SOME RESULTS ON FINITE DRINFELD MODULES

CHIH-NUNG HSU

(Communicated by William W. Adams)

Abstract. Let K be a global function field, \( \infty \) a degree one prime divisor of K and let A be the Dedekind domain of functions in K regular outside \( \infty \). Let H be the Hilbert class field of A, B the integral closure of A in H. Let \( \psi \) be a rank one normalized Drinfeld A-module and let \( P \) be a prime ideal in B. We explicitly determine the finite A-module structure of \( \psi(B/P) \). In particular, if K = \( \mathbb{F}_q(t) \), \( q \) is an odd prime number and \( \psi \) is the Carlitz \( \mathbb{F}_q[t] \)-module, then the finite \( \mathbb{F}_q[t] \)-module \( \psi(\mathbb{F}_q[t]/P) \) is always cyclic.

1. Introduction

Recall that \( \mathbb{G}_m(\mathbb{Z}/p^n\mathbb{Z}) = (\mathbb{Z}/p^n\mathbb{Z})^\times \) is always cyclic except for the case that \( p = 2 \) and \( n \geq 3 \); if \( n \geq 3 \), then \( (\mathbb{Z}/2^n\mathbb{Z})^\times \) is the direct product of two cyclic groups, one of order 2, the other of order \( 2^{n-2} \). Let X be a smooth, projective, geometrically connected curve defined over the finite field \( \mathbb{F}_q \) with \( q \) elements and \( \infty \) be a rational point on X. We set K to be the function field of X over \( \mathbb{F}_q \) and A \( \subset \) K to be the Dedekind domain of functions regular outside \( \infty \). We will consider the Drinfeld A-modules. From the viewpoint of class field theory, these modules are interesting arithmetic objects over function fields. In particular, the rank one Drinfeld A-modules play a role entirely analogous to the important role played by \( \mathbb{G}_m \) over number fields. This naturally leads us to explore an analogous phenomenon for rank one Drinfeld A-modules.

Let \( K_\infty \) be the completion of K with respect to \( \infty \) and let \( C_\infty \) be the completion of the algebraic closure of \( K_\infty \) with respect to \( \infty \). Let \( C_\infty \{ \tau \} = \text{End}_{\mathbb{F}_q}(\mathbb{G}_a/C_\infty) \) be the twisted polynomial ring in the \( q \)th power Frobenius mapping \( \tau \). A rank one Drinfeld A-module \( \psi \) over \( C_\infty \) is an injective ring homomorphism \( \psi : A \rightarrow C_\infty \{ \tau \} \) such that the constant coefficient of \( \psi_a(\tau) \) is equal to \( a \) and \( \deg \psi_a(\tau) = -\text{Ord}_\infty a \) for all \( a \in A \). A sign-function (cf. [5] and [3]) \( \text{sgn} : K_\infty^\times \rightarrow \mathbb{F}_q^\times \) is a co-section of the inclusion map \( \mathbb{F}_q^\times \rightarrow K_\infty^\times \) such that \( \text{sgn}(\alpha) = 1 \) for all \( \alpha \in K_\infty^\times \) with \( \alpha - 1 \) vanishing at \( \infty \). A Drinfeld A-module \( \psi \) of rank one over \( C_\infty \) is said to be \( \text{sgn} \)-normalized if the leading coefficient of \( \psi_a(\tau) \) in \( \tau \) is equal to \( \text{sgn}(a) \) for all \( 0 \neq a \in A \). It is known [5] that any Drinfeld A-module of rank one over \( C_\infty \) is isomorphic to a \( \text{sgn} \)-normalized A-module \( \psi \) over \( H \), where \( H \) is the Hilbert class field of \( A \), i.e., \( H \) is the maximal abelian extension of \( K \) such that the extension \( H/K \) completely splits over \( \infty \) and is unramified over every finite place of \( K \).
Let $B$ be the integral closure of $A$ in $H$ and let $\psi$ be a rank one sgn-normalized Drinfeld $A$-module. Let $\mathfrak{M}$ be an ideal in $B$. Via the action of $\psi \pmod{\mathfrak{M}}$, $B/\mathfrak{M}$ becomes a finite $A$-module. We denote this by $\psi(B/\mathfrak{M})$. This finite $A$-module plays the role of $\mathbb{C}(\mathbb{Z}/p\mathbb{Z})$. The purpose of this note is to determine the $A$-module structure of $\psi(B/\mathfrak{M})$. It is sufficient to consider the case $\psi(B/\mathfrak{P}^N)$, where $\mathfrak{P}$ is a prime ideal in $B$. The structure of $\psi(B/\mathfrak{P}^N)$ is obtained in Theorems 2.1 and 2.2. In particular, if $\psi$ is the Carlitz $\mathbb{F}_q[t]$-module ($[3]$, chapter 3) and $q \neq 2$, then $\psi(B/\mathfrak{P}^N)$ is always cyclic (cf. Corollaries 2.1 and 2.2). The Carlitz case is completely analogous to the classical case.

