Both the index set $I_r$ of a family with a recursive enumeration and the index set $I_{re}$ of a family with a recursively enumerable enumeration are $\Sigma^0_3$ in the arithmetical hierarchy. Here is the coincidence:

**Theorem 2.1.** Let $\mathcal{F}$ be a family. If, for every non-recursive degree $d$, $\mathcal{F}$ has an enumeration recursive in $d$, then its index set $I_r(\mathcal{F})$ is $\Sigma^0_3$.

**Theorem 2.2** (Jockusch). Let $\mathcal{F}$ be a family. If, for every non-recursive degree $d$, $\mathcal{F}$ has an enumeration recursively enumerable in $d$, then its index set $I_{re}(\mathcal{F})$ is $\Sigma^0_3$.

Towards giving a sufficient condition under which the implication of her first question holds, Julia Knight defines an extension function for a family $\mathcal{F}$ to be a (possibly partial) function $f : 2^{<\omega} \to \omega$ such that if $\sigma \in 2^{<\omega}$ and there exists a set $A \in \mathcal{F}$ such that $\chi_A \supseteq \sigma$, then $\varphi_f(\sigma) = \chi_A$ for some such $A$. We mention two facts: Any family with a recursive enumeration has a partial recursive extension function, and so does a family containing all finite sets.

We prove Theorems 2.1 and 2.2 simultaneously with the following two.

**Theorem 2.3** (Knight). Let $\mathcal{F}$ be a family which has, for all non-recursive $d$, an enumeration recursive in $d$. If $\mathcal{F}$ has a partial recursive extension function, then $\mathcal{F}$ has a recursive enumeration.

**Proof (of Theorems 2.3 and 2.1).** Let $\mathcal{F}$ be a family which has, for any non-recursive set $X$, an enumeration recursive in $X$. We construct a generic set $D$, attempting to meet the following requirements and expecting to fail. Below, this failure will be exploited to prove the statements of the two theorems for $\mathcal{F}$ separately.

$R_{e^c}$: $\varphi^D_e$ is not the characteristic function of an enumeration of $\mathcal{F}$.

The set $D$ will be Cohen-generic. The set of forcing conditions is $2^{<\omega}$ and the partial order is given by $\subseteq$. We use the old-fashioned notion of a complete forcing sequence (c.f.s.), where $p_{n+1} \supseteq p_n$, with $p_{n+1}$ entering the $n$th dense set in some countable collection. $D$ is the set with characteristic function $\bigcup_{n<\omega} p_n$.

Fix a condition $p \in 2^{<\omega}$ and $e \in \omega$. We consider the following four possibilities for $p$ and $e$, showing how extensions of $p$ may force satisfaction of $R_e$ in each case.

P1. For some $q \supseteq p$, $n, x \in \omega$, $q \vdash \varphi^D_e(n, x) \upharpoonright \neq 0, 1$, or $q \vdash \varphi^D_e(n, x) \upharpoonright 0, 1$.

We include $q$ in the c.f.s., thereby satisfying $R_e$.

P2. For some $q \supseteq p$ and some $n$, for all $q' \supseteq q$ there exist $x$ and $r_0, r_1 \supseteq q'$ such that $r_i \vdash \varphi^D_e(n, x) = i$.

For each $A \in \mathcal{F}$ the set

$$D^1_A := \{ r : r \supseteq q \Rightarrow (\exists x)(r \vdash \varphi^D_e(n, x) \neq \chi_A(x)) \}$$

is dense. We add $q$ to the c.f.s. and enter the sets $D^1_A$. Then the requirement $R_e$ is satisfied.

We write $q \vdash E_n = A$ if for all $x$ and all $q' \supseteq q$ there is an $r \supseteq q'$ such that $r \vdash \varphi^D_e(n, x) \upharpoonright 0, 1 = \chi_A(x)$. Note that for $q$ and $n$ there is at most one set $A$ such that $q \vdash E_n = A$.

