

## ON SCRAMBLED SETS AND A THEOREM OF KURATOWSKI ON INDEPENDENT SETS

HISAO KATO

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ABSTRACT. The measure of scrambled sets of interval self-maps  $f : I = [0, 1] \rightarrow I$  was studied by many authors, including Smítal, Misiurewicz, Bruckner and Hu, and Xiong and Yang. In this note, first we introduce the notion of “\*-chaos” which is related to chaos in the sense of Li-Yorke, and we prove a general theorem which is an improvement of a theorem of Kuratowski on independent sets. Second, we apply the result to scrambled sets of higher dimensional cases. In particular, we show that if a map  $f : I^k \rightarrow I^k$  ( $k \geq 1$ ) of the unit  $k$ -cube  $I^k$  is \*-chaotic on  $I^k$ , then for any  $\epsilon > 0$  there is a map  $g : I^k \rightarrow I^k$  such that  $f$  and  $g$  are topologically conjugate,  $d(f, g) < \epsilon$  and  $g$  has a scrambled set which has Lebesgue measure 1, and hence if  $k \geq 2$ , then there is a homeomorphism  $f : I^k \rightarrow I^k$  with a scrambled set  $S$  satisfying that  $S$  is an  $F_\sigma$ -set in  $I^k$  and  $S$  has Lebesgue measure 1.

### 1. INTRODUCTION

All spaces considered in this note are assumed to be separable and complete metric spaces. *Maps* are continuous functions. By a *compactum* we mean a compact metric space.

Let  $f : X \rightarrow X$  be a map of a compactum  $X$  with metric  $d$ . A subset  $S$  of  $X$  is a *scrambled set* of  $f$  if there is a positive number  $\tau > 0$  such that for any  $x, y \in S$  with  $x \neq y$ ,

1.  $\limsup_{n \rightarrow \infty} d(f^n(x), f^n(y)) > \tau$ ,
2.  $\liminf_{n \rightarrow \infty} d(f^n(x), f^n(y)) = 0$ .

If there is an uncountable scrambled set  $S$  of  $f$ , then we say that  $f$  is *chaotic in the sense of Li-Yorke*. In the original paper [6] of Li and Yorke, there was the following one more condition: for any  $x \in S$  and any periodic point  $p \in X$ ,  $\limsup_{n \rightarrow \infty} d(f^n(x), f^n(p)) > 0$ . But it is known that this condition is unnecessary.

In [6], it was proved that if a map  $f : I = [0, 1] \rightarrow I$  has a periodic point with period 3, then  $f$  is chaotic in the sense of Li-Yorke. The measure of scrambled sets of interval self-maps was studied by many authors, including Smítal [11], [12], Misiurewicz [7], Bruckner and Hu [1], and Xiong and Yang [13].

In this note, we prove a general theorem which is an improvement of a theorem [4] of Kuratowski on independent sets, and we apply this result to scrambled sets of higher dimensional cases.

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Let  $A$  be a closed subset of a compactum  $X$ . A map  $f : X \rightarrow X$  is *\*-chaotic* on  $A$  (in the sense of Li-Yorke) if the following conditions are satisfied:

1. there is  $\tau > 0$  such that if  $U$  and  $V$  are any nonempty open subsets of  $A$  with  $U \cap V = \emptyset$  and  $N$  is any natural number, then there is a natural number  $n \geq N$  such that  $d(f^n(x), f^n(y)) > \tau$  for some  $x \in U, y \in V$ , and
2. for any nonempty open subsets  $U, V$  of  $A$  and any  $\epsilon > 0$  there is a natural number  $n \geq 0$  such that  $d(f^n(x), f^n(y)) < \epsilon$  for some  $x \in U, y \in V$ .

If  $f : X \rightarrow X$  is \*-chaotic on the total space  $X$ , we say that  $f : X \rightarrow X$  is *everywhere \*-chaotic*.