In section 3, we discuss the relations between $B_\mathfrak{P}$ and $\lim \psi(B/\mathfrak{P}^N)$ via the exponential and logarithm functions of the sgn-normalized Drinfeld module $\psi$.

2. The structure of $\psi(B/\mathfrak{P}^N)$

Let the notation $X, \mathbb{F}_q, \infty, K, A, H, B$ and $\text{sgn}$ be as in the introduction. If $M$ is a commutative $\mathbb{F}_q$-algebra, we let $M\{\tau\}$ denote the composition ring of Frobenius polynomials in $\tau$, where $\tau$ is the $q^th$ power mapping. From now on, we let $\psi$ be a sgn-normalized rank one Drinfeld $A$-module over $H$, i.e., $\psi : A \rightarrow H\{\tau\}$ is a rank one Drinfeld $A$-module over $H$ such that for any $a \in A$, the leading coefficient of $\psi_a(\tau)$ is equal to $\text{sgn}(a)$. It is known that $\psi_a(\tau) \in B\{\tau\}$ for all $a \in A$. Thus, via $\psi$, $B$ becomes an $A$-module. We denote this module by $\psi(B)$ and denote the action $\psi_a(b)$ by $b^a$ for all $a \in A, b \in B$. Let $\mathfrak{M}$ be an ideal in $B$. Since $\psi(A) \subset B\{\tau\}$, it follows that, via $\psi \pmod{\mathfrak{M}}$, $B/\mathfrak{M}$ becomes a finite $A$-module. We denote this finite module by $\psi(B/\mathfrak{M})$. If the decomposition of $\mathfrak{M}$ is equal to $\mathfrak{P}_1^{N_1}\mathfrak{P}_2^{N_2}\cdots\mathfrak{P}_L^{N_L}$, where $\mathfrak{P}_1, \mathfrak{P}_2, \cdots, \mathfrak{P}_L$ are prime ideals in $B$, then, by the Chinese remainder theorem, we have

$$\psi(B/\mathfrak{M}) \cong \bigoplus_{i=1}^L \psi(B/\mathfrak{P}_i^{N_i}).$$

Thus to determine the $A$-module structure of $\psi(B/\mathfrak{M})$, it is sufficient to consider the case $\psi(B/\mathfrak{P}^N)$, where $\mathfrak{P}$ is a prime ideal in $B$. Let $\wp = \mathfrak{P} \cap A$ and let $f$ be the dimension of the vector space $B/\wp$ over $A/\wp$. It follows from class field theory that $\text{Norm}_{A/\wp} = \wp^f$ is a principal ideal in $A$. We let $\wp^f = (\pi_\wp)$ for the unique element $\pi_\wp \in A$ with $\text{sgn}(\pi_\wp) = 1$. It is known that $\psi(B/\wp)$ is a cyclic $A$-module with Euler-Poincaré characteristic $\pi_\wp - 1$ (cf. [3], chapter 4), i.e., as $A$-module,

$$\psi(B/\wp) \cong A/(\pi_\wp - 1).$$

We let $\psi_\wp(\tau)$ be the monic generator of the left ideal of $H\{\tau\}$ generated by $\psi_a(\tau)$ for all $a \in \wp$. We also denote the polynomial $\psi_\wp(\tau)(x)$ in $x$ by $x^\wp$ for all $x \in H$. The important property of the polynomial $\psi_\wp(\tau)$, $\wp \subset A$ a prime ideal, is the following (cf. [5], Proposition 11.4):

$$f(x) = \psi_\wp(\tau)(x)/x = x^\wp/x$$

is an Eisenstein polynomial over $B$ at any prime ideal $\wp$ above $\wp$. Let $c_\wp = f(0) \in B$. Then we have $\deg_{\wp} c_\wp = 1$. If $\wp$ is a rational point on the curve $X$ (i.e., $\wp$ is a prime ideal in $A$ of degree one) defined over $\mathbb{F}_2$, then $\psi_\wp(\tau)(x) = \psi_\wp(\tau) = c_\wp x + 1 \in B\{\tau\}$, i.e., $x^\wp = \psi_\wp(\tau)(x) = c_\wp x + 1$, where $c_\wp \in \wp$ but $c_\wp \not\in \wp^2$.

