ENumerations, Countable Structures and Turing Degrees

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Abstract. It is proven that there is a family of sets of natural numbers which has enumerations in every Turing degree except for the recursive degree. This implies that there is a countable structure which has representations in all but the recursive degree. Moreover, it is shown that there is such a structure which has a recursively represented elementary extension.

1. Introduction

In the following we are concerned with countable structures in a recursive language. Researchers have investigated how one could measure the intuitive idea of information content of such structures and tried to relate each one of them to a Turing degree [2], [3], [1]. The natural starting point is to look at the collection of representations. Let $A$ be a structure. If $B$ is an isomorphic structure with universe $\omega$, then $B$ is called a representation of $A$ (written $B \simeq A$). $D(B)$, its open diagram, can be regarded as a subset of $\omega$ so that it has a Turing degree, and one can look at the collection of degrees $\{\text{deg}(D(B)) : B \simeq A\}$. A first guess for capturing the complexity of $A$ would be to let its degree be the least element of this collection, especially in the light of the following theorem [2, Theorem 4.1]:

Theorem 1.1 (Knight). Let $A$ be a structure in a relational language. Then exactly one of the following holds: (1) For any $d > \text{deg}(D(A))$, there is a representation $B$ of $A$ such that $\text{deg}(D(B)) = d$. (2) There is a finite subset $S$ of the universe of $A$ such that all permutations of the universe which fix $S$ are automorphisms of $A$.

But this idea fails. For example, Richter [3, Theorem 3.3] shows that for any countable order $C$ which has no recursive representation the collection $\{\text{deg}(D(B)) : B \simeq C\}$ has no least element. Therefore more involved concepts have been tried to assign degrees to structures [1].

Now for the particular problems addressed in this paper. Steffen Lempp asked (unpublished): Does a structure with representations in all non-recursive degrees have a recursive representation? Julia Knight asked some related questions: With a binary relation $R \subseteq \omega^2$ associate a family of subsets of $\omega$ given by $F_R := \{R_n : n \in \omega\}$, where $R_n := \{x : (n, x) \in R\}$; say that $R$ is an enumeration of $F_R$. She...
asked (also unpublished): If a family $\mathcal{F}$ has the feature that for every non-recursive set $X$, $\mathcal{F}$ has an enumeration recursive in $X$, does $\mathcal{F}$ have a recursive enumeration? Similarly, if for every non-recursive set $X$, $\mathcal{F}$ has an enumeration r.e. in $X$, does $\mathcal{F}$ have an r.e. enumeration?

In the next section we give some positive results on Knight’s questions under extra hypotheses. In Section 3 we prove that the answer to Knight’s questions is negative, by constructing a single suitable family. This implies that Lempp’s question also has a negative answer, as is shown in the last section. The same finding is obtained in [4], but by another approach. We close by discussing the difference.

The notation is quite standard and follows [5]. All sets considered are subsets of $\omega$, the set of natural numbers. We call countable collections of subsets of $\omega$ families.

Let $\varphi$ be a Gödel-numbering of the partial recursive functions (of varying arity) and $W_i := \text{Rng}(\varphi_i)$ be an enumeration of the recursively enumerable sets as usual. $W_{i,s}$ is the set of numbers enumerated in $W_i$ by stage $s$. We use recursive bijections $\langle \cdot, \cdot \rangle$, $\langle \cdot, \cdot, \cdot \rangle$ between $\omega$ and $\omega^2$, $\omega$ and $\omega^3$, respectively. We also use projections $\langle . \rangle_1$ and $\langle . \rangle_2$ so that, for example, $\langle (a, b) \rangle_1 = a$. Let $\langle x, A \rangle$ be the set $\{\langle x, a \rangle : a \in A \}$. Similarly, $A := \{\langle a, 2 : a \in A \}$. We let $A + x = \{a + x : a \in A \}$ and $A - x = \{b : b + x \in A \}$.

Fix an effective listing $\Omega$ of recursively enumerable enumerations of all families with r.e. enumerations:

$$\Omega^{(e)} := \{(i, x) : (i, x) \in W_e\},$$

and write $\Omega_i^{(e)}$ for $\{x : (i, x) \in W_e\}$, the $i$-th set of the enumeration $\Omega^{(e)}$. Define $C^{(e)} := \Omega_i^{(e)} : i \in \omega$ to be the family enumerated by $\Omega^{(e)}$.