P3. For some $q \supseteq p$, $n \in \omega$ and $B \notin \mathcal{F}$ we have $q \vdash E_n = B$.

By putting $q$ into the c.f.s., we meet $R_e$.

P4. Not P2 but there exist $A \in \mathcal{F}$ and $q \supseteq p$ such that for all $q' \supseteq q$ and all $n$, if $q' \vdash E_n = B$ then $A \neq B$. 


Since $P2$ does not hold, the set

$$D^2_n := \{ q' : q' \supseteq q \Rightarrow (\exists B)(q' \vdash E_n = B) \}$$

is dense for each $n$. By including $q$ in the c.f.s. and entering the sets $D^2_n$, we meet $R_e$.

If for all $p \in 2^{<\omega}$ and $e \in \omega$ one of the cases $P1, \ldots, P4$ holds, then the forcing construction yields a generic (and so non-recursive) set $D$, in which $F$ has no recursive enumeration, contrary to the assumption on $F$.

So let $p \in 2^{<\omega}$ and $e \in \omega$ be such that none of the cases $P1, \ldots, P4$ hold. It follows that if $q \vdash E_n = A$, then $A \in F$ for all $q \supseteq p$, $n$ and $A \subseteq \omega$. Moreover, all elements of $F$ occur in this way.

We first complete the proof of Theorem 2.3. Let $f$ be a partial recursive extension function for $F$. Fix $n \in \omega$ and $p \subseteq q \in 2^{<\omega}$. Say that $\sigma \in 2^{<\omega}$ has a $q$-computation if there is an $r \geq q$ such that $r \vdash \varphi^D_\sigma(n, x) = \sigma(x)$ for all $x \in \text{dom}(\sigma)$. We make the following observations:

- $q \vdash E_n = A$ and $q' \supseteq q$ implies $q' \vdash E_n = A$.
- The set $\{ \sigma \in 2^{<\omega} : \sigma \text{ has a } q\text{-computation} \}$ is r.e. (uniformly in $q$).
- By definition, $q \vdash E_n = A$ if and only if any $\sigma \in 2^{<\omega}$ has a $q$-computation if and only if $\sigma \subseteq \chi_A$.
- Since $p, e$ do not satisfy $P1$, for any $q' \supseteq q$ there is a $\sigma \in 2^{<\omega}$ which has a $q'$-computation.
- The previous two items imply that the predicate $q \supseteq p \Rightarrow (\forall A)(q \not\vdash E_n = A)$ in $q$ and $n$ is r.e.
- Since $p, e$ do not satisfy $P3$, if $q \supseteq p$ and $q \not\vdash E_n = A$ for any $A \in F$, then for any $\sigma \in 2^{<\omega}$ with a $q$-computation there is a set $A \in F$ such that $\sigma \subseteq \chi_A$.

We form a recursive enumeration $R$ of $F$, using pairs $(q, n)$ as indices, where $q \in 2^{<\omega}$ with $q \supseteq p$ and $n \in \omega$. Since there is a recursive bijection between $P \times \omega$ and $\omega$, where $P = \{ q \in 2^{<\omega} : q \supseteq p \}$, this suffices. Let $(\sigma_q^{(n)})_{n \in \omega}$ be an effective enumeration of $\{ \sigma \in 2^{<\omega} : \sigma \text{ has a } q\text{-computation} \}$ by the second observation. We choose this enumeration so that, additionally, for every $\sigma$ with a $q$-computation there are infinitely many $i$ such that $\sigma_q^{(i)} = \sigma$. Define a partial recursive function $a_q$ by $a_q(0) := 0$ and $a_q(n + 1) := \mu m > n.\sigma_q^{(m)} \supseteq \sigma_q^{(n)}$. By the fourth observation, let $h : \omega \rightarrow 2^{<\omega} \times \omega$ be a recursive function with range $\{ (q, n) : q \supseteq p \text{ and } (\forall A)(q \not\vdash E_n = A) \}$. $R(q, n)$ is defined by

$$R(q, n) := \begin{cases} \bigcup_{0 \leq i \leq m} \sigma_q^{(a_q(i))} \cup \varphi_f(\sigma_q^{(a_q(m))}) & \text{if } m \text{ is least such that } h(m) = (q, n), \\ \bigcup_{i \in \omega} \sigma_q^{(a_q(i))} & \text{otherwise.} \end{cases}$$