**Lemma 2.1.** Suppose that $N$ is a positive integer, $\wp$ and $\wp$ are as above. Then for any $b_1, b_2 \in \psi(B)$, if $b_1 \equiv b_2 \pmod{\wp^N}$, then $b_1^N \equiv b_2^N \pmod{\wp^{N+1}}$. 

Proof. We may write $b_1 = b_2 + x$ for some $x \in \mathfrak{P}^N$. Then, by the Eisenstein polynomial property,

$$b_1^p = (b_2 + x)^p = b_2^p + x^p \equiv b_2^p \pmod{\mathfrak{P}^{N+1}}. \quad \square$$

**Lemma 2.2.** Suppose that $\mathfrak{P}$, $\varphi$ and $c_\varphi$ are as above. Then:

1. If $N \geq 2$ is a positive integer, then $x^{\varphi^{N-2}} \equiv x \cdot c_\varphi^{N-2} \pmod{\mathfrak{P}^N}$ for all $x \in \mathfrak{P}$ except for the case that $\varphi$ is a rational point on the curve $X$ defined over $\mathbb{F}_2$.
2. If $\varphi$ is a rational point on the curve $X$ defined over $\mathbb{F}_2$ (i.e., $\varphi$ is a prime ideal of $A$ of degree one) and $N \geq 3$ is a positive integer, then

$$x^{\varphi^{N-3}} \equiv x \cdot c_\varphi^{N-3} \pmod{\mathfrak{P}^N}$$

for all $x \in \mathfrak{P}^2$.
3. If $\varphi$ is a rational point on the curve $X$ defined over $\mathbb{F}_2$, then for all $x \in \mathfrak{P}$, $x \not\equiv 0, c_\varphi \pmod{\mathfrak{P}^2}$, we have $x^p \in \mathfrak{P}^2$ but $x^p \not\in \mathfrak{P}^3$.

Proof. Assertion (1) is obvious for $N = 2$. Now suppose that it is true for $N \geq 2$, i.e.,

$$x^{\varphi^{N-2}} \equiv x \cdot c_\varphi^{N-2} \pmod{\mathfrak{P}^N}.$$

We show that it is true for $N + 1$. Applying Lemma 2.1 and the Eisenstein polynomial property, we have

$$(x^{\varphi^{N-2}})^p \equiv (x \cdot c_\varphi^{N-2})^p \equiv c_\varphi \cdot (x \cdot c_\varphi^{N-2}) + (x \cdot c_\varphi^{N-2})^q^{\deg \varphi} \pmod{\mathfrak{P}^{N+1}}.$$

If $q \neq 2$ or $q = 2$ and $\varphi$ is not a rational point on the curve $X$ over $\mathbb{F}_2$ (i.e., $q^{\deg \varphi} \geq 3$), then we have $(x \cdot c_\varphi^{N-2})^q^{\deg \varphi} \in \mathfrak{P}^{N+1}$. Hence

$$x^{\varphi^{N+1}} \equiv x^{c_\varphi^{N-1}} \pmod{\mathfrak{P}^{N+1}}.$$  

This completes the proof of (1). The proof of (2) is similar.