$D : \omega \rightarrow 2^{\omega}$ denotes the canonical enumeration of the family of finite sets; write $D_n$ for the $n$-th finite set. Then the binary predicates $x \in D_n$ and $x = |D_n|$ are recursive.

### 2. Positive results

In this section we give conditions on a family which ensure that the implications of Knight’s two questions hold. These are given in Theorems 2.3 and 2.4 and are due to Julia Knight. Jockusch gave a proof of Theorem 2.4 which also showed the following: There is a property which families with a recursive (r.e.) enumeration share with families that have, for all non-recursive degrees $d$, an enumeration recursively r.e. (r.e.) in $d$. These seem to be the only positive statements possible about such families.

What is this property? By a rather straightforward forcing construction it follows that if a set of natural numbers is recursive (r.e.) in all non-recursive degrees, then it is recursive (r.e.). Therefore, the members of a family are recursive (recursively enumerable) if the family has, for all non-recursive degrees $d$, an enumeration recursive (r.e.) in $d$. Hence such families $\mathcal{F}$ are fully described by the index set

$$I_r(\mathcal{F}) := \{i : (\exists A \in \mathcal{F})(\varphi_i = \chi_A)\}$$

or, respectively,

$$I_{re}(\mathcal{F}) := \{i : (\exists A \in \mathcal{F})(W_i = A)\}.$$
Both the index set $I_r$ of a family with a recursive enumeration and the index set $I_{r e}$ of a family with a recursively enumerable enumeration are $\Sigma^0_3$ in the arithmetical hierarchy. Here is the coincidence:

**Theorem 2.1.** Let $\mathcal{F}$ be a family. If, for every non-recursive degree $d$, $\mathcal{F}$ has an enumeration recursive in $d$, then its index set $I_r(\mathcal{F})$ is $\Sigma^0_3$.

**Theorem 2.2** (Jockusch). Let $\mathcal{F}$ be a family. If, for every non-recursive degree $d$, $\mathcal{F}$ has an enumeration recursively enumerable in $d$, then its index set $I_{re}(\mathcal{F})$ is $\Sigma^0_3$.

Towards giving a sufficient condition under which the implication of her first question holds, Julia Knight defines an extension function for a family $\mathcal{F}$ to be a (possibly partial) function $f : 2^{<\omega} \to \omega$ such that if $\sigma \in 2^{<\omega}$ and there exists a set $A \in \mathcal{F}$ such that $\chi_A \geq \sigma$, then $\varphi_f(\sigma) = \chi_A$ for some such $A$. We mention two facts: Any family with a recursive enumeration has a partial recursive extension function, and so does a family containing all finite sets.

We prove Theorems 2.1 and 2.2 simultaneously with the following two.

**Theorem 2.3** (Knight). Let $\mathcal{F}$ be a family which has, for all non-recursive $d$, an enumeration recursive in $d$. If $\mathcal{F}$ has a partial recursive extension function, then $\mathcal{F}$ has a recursive enumeration.

**Proof (of Theorems 2.3 and 2.1).** Let $\mathcal{F}$ be a family which has, for any non-recursive set $X$, an enumeration recursive in $X$. We construct a generic set $D$, attempting to meet the following requirements and expecting to fail. Below, this failure will be exploited to prove the statements of the two theorems for $\mathcal{F}$ separately.

- $R_e$: $\varphi^D_e$ is not the characteristic function of an enumeration of $\mathcal{F}$.

The set $D$ will be Cohen-generic. The set of forcing conditions is $2^{<\omega}$ and the partial order is given by $\subseteq$. We use the old-fashioned notion of a complete forcing sequence (c.f.s.), where $p_{n+1} \supseteq p_n$, with $p_{n+1}$ entering the $n$th dense set in some countable collection. $D$ is the set with characteristic function $\bigcup_{n=\omega} P_n$.

Fix a condition $p \in 2^{<\omega}$ and $e \in \omega$. We consider the following four possibilities for $p$ and $e$, showing how extensions of $p$ may force satisfaction of $R_e$ in each case.