Note that if  $S$  is a scrambled set of  $f$ , then  $f$  is \*-chaotic on  $\text{Cl}(S)$ . A map  $f : X \rightarrow X$  of a compactum  $X$  has *sensitive dependence on initial conditions* on a closed subset  $A$  of  $X$  if there is  $\tau > 0$  such that if  $x \in A$  and  $U$  is any neighborhood of  $x$  in  $A$ , then there is  $y \in U$  such that  $d(f^n(x), f^n(y)) > \tau$  for some  $n$ . Suppose that  $A$  is a closed subset of  $X$  and  $A$  has no isolated point (=  $A$  is perfect). Then  $f$  is \*-chaotic on  $A$  if and only if  $f$  has sensitive dependence on initial conditions on  $A$  and the above condition 2 is satisfied.

In the definition of chaos in the sense of Li-Yorke, we only assume the condition that there exists an uncountable scrambled set  $S$  of  $f : X \rightarrow X$ , and in general,  $S$  is an arbitrary subset of  $X$ . Note that if  $S$  is an uncountable set, then  $\text{Cl}(S)$  contains a Cantor set by the Cantor-Bendixon theorem (e.g., see [5, p. 253]).

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## 2. INDEPENDENT SETS AND SCRAMBLED SETS

In this section, first, we prove the following general theorem which is an improvement of a theorem of Kuratowski on independent sets (see [4]). Let  $X$  be a space and  $R$  be any subset of  $X^m$  ( $m \geq 2$ ). A subset  $F \subset X$  is said to be *independent in  $R$*  if for any different  $m$  points  $x_1, \dots, x_m$  of  $F$  (i.e.,  $x_i \neq x_j$  for  $i \neq j$ ), we have  $(x_1, x_2, \dots, x_m) \in X^m - R$ . For a subset  $R \subset X^m$ , consider the following property:

- (\*) If  $U_1, \dots, U_m$  are mutually disjoint nonempty sets of  $X$ , there are nonempty open sets  $V_1, \dots, V_m$  of  $X$  such that  $V_i \subset U_i$  for each  $i$  and  $V_1 \times \dots \times V_m \subset X^m - R$ .

Note that if  $X$  has no isolated point,  $R \subset X^m$  has the property (\*) if and only if  $R$  is nowhere dense (see Lemma 2.2).

A countable union of nowhere dense sets is called a set *of the first category*.

**Theorem 2.1.** *Suppose that  $X$  is a separable and completely metrizable space and  $m \geq 2$  is a fixed natural number. Let  $R \subset X^m$ .*

1. *If  $R = \bigcup_{n=1}^{\infty} R_n$  and each  $R_n$  has the property (\*), then there is an  $F_\sigma$ -set  $S$  of  $X$  such that  $S$  is independent in  $R$ , and  $\text{Cl}(S) = X$ .*
2. *If  $X$  has no isolated point and  $R$  is of the first category, then there is a subset  $S$  of  $X$  such that  $S = \bigcup_{n=1}^{\infty} C_n$ , where  $C_n$  are Cantor sets in  $X$ ,  $S$  is independent in  $R$ , and  $\text{Cl}(S) = X$ .*

To prove this theorem, we need the following lemma by Kuratowski [4, Lemma, p. 66].

**Lemma 2.2.** *Let  $X$  be an arbitrary space and let  $G$  be an open dense set in the space  $X^m$  ( $m \geq 1$ ). Let  $H_1, \dots, H_n$ , where  $n \geq m$ , be open non-empty sets in  $X$ .*

Then there exists a system of nonempty open sets  $U_1, \dots, U_n$  such that  $U_i \subset H_i$  for  $1 \leq i \leq n$ , and

$$U_{i_1} \times \dots \times U_{i_m} \subset G,$$

whenever  $i_1, \dots, i_m$  are different  $m$  elements of  $\{1, 2, \dots, n\}$ .

*Proof of Theorem 2.1.* We prove the second case. The first case can be similarly proved.