To prove (3), since $x \in \mathfrak{P}$ and $\deg_\mathfrak{P} c_\varphi = 1$, $x^p = c_\varphi \cdot x + x^2 = x(x + c_\varphi) \in \mathfrak{P}^2$. The assertion $x^p \not\in \mathfrak{P}^3$ follows from the facts that the characteristic of $B$ is 2 and $x \not\equiv 0, c_\varphi \pmod{\mathfrak{P}^2}$. \quad \square

**Lemma 2.3.** Suppose that $\mathfrak{P}$ and $\varphi$ are as above. Then:

1. If $N \geq 2$ is a positive integer and $x \in \mathfrak{P}, x \not\in \mathfrak{P}^2$, then the submodule $\langle \overline{\varphi} \rangle$ of $\psi(B/\mathfrak{P}^N)$ generated by $\overline{x} = x \pmod{\mathfrak{P}^N} \in \psi(B/\mathfrak{P}^N)$ is isomorphic to $A/\varphi^{N-1}$ except for the case when $\varphi$ is a rational point on the curve $X$ defined over $\mathbb{F}_2$.
2. Suppose that $\varphi$ is a rational point on the curve $X$ defined over $\mathbb{F}_2$. If $N \geq 3$ is a positive integer and $x \in \mathfrak{P}^2, x \not\in \mathfrak{P}^3$, then the submodule $\langle \overline{\varphi} \rangle$ of $\psi(B/\mathfrak{P}^N)$ generated by $\overline{x} = x \pmod{\mathfrak{P}^N} \in \psi(B/\mathfrak{P}^N)$ is isomorphic to $A/\varphi^{N-2}$.
3. Suppose that $\varphi$ is a rational point on the curve $X$ defined over $\mathbb{F}_2$ and $N \geq 2$. Then for all $x \in \mathfrak{P}, x \not\equiv 0, c_\varphi \pmod{\mathfrak{P}^2}$, the submodule $\langle \overline{\varphi} \rangle$ of $\psi(B/\mathfrak{P}^N)$ generated by $\overline{x} = x \pmod{\mathfrak{P}^N} \in \psi(B/\mathfrak{P}^N)$ is isomorphic to $A/\varphi^{N-1}$.

Proof. By Lemma 2.2 (1), $x^{\varphi^{N-1}} \equiv x \cdot c_\varphi^{N-1} \pmod{\mathfrak{P}^{N+1}}$. Since $\deg_\mathfrak{P} c_\varphi = 1$ and $x \in \mathfrak{P}$, we have

$$x^{\varphi^{N-1}} \equiv 0 \pmod{\mathfrak{P}^N}.$$

License or copyright restrictions may apply to redistribution; see http://www.ams.org/journal-terms-of-use
Again by Lemma 2.2 (1), degₚ cₚ = 1 and \( x \in \mathfrak{P}, x \notin \mathfrak{P}^2 \),
\[ x^{\wp^{N-2}} \equiv x \cdot c_{\wp}^{N-2} \neq 0 \pmod{\mathfrak{P}^N}. \]
Since \( \langle \wp \rangle \) is a cyclic \( A \)-module, \( \langle \wp \rangle \) is isomorphic to \( A/\wp^{N-1} \). The proof of (2) is similar, and the proof of (3) follows from Lemma 2.2 (3), (2). □

Let \([H : K] = fr \) be the class number of \( A \), where \( f = [B/\mathfrak{P} : A/\wp] \). Then the main result is

**Theorem 2.1.** Suppose that \( \mathfrak{P}, \wp, \wp \) and \( f \) are as above. Then
\[ \psi(B/\mathfrak{P}^N) \cong \begin{cases} A/(\wp^{N-1}) & \text{if } N = 1, \\ A/(\wp^{N-1}) \oplus (A/\wp^{N-1})^f & \text{if } N > 1, \end{cases} \]
extcept for the case when \( \wp \) is a rational point on the curve \( X \) defined over \( \mathbb{F}_2 \).