P1. For some $q \supseteq p, n, x \in \omega$, $q \vdash \varphi^D_e(n, x) \neq 0, 1$, or $q \vdash \varphi^D_e(n, x) \uparrow$.

We include $q$ in the c.f.s., thereby satisfying $R_e$.

P2. For some $q \supseteq p$ and some $n$, for all $q' \supseteq q$ there exist $x$ and $r_0, r_1 \supseteq q'$ such that $r_0 \vdash \varphi^D_e(n, x) = i$.

For each $A \in \mathcal{F}$ the set

$$D^1_A := \{ r : r \supseteq q \Rightarrow (\exists x)(r \vdash \varphi^D_e(n, x) \neq \chi_A(x)) \}$$

is dense. We add $q$ to the c.f.s. and enter the sets $D^1_A$. Then the requirement $R_e$ is satisfied.

We write $q \vdash E_n = A$ if for all $x$ and all $q' \supseteq q$ there is an $r \supseteq q'$ such that $r \vdash \varphi^D_e(n, x) = \chi_A(x)$. Note that for $q$ and $n$ there is at most one set $A$ such that $q \vdash E_n = A$.

P3. For some $q \supseteq p, n \in \omega$ and $B \notin \mathcal{F}$ we have $q \vdash E_n = B$.

By putting $q$ into the c.f.s., we meet $R_e$.

P4. Not P2 but there exist $A \in \mathcal{F}$ and $q \supseteq p$ such that for all $q' \supseteq q$ and all $n$, if $q' \vdash E_n = B$ then $A \neq B$. 

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Since P2 does not hold, the set

\[ D^2_n := \{ q' : q' \supseteq q \Rightarrow (\exists B)(q' \vdash E_n = B) \} \]

is dense for each \( n \). By including \( q \) in the c.f.s. and entering the sets \( D^2_n \), we meet \( R_e \).

If for all \( p \in 2^{<\omega} \) and \( e \in \omega \) one of the cases P1, …, P4 holds, then the forcing construction yields a generic (and so non-recursive) set \( D \), in which \( \mathcal{F} \) has no recursive enumeration, contrary to the assumption on \( \mathcal{F} \).

So let \( p \in 2^{<\omega} \) and \( e \in \omega \) be such that none of the cases P1, …, P4 hold. It follows that if \( q \vdash E_n = A \), then \( A \in \mathcal{F} \) for all \( q \supseteq p \), \( n \) and \( A \subseteq \omega \). Moreover, all elements of \( \mathcal{F} \) occur in this way.

We first complete the proof of Theorem 2.3. Let \( f \) be a partial recursive extension function for \( \mathcal{F} \). Fix \( n \in \omega \) and \( p \subseteq q \in 2^{<\omega} \). Say that \( \sigma \in 2^{<\omega} \) has a \( q \)-computation if there is an \( r \supseteq q \) such that \( r \vdash \varphi^D_e(n, x) = \sigma(x) \) for all \( x \in \text{dom}(\sigma) \). We make the following observations:

- \( q \vdash E_n = A \) and \( q' \supseteq q \) implies \( q' \vdash E_n = A \).
- The set \( \{ \sigma \in 2^{<\omega} : \sigma \text{ has a } q \text{-computation} \} \) is r.e. (uniformly in \( q \)).
- By definition, \( q \vdash E_n = A \) if and only if any \( \sigma \in 2^{<\omega} \) has a \( q \)-computation if and only if \( \sigma \subseteq \chi_A \).
- Since \( p, e \) do not satisfy P1, for any \( q' \supseteq q \) there is a \( \sigma \in 2^{<\omega} \) which has a \( q' \)-computation.
- The previous two items imply that the predicate \( q \supseteq p \Rightarrow (\forall A)(q \not\vdash E_n = A) \) in \( q \) and \( n \) is r.e.
- Since \( p, e \) do not satisfy P3, if \( q \supseteq p \) and \( q \not\vdash E_n = A \) for any \( A \in \mathcal{F} \), then for any \( \sigma \in 2^{<\omega} \) with a \( q \)-computation there is a set \( A \in \mathcal{F} \) such that \( \sigma \subseteq \chi_A \).