We assume that  $X$  has no isolated points. Also, we may assume that  $R = \bigcup_{k=1}^\infty R_k$ , where  $R_k$  is nowhere dense and a closed subset of  $X^m$  for each  $k$ . Let  $\{U_i\}_{i=1}^\infty$  be a countable open base of  $X$ . Let  $i = 1$ . By the above lemma, we can choose a family

$$\mathcal{V}_1 = \{V_{1,1}, V_{1,2}, \dots, V_{1,n_1}\}$$

of mutually disjoint nonempty closed subsets of  $X$  such that  $\text{Int}(V_{1,j}) \neq \emptyset$  for each  $j$ ,  $(\bigcup \mathcal{V}_1) \cap U_1 \neq \emptyset$ , and for any different  $m$  elements  $V_{1,j_1}, \dots, V_{1,j_m}$  of  $\mathcal{V}_1$ ,

$$(V_{1,j_1} \times \dots \times V_{1,j_m}) \cap (R_1) = \emptyset,$$

where  $\bigcup \mathcal{V}_1 = \bigcup \{V \mid V \in \mathcal{V}_1\}$ .

Inductively, by the above lemma we can choose a sequence

$$\mathcal{V}_i = \{V_{i,1}, \dots, V_{i,n_i}\} \quad (i \geq 1)$$

of families of mutually disjoint nonempty closed subsets of  $X$  such that

1.  $\text{Int}(V_{i,j}) \neq \emptyset$  and  $\text{mesh}(\mathcal{V}_i) < 1/i$ ,
2. for each  $V_{i,j}$ , there are at least two elements  $V_{i+1,p}, V_{i+1,q}$  ( $p \neq q$ ) of  $\mathcal{V}_{i+1}$  such that  $V_{i+1,p} \cup V_{i+1,q} \subset \text{Int}(V_{i,j})$ ,
3.  $U_i \cap (\bigcup \mathcal{V}_i) \neq \emptyset$ , and
4. for any different  $m$  elements  $V_{i,j_1}, \dots, V_{i,j_m}$  of  $\mathcal{V}_i$ ,

$$(V_{i,j_1} \times \dots \times V_{i,j_m}) \cap \left( \bigcup_{k=1}^i R_k \right) = \emptyset.$$

For each  $i$ , by induction we can obtain the sequence  $\{i,j \mid (j = i, i + 1, i + 2, \dots)\}$  of subfamilies of  $\mathcal{V}_j$  as follows:

$$\mathcal{F}_{i,i} = \mathcal{V}_i, \quad \mathcal{F}_{i,i+1} = \{V \in \mathcal{V}_{i+1} \mid V \text{ is contained in some element of } \mathcal{F}_{i,i}\} \quad (j \geq i).$$

Put  $C_i = \bigcap_{j=i}^\infty (\bigcup \mathcal{F}_{i,j})$ . Then we see that  $C_i$  is a Cantor set and  $\{C_i\}_{i=1}^\infty$  is increasing. Then by the condition 4,  $S = \bigcup_{i=1}^\infty C_i$  is independent in  $R = \bigcup_{k=1}^\infty R_k$  and by the condition 3,  $S$  is dense in  $X$ . This completes the proof.

Let  $I = [0, 1]$  be the unit interval and  $I^k$  ( $k \geq 1$ ) the unit  $k$ -cube. A space homeomorphic to  $I^k$  is a  $k$ -cell. A  $k$ -cell  $B$  in  $I^k - \partial I^k$  is flat if there is a homeomorphism  $h : I^k \rightarrow I^k$  such that  $h(B) = J^k$ , where  $J = [1/3, 2/3] \subset I$ . A 0-dimensional compactum  $D$  in  $I^k - \partial I^k$  is flat in  $I^k$  if for any neighborhood  $V$  of  $D$  in  $I^k$ , there is a neighborhood  $U$  of  $D$  in  $I^k$  such that  $U \subset V$  and  $U = B_1 \cup \dots \cup B_p$ , where  $B_i$  ( $i = 1, 2, \dots, p$ ) are mutually disjoint  $k$ -cells. Then we may assume that  $B_i$  is flat in  $I^k$  for each  $i$ , because we can choose a  $k$ -cell  $B'$  in  $\text{Int}(B_i)$  such that  $\partial B'$  is locally flat and hence  $B'$  is flat by Generalized Schoenflies Theorem (e.g. see [10, p. 48]). Note that if  $C$  and  $C'$  are flat Cantor sets in  $I^k$  and  $k \geq 2$ , then any homeomorphism  $f : C \cup \partial I^k \rightarrow C' \cup \partial I^k$  can be extended to a homeomorphism  $F : I^k \rightarrow I^k$  (e.g., see the proof of [8, p. 93, Theorem 7]). Also, note that any