**Proof.** The case \( N = 1 \) follows from the theory of Drinfeld modules over finite fields (cf. [3], Chapter 4). We suppose that \( N \geq 2 \). Given \( x \in \psi(B) \). Since \( \psi(B/\mathfrak{P}) \cong A/(\wp^{N-1}) \), it follows that \( x^{\wp^{N-1}} \equiv 0 \pmod{\mathfrak{P}} \). By Lemma 2.2 (1), we have
\[ (x^{\wp^{N-1}})^{\wp^{N-1}} \equiv 0 \pmod{\mathfrak{P}^N}. \]
This implies that the Euler-Poincaré characteristic of any \( A \)-cyclic submodule of \( \psi(B/\mathfrak{P}^N) \) divides \( (\wp^{N-1})^f \).
Since \( \psi(A) \subset B\{x\} \), \( \psi(\mathfrak{P}/\mathfrak{P}^N) \) is a submodule of \( \psi(B/\mathfrak{P}^N) \). We have
\[ \dim_{\mathbb{F}_2} \psi(\mathfrak{P}/\mathfrak{P}^N) = (N - 1) \dim_{\mathbb{F}_2} B/\mathfrak{P} = f(N - 1) \dim_{\mathbb{F}_2} A/\wp. \]
Since \( x^{\wp^{N-1}} \equiv 0 \pmod{\mathfrak{P}^N} \) for all \( x \in \mathfrak{P} \) (by Lemma 2.2 (1)), as \( A \)-module
\[ \psi(\mathfrak{P}/\mathfrak{P}^N) \cong \bigoplus_{i=1}^l A/\wp^{n_i} \]
for suitable positive integers \( 1 \leq n_1 \leq n_2 \leq \cdots \leq n_l \leq N - 1 \) such that
\[ n_1 + n_2 + \cdots + n_l = (N - 1)f. \]
By Lemma 2.2 (1) and Lemma 2.3 (1), the subset of elements \( \bar{x} \) in \( \psi(\mathfrak{P}/\mathfrak{P}^N) \) such that \( x^{\wp^{N-1}} \equiv 0 \pmod{\mathfrak{P}^N} \) is equal to \( \psi(\mathfrak{P}^2/\mathfrak{P}^N) \). Counting cardinalities, we must have
\[ l = f, \quad n_1 = n_2 = \cdots = n_f = N - 1. \]
Thus we get
\[ \psi(\mathfrak{P}/\mathfrak{P}^N) \cong (A/\wp^{N-1})^f. \]

Next, we take \( g \in B \) such that \( g \pmod{\mathfrak{P}} \) is a generator of \( \psi(B/\mathfrak{P}) \). We let \( \langle \bar{g} \rangle \) be the \( A \)-submodule of \( \psi(B/\mathfrak{P}^N) \) generated by \( \bar{g} = g \pmod{\mathfrak{P}^N} \) in \( \psi(B/\mathfrak{P}^N) \). We define the \( A \)-module homomorphism \( \chi : \langle \bar{g} \rangle \rightarrow \psi(B/\mathfrak{P}) \) by \( \chi(x \pmod{\mathfrak{P}^N}) = x \pmod{\mathfrak{P}} \) for all \( x \pmod{\mathfrak{P}^N} \) in \( \langle \bar{g} \rangle \). Since \( g \pmod{\mathfrak{P}} \) is a generator of \( \psi(B/\mathfrak{P}) \) and \( \chi(g \pmod{\mathfrak{P}^N}) = g \pmod{\mathfrak{P}} \), \( \chi \) is a surjective homomorphism of \( \langle \bar{g} \rangle \) onto \( \psi(B/\mathfrak{P}) \). This implies that \( (\wp^{N-1}) \) divides the Euler-Poincaré characteristic of \( \psi(B/\mathfrak{P}^N) \), because \( \langle \bar{g} \rangle \) is a submodule of \( \psi(B/\mathfrak{P}^N) \). Combining these, we obtain that \( (\wp^{N-1})^f \) divides the Euler-Poincaré characteristic of \( \psi(B/\mathfrak{P}^N) \); this implies that the Euler-Poincaré characteristic of \( \psi(B/\mathfrak{P}^N) \) is equal to
(\pi_\varphi - 1)\psi^{f(N-1)}; hence \langle \psi \rangle (or \psi(\mathcal{B}/\mathcal{P}^N)) contains an A-submodule which is isomorphic to \( A/(\pi_\varphi - 1) \). Therefore, we obtain that
\[
\psi(\mathcal{B}/\mathcal{P}^N) \cong A/(\pi_\varphi - 1) \oplus (A/\varphi^{N-1})^f. \]

**Theorem 2.2.** Suppose that \( \varphi \) is a rational point on the curve \( X \) defined over \( \mathbb{F}_2 \). Then
\[
\psi(\mathcal{B}/\mathcal{P}^N) \cong \begin{cases} 
A/(\pi_\varphi - 1), & \text{if } N = 1; \\
A/(\pi_\varphi - 1) \oplus (A/\varphi)^f, & \text{if } N = 2; \\
A/(\pi_\varphi - 1) \oplus A/\varphi \oplus (A/\varphi^{N-1})^f - 1 \oplus A/\varphi^{N-2}, & \text{if } N \geq 3.
\end{cases}
\]