We form a recursive enumeration \( R \) of \( \mathcal{F} \), using pairs \( (q, n) \) as indices, where \( q \in 2^{<\omega} \) with \( q \supseteq p \) and \( n \in \omega \). Since there is a recursive bijection between \( P \times \omega \) and \( \omega \), where \( P = \{ q \in 2^{<\omega} : q \supseteq p \} \), this suffices. Let \( (\sigma^{(n)}_q)_{n \in \omega} \) be an effective enumeration of \( \{ \sigma \in 2^{<\omega} : \sigma \text{ has a } q \text{-computation} \} \) by the second observation. We choose this enumeration so that, additionally, for every \( \sigma \) with a \( q \)-computation there are infinitely many \( i \) such that \( \sigma^{(i)}_q = \sigma \). Define a partial recursive function \( a_q \) by \( a_q(0) := 0 \) and \( a_q(n+1) := \mu m > n. \sigma^{(m)}_q \supseteq \sigma \) with \( (\forall A)(q \not\vdash E_n = A) \).

Here, \( R_{(q,n)} \) is defined by

\[
R_{(q,n)} := \begin{cases} 
\bigcup_{0 \leq i \leq m} \sigma^{(a_q(i))}_q \cup \varphi_f(\sigma^{(a_q(m))}_q) & \text{if } m \text{ is least such that } h(m) = (q, n), \\
\bigcup_{i \in \omega} \sigma^{(a_q(i))}_q & \text{otherwise.}
\end{cases}
\]

Fix \( q \supseteq p \) and \( n \in \omega \). By definition of the functions \( h \), there is \( m \) satisfying the first case if and only if \( (\forall A)(q \not\vdash E_n = A) \). If \( q \vdash E_n = A \), then the choice of the enumeration \( (\sigma^{(n)}_q)_{n \in 2^{<\omega} \cap \omega} \) and the function \( a \) guarantees \( R_{(q,n)} = A \). If \( q \not\vdash E_n = A \) for any \( A \), then by the fact that \( f \) is an extension function for \( \mathcal{F} \), it follows that \( R_{(q,n)} = \mathcal{F} \).

Since for every \( A \in \mathcal{F} \) there are \( q \) and \( n \) such that \( q \vdash E_n = A \), it follows that \( R \) is an enumeration of \( \mathcal{F} \), and Theorem 2.3 is proved.
We turn to the proof of Theorem 2.1. As mentioned above, if none of the cases holds for $p$ and $e$, then we have

$$\mathcal{F} = \{X : (\exists q \supseteq p)(\exists n)(q \vdash E_n = X)\}.$$  

The ternary relation $(q \vdash E_n = X) \land (X = \varphi_i)$ (in $q$, $n$ and $i$), when restricted to $i$ such that $\varphi_i$ is the characteristic function of a set, is $\Pi^0_2$ so that $I_r(\mathcal{F})$ is $\Sigma^0_3$. \qed

**Theorem 2.4** (Knight). Let $\mathcal{F}$ be a family such that for all non-recursive $X$, $\mathcal{F}$ has an enumeration r.e. in $X$. If $\mathcal{F}$ contains all finite sets, then $\mathcal{F}$ has an r.e. enumeration.

**Proof of Theorems 2.4 and 2.2.** As above, we construct a generic set $D$, attempting to meet the following requirements and expecting to fail.

$R_e$. $W^D_e$ is not an enumeration of $\mathcal{F}$.

We use the same forcing notion as was used for the previous proof. Fix $p \in 2^{< \omega}$ and $e \in \omega$. We consider the following three cases.

C1. For some $q \supseteq p$ and some $n \in \omega$, for all $q' \supseteq q$, there exist $x \in \omega$ and $r_0, r_1 \supseteq q'$ such that $r_0 \vdash (n, x) \in W^D_e$ and $r_1 \vdash (n, x) \notin W^D_e$.

For each $A \in \mathcal{F}$ the set

$$D^1_A := \{r : r \supseteq q \Rightarrow (\exists x)[(r \vdash (n, x) \in W^D_e \land x \notin A) \lor (r \vdash (n, x) \notin W^D_e \land x \in A)]\}$$

is dense. We put $q$ into the c.f.s. and enter the sets $D^1_A$. Requirement $R_e$ is satisfied.