closed subset of a flat 0-dimensional compactum is also flat. It is well known that if  $k \leq 2$ , any Cantor set in  $I^k - \partial I^k$  is flat, but if  $k \geq 3$ , there are Cantor sets in  $I^k - \partial I^k$  which are not flat (see [8, p. 127]).

**Proposition 2.3.** *Suppose that  $X$  is the unit  $k$ -cube  $I^k$  ( $k \geq 1$ ). Let  $m \geq 2$  be a fixed natural number and let  $R \subset X^m$  be of the first category in  $X^m$ . Then there is a subset  $S$  of  $X$  such that  $S = \bigcup_{n=1}^{\infty} C_n$ , where  $C_n$  are flat Cantor sets in the unit  $k$ -cube  $X$ ,  $S$  is independent in  $R$ , and  $Cl(S) = X$ .*

*Proof.* In the proof of the above theorem, we may assume that each  $V_{i,j}$  is a  $k$ -cell (see Lemma 2.2).

**Theorem 2.4.** *If  $f : X \rightarrow X$  is a map of a compactum  $X$  and  $f$  is  $*$ -chaotic on a closed set  $A$ , then there is an  $F_{\sigma}$ -set  $S \subset A$  such that  $S$  is a scrambled set of  $f$  and  $Cl(S) = A$ . If  $A$  has no isolated points, we can choose  $S$  such that  $S$  is a countable union of Cantor sets  $C_n$ . Moreover if  $X = I^k$ , then the Cantor sets  $C_n$  can be chosen as flat Cantor sets in  $X$ .*

*Proof.* Suppose that  $\tau$  is a positive number as in the definition of scrambled set  $S$ . Consider the following sets:

$$R_1 = \{(x, y) \in A^2 \mid \limsup_{i \rightarrow \infty} d(f^i(x), f^i(y)) < \tau\},$$

$$R_2 = \{(x, y) \in A^2 \mid \liminf_{i \rightarrow \infty} d(f^i(x), f^i(y)) > 0\}.$$

Let  $\epsilon_n = 1/n$  ( $n = 1, 2, \dots$ ). Then  $R_1 = \bigcup_{n=1}^{\infty} T_n$ , where

$$T_n = \{(x, y) \in A^2 \mid d(f^i(x), f^i(y)) \leq \tau - \epsilon_n \text{ for every } i \geq n\}.$$

Also,  $R_2 = \bigcup_{n=1}^{\infty} W_n$ , where

$$W_n = \{(x, y) \in A^2 \mid d(f^i(x), f^i(y)) \geq \epsilon_n \text{ for every } i \geq n\}.$$

Note that  $T_n, W_n \subset A^2$  are closed. Since  $f$  is  $*$ -chaotic on  $A$ , they have the property (\*). By Theorem 2.1 and Proposition 2.3, we obtain a desired scrambled set  $S$ .

**Corollary 2.5.** *Let  $f : X \rightarrow X$  be a map of a compactum  $X$  and  $S$  an uncountable scrambled set of  $f$ . Then there is an  $F_{\sigma}$ -set  $S'$  of  $X$  such that  $S'$  is a scrambled set of  $f$ ,  $Cl(S') = Cl(S)$ , and  $S'$  contains a Cantor set. Moreover, if  $Cl(S)$  has no isolated point,  $S'$  can be chosen so that  $S'$  is a countable union of Cantor sets  $C_n$ . Moreover if  $X = I^k$ , then the Cantor sets  $C_n$  can be chosen as flat Cantor sets in  $I^k$ .*

Let  $\mu$  be the Lebesgue measure on  $I^k$ . Note that there are subsets  $E$  of  $I^k - \partial I^k$  with  $\mu(E) = 1$  which are countable union of flat Cantor sets in  $I^k$ .