**Proof.** The case \( N = 1 \) is standard. We suppose that \( N \geq 2 \). Using Lemmas 2.2 and 2.3, the proof is almost the same as the proof of Theorem 2.1. We obtain that the Euler-Poincaré characteristic of any A-cyclic submodule of \( \psi(\mathcal{B}/\mathcal{P}^N) \) divides \((\pi_\varphi - 1)\varphi^{N-1}\), the finite A-module \( \psi(\mathcal{B}/\mathcal{P}^N) \) is annihilated by \( \varphi^{N-1} \), and \( \psi(\mathcal{B}/\mathcal{P}^N) \) contains an A-submodule which is isomorphic to \( A/(\pi_\varphi - 1) \). From these, we deduce that the Euler-Poincaré characteristic of \( \psi(\mathcal{B}/\mathcal{P}^N) \) is equal to \((\pi_\varphi - 1)\psi^{f(N-1)}\) and
\[
\psi(\mathcal{B}/\mathcal{P}^N) \cong A/(\pi_\varphi - 1) \oplus \psi(\mathcal{P}/\mathcal{P}^N).
\]

Next, we deal with the A-module structure of \( \psi(\mathcal{P}/\mathcal{P}^N) \). For \( N = 2 \), since \( \psi(\mathcal{P}/\mathcal{P}^2) \) is annihilated by \( \varphi \), counting the dimension of \( \psi(\mathcal{P}/\mathcal{P}^2) \) over \( A/\varphi \), we obtain that \( \psi(\mathcal{P}/\mathcal{P}^2) \cong (A/\varphi)^f \). For \( N \geq 3 \), since \( \varphi \) is a rational point on the curve \( X \) defined over \( \mathbb{F}_2 \) and \([B/\mathcal{P}: A/\varphi] = f\), as abelian group \( \mathcal{P}/\mathcal{P}^2 \cong (A/\varphi)^f \cong \mathbb{F}_2^f \). Let \( S \) be the A-submodule of \( \psi(\mathcal{P}/\mathcal{P}^N) \) generated by elements \( x \pmod{\mathcal{P}^N} \) is such that \( x \in \mathcal{P} \) but \( x \not\in \mathcal{P}^2 \). We define the abelian group homomorphism \( \chi : S \to \psi(\mathcal{P}^2/\mathcal{P}^3) \) by \( \chi(x \pmod{\mathcal{P}^N}) = x^\varphi \pmod{\mathcal{P}^3} \) for all \( x \pmod{\mathcal{P}^N} \in S \). From Lemma 2.2 (3), we know that if \( x \pmod{\mathcal{P}^N} \in S \) is such that \( \chi(x \pmod{\mathcal{P}^N}) = 0 \), then \( x \equiv 0, c_{\varphi} \pmod{\mathcal{P}^2} \). This implies that \( \dim_{\mathbb{F}_2} \chi(S) = f - 1 \). Combining this with Lemma 2.3 (3) and the fact that \( S \) is annihilated by \( \varphi^{N-1} \), we obtain that
\[
S \cong A/\varphi \oplus (A/\varphi^{N-1})^{f-1}.
\]

Since \( \chi(c_{\varphi} \pmod{\mathcal{P}^N}) = 0 \) and \( c_{\varphi} \pmod{\mathcal{P}^N} \) \( \in S \), there exists an element \( x \in \mathcal{P}^2, x \not\in \mathcal{P}^3 \), such that \( x \pmod{\mathcal{P}^3} \not\in \chi(S) \). By Lemma 2.2 (2), we know that \( \langle x \pmod{\mathcal{P}^N} \rangle \) is a submodule of \( \psi(\mathcal{P}/\mathcal{P}^N) \) which is isomorphic to \( A/\varphi^{N-2} \). Combining these and counting the dimension of \( \psi(\mathcal{P}/\mathcal{P}^N) \) over \( A/\varphi \cong \mathbb{F}_2 \), we obtain that
\[
\psi(\mathcal{B}/\mathcal{P}^N) \cong A/(\pi_\varphi - 1) \oplus A/\varphi \oplus (A/\varphi^{N-1})^{f-1} \oplus A/\varphi^{N-2}.
\]

This completes the proof.