We write $q \vdash E_n = A$ if for all $x$, if $x \in A$, then for all $q' \supseteq q$ there is $r \supseteq q'$ such that $r \vdash (n, x) \in W^D_e$, and if $x \notin A$ then $q \vdash (n, x) \notin W^D_e$.

C2. Case C1 does not hold, but $q \vdash E_n = B$ for some $q \supseteq p$, $n \in \omega$ and $B \notin \mathcal{F}$.

Requirement $R_e$ is met by putting $q$ in the c.f.s.

C3. Case C1 does not hold, but there exists $A \in \mathcal{F}$ such that for all $q' \supseteq q$ and all $n$, if $q' \vdash E_n = B$, then $A \neq B$.

Since Case C1 does not hold, the set

$$D^2_n := \{q' : q' \supseteq q \Rightarrow (\exists B)(q' \vdash E_n = B)\}$$

is dense for all $n$. We meet the requirement $R_e$ by including $q$ in the c.f.s. and entering all sets $D^2_n$.

If for all $p \in 2^{< \omega}$ and $e \in \omega$ one of the cases C1,C2, or C3 holds, then the forcing construction yields a generic (and hence non-recursive) set $D$, in which $\mathcal{F}$ has no r.e. enumeration. This contradicts the assumption of both Theorem 2.2 and Theorem 2.4. So let $p \in 2^{< \omega}$ and $e \in \omega$ be such that none of the cases C1,C2,C3 holds. We show that the index set $I_{re}$ of $\mathcal{F}$ is $\Sigma^0_3$. As in the proof of Theorem 2.3, for all $q \supseteq p$, $n$, and $A \subseteq \omega$, if $q \vdash E_n = A$, then $A \in \mathcal{F}$, and all members of $\mathcal{F}$ occur in this way. Therefore we have

$$\{i : W_i \in \mathcal{F}\} = \{i : (\exists q \supseteq p)(\exists n)(q \vdash E_n = W_i)\}.$$  

The relation “$q \vdash E_n = W_i$” is $\Pi^0_2$, and so the index set of $\mathcal{F}$ is $\Sigma^0_3$. This completes the proof of Theorem 2.2. To complete the proof of Theorem 2.4, note that since $\mathcal{F}$ includes all finite sets and only contains r.e. sets, by a theorem of Yates [7, Theorem 8], $\mathcal{F}$ has an r.e. enumeration. \qed
3. A FAMILY OF FINITE SETS

In this section we first define a family $\mathcal{C}$ which has no r.e. enumeration. Then we show that for every non-recursive set $X$ there is an enumeration of $\mathcal{C}$ which is recursive in $X$, and finally that every non-recursive degree contains an enumeration of $\mathcal{C}$. This corrects an earlier statement in [6, p. 187]. In particular, we apologize for connecting the error with Martin Kummer.

Let $r$ be the partial recursive function defined by

$$r(e) := (\mu(i, x, s). (e, x) \in \Omega^{(e)}_{i, x}).1.$$  

Informally, the value of $r(e)$ is the first index $i$ to be found such that there is a number of the form $\langle e, x \rangle \in \Omega^{(e)}_{i, x}$; if there is no such $i$ then $r(e)$ is not defined. Let the family $\mathcal{C}$ be defined by

$$\mathcal{C} := \{\langle e, A \rangle : A \text{ is finite, } e \in \omega \} \setminus \{\langle e, \omega \rangle \cap \Omega^{(e)}_{r(e)} : r(e) \downarrow\}.$$  

$\mathcal{C}$ does not have an r.e. enumeration; for suppose $\Omega^{(e_0)}$ is an enumeration of $\mathcal{C}$. Then $r(e_0)$ is defined, and the set $\Omega^{(e_0)}_{r(e_0)} = \langle e, \omega \rangle \cap \Omega^{(e_0)}_{r(e_0)}$ is not a member of $\mathcal{C}$.

Let $X$ be an arbitrary non-recursive set. To see how to construct an enumeration $S^X$ of $\mathcal{C}$ such that $S^X \leq_T X$ we use the following lemma.

**Lemma 3.1.** Uniformly in $i$ and recursively in $X$ there is a finite set $A^X_i \leq_T X$ such that $W_i \neq A^X_i$.