**Theorem 2.6.** *Suppose that  $E$  is a countable union of flat Cantor sets of  $I^k$  ( $k \geq 1$ ) such that  $\mu(E) = 1$ . If  $f : I^k \rightarrow I^k$  is everywhere  $*$ -chaotic, then for any  $\epsilon > 0$  there is a map  $g : I^k \rightarrow I^k$  such that  $f$  and  $g$  are topologically conjugate,  $d(f, g) < \epsilon$  and  $E$  is a scrambled set of  $g$ , where  $d(f, g) = \sup\{d(f(x), g(x)) \mid x \in X\}$ . In particular,  $g$  has a scrambled set which has Lebesgue measure 1.*

*Proof.* First, we assume  $k \geq 2$ . Note that the space  $H(X)$  of all homeomorphisms of a compactum  $X$  has a complete metric, i.e.,  $\rho(f, g) = d(f, g) + d(f^{-1}, g^{-1})$ . By Theorem 2.4, we can choose a scrambled set  $S$  of  $f$  such that  $S$  is a countable union of Cantor sets which are flat in  $I^k$ , and  $S$  is dense in  $I^k$ . Let  $\epsilon_1 > \epsilon_2 > \dots$  be a sequence of positive numbers with  $\sum_{n=1}^{\infty} \epsilon_n < \epsilon$ . Since any closed subset of a flat Cantor set is also flat, we can choose mutually disjoint flat Cantor sets  $D_n$  ( $n = 1, 2, \dots$ ) such that  $E = \bigcup_{n=1}^{\infty} D_n$ . For  $n = 1$ , we choose mutually disjoint  $k$ -cells  $B_j$  ( $j = 1, 2, \dots, p$ ) such that  $U = \bigcup_{j=1}^p B_j$  is a neighborhood of  $D_1$  and  $\text{diam}(B_j) < \epsilon_1$ . Also, since any closed subset of a flat Cantor set is flat, we can choose a flat Cantor set  $C_1 \subset S \cap \text{Int}(U)$  such that  $C_1 \cap B_j \neq \emptyset$  for each  $j$ . Then there is a homeomorphism  $h_1 : I^k \rightarrow I^k$  such that  $h_1(D_1) = C_1 \subset S, h_1|I^k - U = id$  (see the proof of [8, p. 93, Theorem 7]). Then  $\rho(id, h_1) < \epsilon_1$ . Put  $D' = h_1(D_2)$ . Note that  $D'$  is flat. We can choose a neighborhood  $V$  of  $D'$  such that  $V \cap h_1(D_1) = \emptyset$ . By the above argument, we obtain a homeomorphism  $h' : I^k \rightarrow I^k$  such that  $h'(D') \subset S, h'|I^k - V = id, \rho(id, h') < \epsilon_2$ . Put  $h_2 = h' \cdot h_1$ .

If we continue this procedure, we obtain a sequence  $h_n : I^k \rightarrow I^k$  of homeomorphisms such that

1.  $\rho(h_1, 1) < \epsilon_1, \rho(h_n, h_{n+1}) < \epsilon_n$  for each  $n$ ,
2.  $h_{n+1}(\bigcup_{j=1}^n D_j) = h_n(\bigcup_{j=1}^n D_j)$ , and  $h_n(D_n) \subset S$ .

Then we obtain a homeomorphism  $h = \lim_{n \rightarrow \infty} h_n$ , since  $H(X)$  is complete and the sequence  $\{h_n\}_{n=1}^{\infty}$  is a Cauchy sequence. By the condition 2, we see that  $h(E) \subset S$ . Put  $g = h^{-1} \cdot f \cdot h : I^k \rightarrow I^k$ . Then  $g$  is a desired map. The case  $k = 1$  is similarly proved by using Theorem 2.4 and [2]. This completes the proof.