Fix $q \supseteq p$ and $n \in \omega$. By definition of the functions $h$, there is $m$ satisfying the first case if and only if $(\forall A)(q \not\vdash E_n = A)$. If $q \vdash E_n = A$, then the choice of the enumeration $(\sigma_q^{(n)})_{n \in 2^{<\omega} \cap \omega}$ and the function $a$ guarantees $R(q, n) = A$. If $q \not\vdash E_n = A$ for any $A$, then by the fact that $f$ is an extension function for $F$, it follows that $R(q, n) \in F$.

Since for every $A \in F$ there are $q$ and $n$ such that $q \vdash E_n = A$, it follows that $R$ is an enumeration of $F$, and Theorem 2.3 is proved.
We write $q$ that $r_i$ to $q$ for all $i$. We use the same forcing notion as was used for the previous proof. Fix $e$ such that the characteristic function of a set, is $\Pi_2^0$ so that $I_e (F)$ is $\Sigma_3^0$. 

**Theorem 2.4** (Knight). Let $F$ be a family such that for all non-recursive $X$, $F$ has an enumeration r.e. in $X$. If $F$ contains all finite sets, then $F$ has an r.e. enumeration.

**Proof of Theorems 2.4 and 2.2.** As above, we construct a generic set $D$, attempting to meet the following requirements and expecting to fail.

$R_e$. $W_e^D$ is not an enumeration of $F$.

We use the same notion as was used for the previous proof. Fix $p \in 2^{<\omega}$ and $e \in \omega$. We consider the following three cases.

C1. For some $q \supseteq p$ and some $n \in \omega$, for all $q' \supseteq q$, there exist $x \in \omega$ and $r_0, r_1 \supseteq q'$ such that $r_0 \vdash (n, x) \in W_e^D$ and $r_1 \vdash (n, x) \notin W_e^D$.

For each $A \in F$ the set

$$D_A^1 := \{ r : r \supseteq q \Rightarrow (\exists x)[(r \vdash (n, x) \in W_e^D \land x \notin A) \lor (r \vdash (n, x) \notin W_e^D \land x \in A)] \}$$

is dense. We put $q$ into the c.f.s. and enter the sets $D_A^1$. Requirement $R_e$ is satisfied.

We write $q \vdash E_n = A$ if for all $x$, if $x \in A$, then for all $q' \supseteq q$ there is $r \supseteq q'$ such that $r \vdash (n, x) \in W_e^D$, and if $x \notin A$ then $q' \vdash (n, x) \notin W_e^D$.

C2. Case C1 does not hold, but $q \vdash E_n = B$ for some $q \supseteq p$, $n \in \omega$ and $B \notin F$.

Requirement $R_e$ is met by putting $q$ in the c.f.s.

C3. Case C1 does not hold, but there exists $A \in F$ such that for all $q' \supseteq q$ and all $n$, if $q' \vdash E_n = B$, then $A \neq B$.

Since Case C1 does not hold, the set

$$D_n^2 := \{ q' : q' \supseteq q \Rightarrow (\exists B)(q' \vdash E_n = B) \}$$

is dense for all $n$. We meet the requirement $R_e$ by including $q$ in the c.f.s. and entering all sets $D_n^2$.