As an application, we let \( A = \mathbb{F}_q[t] \) and let \( \phi \) be the Carlitz A-module, i.e., \( \phi : A \to \mathbb{F}_2(t)\{\tau\} \) is given by
\[
\phi(t) = t\tau^0 + \tau^1.
\]

Then we have

**Corollary 2.1.** If \( N \) is a positive integer and \( \varphi = (p) \) is a prime ideal in \( A \) generated by the monic polynomial \( p \), then the finite A-module
\[
\psi(A/\mathcal{P}^N) \cong A/(p^N - p^{N-1})
\]
is cyclic except for the case when \( \mathbb{F}_q \) equals \( \mathbb{F}_2 \) and \( p \mid t(t + 1) \).
Corollary 2.2. If \( N \) is a positive integer, \( A = \mathbb{F}_2[t] \) with \( p = t \) or \( t + 1 \in A \), then the finite \( A \)-module \( \phi(A / \varphi^N) \) is isomorphic to
\[
\begin{cases}
  A / (p - 1), & \text{if } N = 1; \\
  A / (t^2 + t), & \text{if } N = 2; \\
  A / (t^2 + t) \oplus A / (p^{N-2}) & \text{if } N \geq 3.
\end{cases}
\]

3. Passage to the limit

Let the notation \( X, \mathbb{F}_q, \infty, K, A, H, f, \pi, sgn \) be as before. Let \( \psi \) be a \( sgn \)-normalized rank one Drinfeld \( A \)-module over \( H \). Suppose that \( P \) and \( \wp \) are as in section 2 and \( \wp \) does not correspond to a rational point on \( X \) if \( q = 2 \). It is well-know that there exists a lattice \( A_\zeta, \zeta \in C_\infty, \mathfrak{A} \) an ideal of \( A \), of rank one such that \( \psi \) is determined by this lattice. The exponential function \( e_\psi \) associated to \( A_\zeta \) is defined to be
\[
e_\psi(x) = z \prod_{a \in \mathfrak{A}} \left(1 - \frac{x}{a \cdot \zeta}\right) \in H[[\tau]].
\]

Let \( H_\wp \) (resp. \( K_\wp \)) be the completion fields associated to \( \wp \) (resp. \( \varphi \)). Let \( B_\wp \subset H_\wp \) and \( A_\wp \subset K_\wp \) be the rings of integers.

It follows from theorem 2.1 that as \( A \)-module
\[
\psi(B_\wp) = \psi(\lim_{\leftarrow} B / \wp^N) = \lim_{\leftarrow} \psi(B / \wp^N) = \lim_{\leftarrow} A / (\pi_\wp - 1) \oplus (A / \varphi^{N-1})^f \\
= A / (\pi_\wp - 1) \oplus A_\wp^f \\
= A / (\pi_\wp - 1) \oplus B_\wp.
\]

We know that the coefficients of \( e_\psi \) are in \( H \) and these coefficients are obtained by solving a recursion equation via any \( \psi_a, a \in A, a \not\in \mathbb{F}_q \) (cf. [3], Lemma 4.6.5). We can deduce from this recursion that \( e_\psi \) converges in a neighborhood of 0. Thus there exist element \( \alpha \in H_\wp \) such that \( e(x) = e_\psi(\alpha \cdot x) \) is an analytic injective function of \( B_\wp \) into \( B_\wp \). By the property of the exponential function \( e_\psi \), we obtain that \( e(ax) = \psi_a(e(x)) \) for all \( a \in A \). Combining these, we have

Theorem 3.1. As \( A \)-module,
\[
\psi(B_\wp) \cong \begin{cases}
  A / (\pi_\wp - 1) \oplus A / \varphi \oplus B_\wp, & \text{if } q = 2 \text{ and } \varphi \text{ is a rational point}; \\
  A / (\pi_\wp - 1) \oplus B_\wp, & \text{otherwise}.
\end{cases}
\]

Moreover, one has an analytic map \( e : B_{\wp} \to \psi(B_{\wp}) \) satisfying the following commutative diagram:
\[
\begin{array}{ccc}
B_{\wp} & \xrightarrow{e} & \psi(B_{\wp}) \\
\alpha \downarrow & & \downarrow \psi_a \\
B_{\wp} & \xrightarrow{e} & \psi(B_{\wp})
\end{array}
\]
References


Department of Mathematics, National Taiwan Normal University, 88 Sec. 4 Ting-Chou Road, Taipei, Taiwan
E-mail address: maco@math.ntnu.edu.tw