Let $g$ be a partial recursive function such that $g(e) \downarrow$ if and only if $r(e) \downarrow$ and if $r(e) \downarrow$ then $W_{g(e)} = (\langle e, \omega \rangle \cap \Omega^{(e)}_{r(e)})$. Let $h$ be a partial recursive function such that $h(e, a) \downarrow$ if and only if $g(e) \downarrow$ and if $g(e) \downarrow$ then $W_{h(e, a)} = W_{g(e)} - a$.

Define

$$S_{(n, t, e)}^X := \langle e, B^X(n, t, e) \rangle,$$

where

$$B^X(n, t, e) := \begin{cases} D_n \cup (s_0 + A^X_{h(e, s_0)}) & \text{if there is } s > t, \max(D_n) + 1 \\
 & \text{such that } g(s) \downarrow \text{ and } W_{g(s), s} = D_n, \\
 & \text{and } s_0 \text{ is the least such,} \\
D_n & \text{otherwise.} \end{cases}$$

By the lemma, $S^X$ is recursive in $X$. We claim that $S^X$ is an enumeration of $\mathcal{C}$.

"$\subseteq"$. First of all, it follows from the lemma that all sets $B^X(n, t, e)$ are finite and so the family enumerated by $S^X$ is contained in $\{\langle e, A \rangle : A \text{ is finite}\}$.

Suppose $r(e) \downarrow$ and $S_{(n, t, e)}^X = \langle e, \omega \rangle \cap \Omega^{(e)}_{r(e)}$. It follows that $B^X(n, t, e) = W_{g(e)}$. There has to be a stage $s > t, \max(D_n) + 1$ such that $W_{g(e), s} = D_n$, because otherwise $B^X(n, t, e) = D_n \neq W_{g(e)}$, a contradiction. Let $s_0$ be the least such $s$. Then $B^X(n, t, e) = D_n \cup (s_0 + A^X_{h(e, s_0)}) = W_{g(e)}$, so that $A^X_{h(e, s_0)} = W_{h(e, s_0)}$, a contradiction.

"$\supseteq"$. Let $e, n \in \omega$ such that $C = \langle e, D_n \rangle \in \mathcal{C}$. We want to find a number $z$ such that $S^X_z = C$. Let $t$ be such that $W_{g(e), s} \neq D_n$ for all $s \geq t$. (If there is no such $t$ then $C \not\in \mathcal{C}$.) By definition, $B^X(n, t, e) = D_n$, so that $S_{(n, t, e)}^X = C$.

We have constructed an enumeration of $\mathcal{C}$ which is recursive in $X$. Simple coding is sufficient to obtain an enumeration $T$-equivalent to $X$: Choose $A, B \in \mathcal{C}$ so that
$A - B \neq \emptyset$. Define another enumeration of $C$,

$$P_{2n}^X := S_n^X, \quad P_{2n+1}^X := \begin{cases} A & \text{if } n \in X, \\ B & \text{if } n \notin X. \end{cases}$$

Clearly, $P^X$ is recursive in $X$ and enumerates $C$. Let $x \in A - B$. Then $z \in X$ if and only $x \in P_{2z+1}^X$, so that $X$ is recursive in $P^X$.

**Proof of Lemma 3.1.** Let $y, n : \omega^2 \to \omega$ be two recursive one-one functions such that their ranges are disjoint and cover $\omega$. Let

$$\alpha(s, x) := \begin{cases} y(s, x) & \text{if } x \in X, \\ n(s, x) & \text{otherwise,} \end{cases}$$

so that $\alpha \leq_T X$. We construct the set $A^X_i$ in stages uniformly in $i$ and recursively in $X$. In the course of the construction, numbers $a_x$ may become defined.

Stage 0. Set $x := 0$. $A^X_{i,0}$ is empty.

Stage $s + 1$. If $W_{i,s} \neq A^X_{i,s}$, pass to the next stage. Otherwise enumerate $\alpha(s, x)$ in $A^X_i$, let $a_x := s$ and increase $x$ by one.

End of construction.

The set $A^X_i$ is (by construction) r.e. in $X$. It is also recursive in $X$: By inspection, $A^X_i$ only contains numbers $\alpha(a, b)$. If $y(s, x) \in A^X_i$, then $y(s, x) \in A^X_{i,s+1}$ and the same holds for numbers $n(s, x)$. This together with the choice of $y$ and $n$ is sufficient.