If for all $p \in 2^{<\omega}$ and $e \in \omega$ one of the cases C1,C2, or C3 holds, then the forcing construction yields a generic (and hence non-recursive) set $D$, in which $F$ has no r.e. enumeration. This contradicts the assumption of both Theorem 2.2 and Theorem 2.4. So let $p \in 2^{<\omega}$ and $e \in \omega$ be such that none of the cases C1,C2,C3 holds. We show that the index set $I_{re}$ of $F$ is $\Sigma_3^0$. As in the proof of Theorem 2.3, for all $q \supseteq p$, $n$, and $A \subseteq \omega$, if $q \vdash E_n = A$, then $A \in F$, and all members of $F$ occur in this way. Therefore we have

$$\{ i : W_i \in F \} = \{ i : (\exists q \supseteq p)(\exists n)(q \vdash E_n = W_i) \}.$$ 

The relation “$q \vdash E_n = W_i$” is $\Pi_2^0$, and so the index set of $F$ is $\Sigma_3^0$. This completes the proof of Theorem 2.2. To complete the proof of Theorem 2.4, note that since $F$ includes all finite sets and only contains r.e. sets, by a theorem of Yates [7, Theorem 8], $F$ has an r.e. enumeration.
3. A family of finite sets

In this section we first define a family $\mathcal{C}$ which has no r.e. enumeration. Then we show that for every non-recursive set $X$ there is an enumeration of $\mathcal{C}$ which is recursive in $X$, and finally that every non-recursive degree contains an enumeration of $\mathcal{C}$. This corrects an earlier statement in [6, p 187]. In particular, we apologize for connecting the error with Martin Kummer.

Let $r$ be the partial recursive function defined by

$$ r(e) := (\mu(i, x, s). (e, x) \in \Omega^{(e)}_{i,s}), $$

Informally, the value of $r(e)$ is the first index $i$ to be found such that there is a number of the form $\langle e, x \rangle \in \Omega^{(e)}_{i,s}$; if there is no such $i$ then $r(e)$ is not defined. Let the family $\mathcal{C}$ be defined by

$$ \mathcal{C} := \{ \langle e, A \rangle : A \text{ is finite, } e \in \omega \} - \{ (e, \omega) \cap \Omega^{(e)}_{r(e)} : r(e) \downarrow \}. $$

$\mathcal{C}$ does not have an r.e. enumeration; for suppose $\Omega^{(e_0)}$ is an enumeration of $\mathcal{C}$. Then $r(e_0)$ is defined, and the set $\Omega^{(e_0)}_{r(e_0)} = (e, \omega) \cap \Omega^{(e_0)}_{r(e_0)}$ is not a member of $\mathcal{C}$.

Let $X$ be an arbitrary non-recursive set. To see how to construct an enumeration $S^X$ of $\mathcal{C}$ such that $S^X \preceq_T X$ we use the following lemma.

**Lemma 3.1.** Uniformly in $i$ and recursively in $X$ there is a finite set $A^X_i \preceq_T X$ such that $W_i \neq A^X_i$.

Let $g$ be a partial recursive function such that $g(e) \downarrow$ if and only if $r(e) \downarrow$ and if $r(e) \downarrow$ then $W_{g(e)} = (e, \omega) \cap \Omega^{(e)}_{r(e)}$. Let $h$ be a partial recursive function such that $h(e, a) \downarrow$ if and only if $g(e) \downarrow$ and if $g(e) \downarrow$ then $W_{h(e,a)} = W_{g(e)} - a$. Define

$$ S^X_{(n,t,e)} := (e, B^X(n, t, e)), $$

where

$$ B^X(n, t, e) := \begin{cases} 
D_n \cup (s_0 + A^X_{h(e,s_0)}) & \text{if there is } s > t, \max(D_n) + 1 \\
D_n & \text{such that } g_s(e) \downarrow \text{ and } W_{g(e),s} = D_n, \\
d_0 & \text{and } s_0 \text{ is the least such,} \\
D_n & \text{otherwise.}
\end{cases} $$

By the lemma, $S^X$ is recursive in $X$. We claim that $S^X$ is an enumeration of $\mathcal{C}$.

"$\subseteq"$. First of all, it follows from the lemma that all sets $B^X(n, t, e)$ are finite and so the family enumerated by $S^X$ is contained in $\{ \langle e, A \rangle : A \text{ is finite} \}$.