*Remark 2.7.* For the proof of a weak version of Theorem 2.6, we can use the following theorem [9, Theorem 9] by Oxtoby and Ulam: for any subset  $B$  of any  $k$ -cube  $R$  ( $k \geq 1$ ), there is a homeomorphism  $h : R \rightarrow R$  with  $h|\partial R = id$  such that  $\mu(h(B)) = 0$  if and only if  $R - B$  contains a sequence of perfect sets whose union is dense in  $R$ . In fact, for any  $\delta > 0$ , we decompose  $I^k$  into  $k$ -cubes  $R_1, \dots, R_p$  such that  $\text{Int}(R_i) \cap \text{Int}(R_j) = \emptyset$  ( $i \neq j$ ),  $\text{diam}(R_i) < \delta$ , and  $I^k = R_1 \cup \dots \cup R_p$ . If  $S$  is a scrambled set of  $f$  such that  $S$  is dense in  $I^k$  and  $S$  is a countable union of Cantor sets, then for each  $R_i$  there is a homeomorphism  $h_i : R_i \rightarrow R_i$  such that  $h_i|\partial R_i = id, \mu(h_i(R_i - S)) = 0$ . Define a map  $h : I^k \rightarrow I^k$  by  $h|R_i = h_i$ . Put  $S' = h(S)$  and  $g = h \cdot f \cdot h^{-1}$ . Then  $\mu(S') = 1$ . If we choose a sufficiently small positive number  $\delta$ , then  $g$  has a scrambled set  $S'$  which has Lebesgue measure 1, and  $d(f, g) < \epsilon$ .

By [3, Corollary 3.3], every compact connected  $k$ -manifold ( $k \geq 2$ ) admits an everywhere  $*$ -chaotic homeomorphism. Hence we obtain

**Corollary 2.8.** *There is a homeomorphism  $f : I^k \rightarrow I^k$  ( $k \geq 2$ ) such that there is a scrambled set  $S$  of  $f$  with  $\mu(S) = 1$ .*

*Remark 2.9.* There is no homeomorphism  $f : I \rightarrow I$  with a nonempty scrambled set.

By the similar proof of Theorem 2.6, we obtain the following.

**Theorem 2.10.** *Let  $f : I^k \rightarrow I^k$  be a map of the unit  $k$ -cube  $I^k$  and suppose that there is an uncountable scramble set  $S$  of  $f$  with  $S \subset I^k - \partial I^k$ . Then for any  $\epsilon > 0$  there is a map  $g : I^k \rightarrow I^k$  such that  $d(f, g) < \epsilon$ ,  $g$  is topologically conjugate to*

$f$ , and there is a scrambled set  $S'$  of  $g$  such that  $S'$  is a Cantor set and  $S'$  has a positive Lebesgue measure.

Let  $f : X \rightarrow X$  be a map of a compactum  $X$ . A point  $x \in X$  is *recurrent* if for any neighborhood  $U$  of  $x$  in  $X$ , there is a natural number  $N > 0$  such that  $f^N(x) \in U$ . A point  $x \in X$  is *nonwandering* if for any neighborhood  $U$  of  $x$  in  $X$ , there is a natural number  $N > 0$  such that  $f^N(U) \cap U \neq \emptyset$ . If there is a point of  $X$  whose orbit is dense, then  $f$  is called *transitive*. It is well known that the set  $\Omega(f)$  of all nonwandering points of  $f$  is closed, and the set  $R(f)$  of all recurrent points of  $f$  is  $G_\delta$ . Put  $T(f) = \{x \in X \mid \text{the orbit of } x \text{ is dense in } X\}$ . Then  $T(f)$  is  $G_\delta$ -dense if  $f$  is transitive. Note that if  $X = \Omega(f)$ , then  $R(f)$  is dense in  $X$ . Note that, in general, if a set  $F$  is of the first category in  $X$ , then  $(F \times X) \cup (X \times F)$  is of the first category in  $X^2$ . Hence by the same argument as above, we can add the following restriction to scrambled sets.