**Claim 1.** The set $A^X_i$ is finite.

Suppose $A^X_i$ is infinite. During the construction of $A^X_i$ infinitely many numbers $a_x$ are defined. By induction on $x$, it follows that $A_{i,a_x}^X = \{ \alpha(a_j, j) : j \leq x \}$, and therefore $W_{i,a_x} = \{ \alpha(a_j, j) : j < x \}$ for all $x \in \omega$. Thus, $W_i = \{ \alpha(a_x, x) : x \in \omega \}$. Now, $x \in X$ if and only if there exists $y(s, x) \in W_i$, and $x \notin X$ if and only if there exists $n(s, x) \in W_i$. This means that $X$ is recursive, a contradiction.

**Claim 2.** $A^X_i$ is different from $W_i$.

By the previous claim it is sufficient to consider the case when $W_i$ is finite. Let $s_0$ be the least number such that $W_{i,s_0} = W_i$. Then either $W_{i,s_0} \neq A^X_{i,s_0}$, in which case $W_{i,s_0} \neq A^X_{i,s_0} = A^X_i$; or $W_{i,s_0} = A^X_{i,s_0}$, so that at stage $s_1 = s_0 + 1$ a new number is enumerated in $A^X_i$, whence $W_i = W_{i,s_1} = A^X_{i,s_0} \neq A^X_i$.

Lemma 3.1 is proved.

4. **APPLICATION TO LEMPP’S QUESTION**

Let $\mathcal{F}$ be a family. With $\mathcal{F}$ we associate the following countable structure $\mathfrak{A}_F$ in the language $L = (S, Z, I)$, where $S$ is a binary predicate symbol and $Z$ and $I$ are unary predicate symbols. The universe of $\mathfrak{A}_F$ is $F \times \omega \times \omega$. For every $A \in \mathcal{F}$, set $Z((A, x, 0))$ and $S((A, x, n), (A, x, n + 1))$. Set $I((A, x, n))$ if and only if $n \in A$. Thus, countably many $S$-chains $(A, x, 0), (A, x, 1), (A, x, 2), \ldots$ are associated with every $A \in \mathcal{F}$, and in every chain $I$ holds of the $n$-th member if $n \in A$.

**Theorem 4.1.** Let $d$ be a Turing degree and $\mathcal{F}$ be a family. Then $\mathcal{F}$ has an enumeration recursive in $d$ if and only if the structure $\mathfrak{A}_F$ has a representation recursive in $d$. 

Proof. “⇒”. Let $Q^X$ be an enumeration of $F$ which is recursive in $X \in d$. Define $R^X$ to be another $X$-recursive enumeration of $F$ by $R^X_{(n,i)} := Q^X_i$. A representation of $\mathfrak{A}_F$ is given by $\mathfrak{B}$, where $Z_{\mathfrak{B}}(x,0)$, $S_{\mathfrak{B}}(x,n,\langle x, n + 1 \rangle)$, and $I_{\mathfrak{B}}(x,n)$ if and only if $x \in R^X_n$. These predicates are recursive in $R^X$, and therefore $D(\mathfrak{B})$ is recursive in $X$.

“⇐”. Let $\mathfrak{B}$ be a representation of $\mathfrak{A}_F$ whose open diagram is recursive in $X \in d$. The set $Z := \{ x : Z_{\mathfrak{B}}(x) \}$ is recursive in $X$. With each $x \in Z$ we associate the set $S_x := \{ n : I_{\mathfrak{B}}(f(n)(x)) \}$, where $f(x)$ is the unique $y$ such that $S_{\mathfrak{B}}(x,y)$, and $f^{(n)}(x)$ denotes the $n$-fold application of $f$ to $x$. The sets $S_x$ are uniformly in $x \in Z$ recursive in $X$, and, by definition of $\mathfrak{A}_F$, $F = \{ S_x : x \in Z \}$. This suffices.

Corollary 4.2 (Slaman [4]). There is a structure which has representations only in the non-recursive degrees.

Proof. Apply the theorem to the family $C$ defined in the previous section to obtain representations of $\mathfrak{A}_C$ below any non-recursive $T$-degree. Apply Theorem 1.1 to obtain representations in all non-recursive $T$-degrees.