Suppose $r(e) \downarrow$ and $S^X_{(n,t,e)} = \langle e, \omega \rangle \cap \Omega^{(e)}_{r(e)}$. It follows that $B^X(n, t, e) = W_{g(e)}$. There has to be a stage $s > t, \max(D_n) + 1$ such that $W_{g(e),s} = D_n$, because otherwise $B^X(n, t, e) = D_n \neq W_{g(e)}$, a contradiction. Let $s_0$ be the least such $s$. Then $B^X(n, t, e) = D_n \cup (s_0 + A^X_{h(e,s_0)}) = W_{g(e)}$, so that $A^X_{h(e,s_0)} = W_{h(e,s_0)}$, a contradiction.

"$\supseteq"$. Let $e, n \in \omega$ such that $C = \langle e, D_n \rangle \in \mathcal{C}$. We want to find a number $z$ such that $S^X_z = C$. Let $t$ be such that $W_{g(e),s} \neq D_n$ for all $s > t$. (If there is no such $t$ then $C \notin \mathcal{C}$.) By definition, $B^X(n, t, e) = D_n$, so that $S^X_{(n,t,e)} = C$.

We have constructed an enumeration of $\mathcal{C}$ which is recursive in $X$. Simple coding is sufficient to obtain an enumeration $T$-equivalent to $X$: Choose $A, B \in \mathcal{C}$ so that

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\[ A - B \neq \emptyset. \] Define another enumeration of \( C, \)
\[
P^X_{2n} := S^X_n, \quad P^X_{2n+1} := \begin{cases} A & \text{if } n \in X, \\ B & \text{if } n \notin X. \end{cases}
\]
Clearly, \( P^X \) is recursive in \( X \) and enumerates \( C. \) Let \( x \in A - B. \) Then \( z \in X \) if and only \( x \in P^X_{2z+1}, \) so that \( X \) is recursive in \( P^X. \)

**Proof of Lemma 3.1.** Let \( y, n : \omega^2 \to \omega \) be two recursive one-one functions such that their ranges are disjoint and cover \( \omega. \) Let
\[
\alpha(s, x) := \begin{cases} y(s, x) & \text{if } x \in X, \\ n(s, x) & \text{otherwise,} \end{cases}
\]
so that \( \alpha \leq_T X. \) We construct the set \( A^X_i \) in stages uniformly in \( i \) and recursively in \( X. \) In the course of the construction, numbers \( a_x \) may become defined.

Stage 0. Set \( x := 0. \) \( A^X_{i,0} \) is empty.
Stage \( s + 1. \) If \( W_{i, s} \neq A^X_{i, s}, \) pass to the next stage. Otherwise enumerate
\[
\alpha(s, x) \in A^X_i, \quad \text{let } a_x := s \text{ and increase } x \text{ by one.}
\]
End of construction.

The set \( A^X_i \) is (by construction) r.e. in \( X. \) It is also recursive in \( X: \) By inspection, \( A^X_i \) only contains numbers \( \alpha(a, b). \) If \( y(s, x) \in A^X_i, \) then \( y(s, x) \in A^X_{i,s+1} \) and the same holds for numbers \( n(s, x). \) This together with the choice of \( y \) and \( n \) is sufficient.

**Claim 1.** The set \( A^X_i \) is finite.

Suppose \( A^X_i \) is infinite. During the construction of \( A^X_i \) infinitely many numbers \( a_x \) are defined. By induction on \( x, \) it follows that \( A^X_{i,a_x} = \{ \alpha(a_j, j) : j \leq x \}, \) and therefore \( W_{i,a_x} = \{ \alpha(a_j, j) : j < x \} \) for all \( x \in \omega. \) Thus, \( W_i = \{ \alpha(a_x, x) : x \in \omega \}. \) Now, \( x \in X \) if and only if \( (\exists t)(y(t, x) \in W_i), \) and \( x \notin X \) if and only if \( (\exists t)(n(t, x) \in W_i). \) This means that \( X \) is recursive, a contradiction.