**Corollary 2.11.** *Let  $f : I^k \rightarrow I^k$  be an everywhere \*-chaotic map.*

1. *If  $f$  is transitive, then for any  $\epsilon > 0$  there is a map  $g : I^k \rightarrow I^k$  such that  $f$  and  $g$  are topologically conjugate,  $d(f, g) < \epsilon$  and there is a scrambled set  $S \subset T(g)$  of  $g$  which has Lebesgue measure 1.*
2. *If  $\Omega(f) = X$ , then for any  $\epsilon > 0$  there is a map  $g : I^k \rightarrow I^k$  such that  $f$  and  $g$  are topologically conjugate,  $d(f, g) < \epsilon$  and there is a scrambled set  $S \subset R(g)$  of  $g$  which has Lebesgue measure 1.*

A map  $f : X \rightarrow X$  is *mixing* if for nonempty open sets  $U, V$  of  $X$ , there is a natural number  $N$  such that if  $n > N$ , then  $f^n(U) \cap V \neq \emptyset$ .

**Corollary 2.12** (cf. [13]). *Let  $f : I^k \rightarrow I^k$  be a mixing map. Then for any  $\epsilon > 0$  there is a map  $g : I^k \rightarrow I^k$  such that  $f$  and  $g$  are topologically conjugate,  $d(f, g) < \epsilon$  and there is a scrambled set  $S \subset T(g)$  of  $g$  which has Lebesgue measure 1.*

*Remark 2.13* (cf. [4, Applications]). 1. There is a Vitali set in the space  $E$  of reals containing a countable union  $S$  of Cantor sets such that  $S$  is dense in  $E$ .

2. If  $X$  is an indecomposable continuum, then  $X$  contains a subset  $S$  such that  $S$  is a countable union of Cantor sets,  $S$  is dense in  $X$ , and no two of points of  $S$  belong to the same component of  $X$ .

#### REFERENCES

1. B. A. M. Bruckner and T. Hu, On scrambled sets for chaotic functions, *Trans. Amer. Math. Soc.* 301 (1987), 289-297. MR **88f**:26003
2. W. J. Gorman III, The homeomorphic transformation of c-sets into d-sets, *Proc. Amer. Math. Soc.* 17 (1966), 825-830. MR **34**:7734
3. H. Kato, Attractors and everywhere chaotic homeomorphisms in the sense of Li-Yorke on manifolds and  $k$ -dimensional Menger manifolds, *Topology Appl.* 72 (1996), 1-17.
4. Ku K. Kuratowski, Applications of Baire-category method to the problem of independent sets, *Fund. Math.* 81 (1973), 65-72. MR **49**:3855
5. K. Kuratowski, *Topology I*, PWN, Warsaw, 1966. MR **36**:840
6. T. Y. Li and J. A. Yorke, Period three implies chaos, *Amer. Math. Monthly*, 82 (1975), 985-992. MR **52**:5898
7. M. Misiurewicz, Chaos almost everywhere, *Lecture Notes in Math.* 1163 (1985), 125-130. MR **87e**:58152
8. E. E. Moise, *Geometric topology in dimension 2 and 3*, Springer, 1977. MR **58**:7631
9. J. C. Oxtoby and S. M. Ulam, Measure-preserving homeomorphisms and metrical transitivity, *Ann. of Math.* 42 (1941), 873-920. MR **3**:211b

10. T. B. Rushing, Topological embeddings, Academic Press, New York, 1973. MR **50**:1247
11. J. Smítal, A chaotic function with some extremal properties, Proc. Amer. Math. Soc. 87 (1983), 54-56. MR **84h**:26008
12. J. Smítal, A chaotic function with a scrambled set of positive Lebesgue measure, Proc. Amer. Math. Soc. 92 (1984), 50-54. MR **86b**:26009b
13. J. Xiong and Z. Yang, Chaos caused by a topologically mixing map, World Scientific, Advanced Series in Dynamical Systems Vol 9, 550-572. MR **93c**:58153

INSTITUTE OF MATHEMATICS, UNIVERSITY OF TSUKUBA, IBARAKI, 305 JAPAN  
*E-mail address:* hisakato@sakura.cc.tsukuba.ac.jp