Slaman remarked (private communication) that the construction given in [4] yields a structure which is not elementarily equivalent to any recursively represented structure. Informally, the reason for this is as follows. Essentially, the construction proceeds in such a way that the final outcome of the actions taken to diagonalize against the recursive representations can be read off the theory of the structure (which is called $\mathfrak{M}$):

He writes ([4, Section 2.1]): “We will ensure that either $R^{-1}(T_i)^{\mathfrak{M}} = \emptyset$, or $\langle T_i, <_L | T_i \rangle^{\mathfrak{M}}$ is not isomorphic to $\langle T_i, <_L | T_i \rangle^{\mathfrak{M}}$, or there is a $p$ in $R^{-1}(T_i)^{\mathfrak{M}}$ such that $\zeta(p)^{\mathfrak{M}}$ is not maximal, or there is a $p$ in $R^{-1}(T_i)^{\mathfrak{M}}$ such that $\zeta(p)^{\mathfrak{M}}$ is infinite. Since none of these disjuncts apply to $\mathfrak{M}$, we will thus ensure that $\mathfrak{M}$ has no recursive presentation.”

Fix $i \in \omega$. The first and third disjunct can be directly formalized in the language $\mathcal{L}$ of the structure. If the second disjunct holds, then the construction [4, Section 2.2.1] yields a tree $\langle T_i, <_L | T_i \rangle^{\mathfrak{M}}$ which is finite. Therefore this tree is described by a sentence in the theory of $\mathfrak{M}$. The fourth disjunct also cannot be formalized in $\mathcal{L}$, but, provided the first three disjuncts are not true, the strategy used in [4, Section 2.2.2] results in

$$(\exists p)(R(p,s^i(0)) \land (\forall x)(R(p,x) \rightarrow (\exists y)(x <_T y \land R(p,y))))$$

being true in $\mathfrak{M}_i$. This is a formula in $\mathcal{L}$, and (by the same strategy) not true in $\mathfrak{M}$. Hence there is no recursively represented structure which is elementarily equivalent to $\mathfrak{M}$.

The structure obtained from the family $C$ by Theorem 4.1 is of a different kind:

Theorem 4.3. There is a structure which has representations only in the non-recursive degrees and has a recursively represented elementary extension.

Proof. Let us look at the following family:

$$D := \{ (e,A) : A \text{ is finite, } e \in \omega \}.$$  

Obviously, $D$ has a recursive enumeration, and $\mathfrak{A}_D$ has a recursive representation. At the same time, the structure $\mathfrak{A}_C$ from the proof of Corollary 4.2 is contained in $\mathfrak{A}_D$, and they are elementarily equivalent:
It suffices to show that for any formula \((\exists x)(\phi)\) in variables \(x_1, \ldots, x_n\), if
\[ A_D \models (\exists x)(\phi)[x_1 := c_1, \ldots, x_n := c_n] \]
and \(c_1, \ldots, c_n \in A_C\), then there is \(c \in A_C\) such that
\[ A_D \models \phi[x := c, x_1 := c_1, \ldots, x_n := c_n]. \]
Note that if \(A_D \models (\exists x)(\phi)[x_1 := c_1, \ldots, x_n := c_n]\), then there is a \(d \in A_D\) such that either
\[ A_D \models \phi[x := d, x_1 := c_1, \ldots, x_n := c_n]\]
\[ \wedge \neg S(d, c_1) \wedge \neg S(c_1, d) \wedge \ldots \wedge \neg S(d, c_n) \wedge \neg S(c_n, d), \]
or
\[ A_D \models \phi[x := d, x_1 := c_1, \ldots, x_n := c_n]\]
\[ \wedge (S(d, c_1) \vee S(c_1, d) \vee \ldots \vee S(d, c_n) \vee S(c_n, d)). \]
In the former case, choose \(c\) from \(A_C\) outside of the \(S\)-chains of \(A_C\) which \(c_1, \ldots, c_n\) belong to such that it satisfies \(Z\) and \(I\) in the same way \(d\) does. In the latter case, \(d\) is already part of an \(S\)-chain which is contained in \(A_C\), and so an element of \(A_C\). ⚫

References


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