**Claim 2.** \( A^X_i \) is different from \( W_i. \)

By the previous claim it is sufficient to consider the case when \( W_i \) is finite. Let \( s_0 \) be the least number such that \( W_{i,s_0} = W_i. \) Then either \( W_{i,s_0} \neq A^X_{i,s_0,0}, \) in which case \( W_{i,s_0} \neq A^X_{i,s_0} = A^X_i \); or \( W_{i,s_0} = A^X_{i,s_0}, \) so that at stage \( s_1 = s_0 + 1 \) a new number is enumerated in \( A^X_i, \) whence \( W_i = W_{i,s_1} = A^X_{i,s_0} \neq A^X_i. \)

Lemma 3.1 is proved.

4. Application to Lempp’s Question

Let \( F \) be a family. With \( F \) we associate the following countable structure \( A_F \)
in the language \( L = (S, Z, I), \) where \( S \) is a binary predicate symbol and \( Z \) and \( I \) are unary predicate symbols. The universe of \( A_F \) is \( F \times \omega \times \omega. \) For every \( A \in F, \)
set \( Z((A, x, 0)) \) and \( S((A, x, n), (A, x, n + 1)). \) Set \( I((A, x, n)) \) if and only if \( n \in A. \)
Thus, countably many \( S \)-chains \( (A, x, 0), (A, x, 1), (A, x, 2), \ldots \) are associated with every \( A \in F, \) and in every chain \( I \) holds of the \( n \)-th member if \( n \in A. \)

**Theorem 4.1.** Let \( d \) be a Turing degree and \( F \) be a family. \( F \) then has an enumeration recursive in \( d \) if and only if the structure \( A_F \) has a representation recursive in \( d. \)
Proof. “⇒”. Let $Q^X$ be an enumeration of $F$ which is recursive in $X \in d$. Define $R^X$ to be another $X$-recursive enumeration of $F$ by $R^X_{(n,i)} := Q^X_i$. A representation of $\mathfrak{A}_F$ is given by $\mathfrak{B}$, where $Z_{\mathfrak{B}}(x,0)$, $S_{\mathfrak{B}}(x,n)$, $\langle x, n+1 \rangle$, and $I_{\mathfrak{B}}(x,n)$ if and only if $x \in R^X_n$. These predicates are recursive in $R^X$, and therefore $D(\mathfrak{B})$ is recursive in $\mathfrak{X}$.

“⇐”. Let $\mathfrak{B}$ be a representation of $\mathfrak{A}_F$ whose open diagram is recursive in $X \in d$. The set $Z := \{ x : Z_{\mathfrak{B}}(x) \}$ is recursive in $X$. With each $x \in Z$ we associate the set $S_x := \{ n : I_{\mathfrak{B}}(f(n)(x)) \}$, where $f(x)$ is the unique $y$ such that $S_{\mathfrak{B}}(x,y)$, and $f^{(n)}(x)$ denotes the $n$-fold application of $f$ to $x$. The sets $S_x$ are uniformly in $x \in Z$ recursive in $X$, and, by definition of $\mathfrak{A}_F$, $F = \{ S_x : x \in Z \}$. This suffices.

Corollary 4.2 (Slaman [4]). There is a structure which has representations only in the non-recursive degrees.

Proof. Apply the theorem to the family $C$ defined in the previous section to obtain representations of $\mathfrak{A}_C$ below any non-recursive $T$-degree. Apply Theorem 1.1 to obtain representations in all non-recursive $T$-degrees.

Slaman remarked (private communication) that the construction given in [4] yields a structure which is not elementarily equivalent to any recursively represented structure. Informally, the reason for this is as follows. Essentially, the construction proceeds in such a way that the final outcome of the actions taken to diagonalize against the recursive representations can be read off the theory of the structure (which is called $\mathfrak{M}$):

He writes ([4, Section 2.1]): “We will ensure that either $R^{-1}(T_i)_{\mathfrak{M}} = \emptyset$, or $\langle T_i, <_L \upharpoonright T_i \rangle_{\mathfrak{M}}$ is not isomorphic to $\langle T_i, <_L \upharpoonright T_i \rangle_{\mathfrak{M}}$, or there is a $p$ in $R^{-1}(T_i)_{\mathfrak{M}}$ such that $\zeta(p)_{\mathfrak{M}}$ is not maximal, or there is a $p$ in $R^{-1}(T_i)_{\mathfrak{M}}$ such that $\zeta(p)_{\mathfrak{M}}$ is infinite. Since none of these disjuncts apply to $\mathfrak{M}$, we will thus ensure that $\mathfrak{M}$ has no recursive presentation.”

Fix $i \in \omega$. The first and third disjunct can be directly formalized in the language $\mathfrak{L}$ of the structure. If the second disjunct holds, then the construction [4, Section 2.2.1] yields a tree $\langle T_i, <_L \upharpoonright T_i \rangle_{\mathfrak{M}}$ which is finite. Therefore this tree is described by a sentence in the theory of $\mathfrak{M}$. The fourth disjunct also cannot be formalized in $\mathfrak{L}$, but, provided the first three disjuncts are not true, the strategy used in [4, Section 2.2.2] results in

$$(\exists p)(R(p,s^i(0)) \land (\forall x)(R(p,x) \rightarrow (\exists y)(x <_T y \land R(p,y))))$$

being true in $\mathfrak{R}_i$. This is a formula in $\mathfrak{L}$, and (by the same strategy) not true in $\mathfrak{M}$. Hence there is no recursively represented structure which is elementarily equivalent to $\mathfrak{M}$. The structure obtained from the family $C$ by Theorem 4.1 is of a different kind:

Theorem 4.3. There is a structure which has representations only in the non-recursive degrees and has a recursively represented elementary extension.

Proof. Let us look at the following family:

$$D := \{ (e,A) : A \text{ is finite}, e \in \omega \}.$$ 

Obviously, $D$ has a recursive enumeration, and $\mathfrak{A}_D$ has a recursive representation. At the same time, the structure $\mathfrak{A}_C$ from the proof of Corollary 4.2 is contained in $\mathfrak{A}_D$, and they are elementarily equivalent:
It suffices to show that for any formula \((\exists x)(\phi)\) in variables \(x_1, \ldots, x_n\), if 
\[ \mathfrak{A}_D \models (\exists x)(\phi)[x_1 := c_1, \ldots, x_n := c_n] \]
and \(c_1, \ldots, c_n \in \mathfrak{A}_C\), then there is \(c \in \mathfrak{A}_C\) such that 
\[ \mathfrak{A}_D \models \phi[x := c, x_1 := c_1, \ldots, x_n := c_n]. \]
Note that if \(\mathfrak{A}_D \models (\exists x)(\phi)[x_1 := c_1, \ldots, x_n := c_n]\), then there is a \(d \in \mathfrak{A}_D\) such that either 
\[ \mathfrak{A}_D \models \phi[x := d, x_1 := c_1, \ldots, x_n := c_n] \]
\[ \wedge \neg S(d, c_1) \wedge \neg S(c_1, d) \wedge \ldots \wedge \neg S(d, c_n) \wedge \neg S(c_n, d), \]
or 
\[ \mathfrak{A}_D \models \phi[x := d, x_1 := c_1, \ldots, x_n := c_n] \]
\[ \wedge (S(d, c_1) \lor S(c_1, d) \lor \ldots \lor S(d, c_n) \lor S(c_n, d)). \]
In the former case, choose \(c\) from \(\mathfrak{A}_C\) outside of the \(S\)-chains of \(\mathfrak{A}_C\) which \(c_1, \ldots, c_n\) belong to such that it satisfies \(Z\) and \(I\) in the same way \(d\) does. In the latter case, \(d\) is already part of an \(S\)-chain which is contained in \(\mathfrak{A}_C\), and so an element of \(\mathfrak{A}_C\). □

References