ON SCRAMBLED SETS AND A THEOREM OF KURATOWSKI ON INDEPENDENT SETS

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Abstract. The measure of scrambled sets of interval self-maps $f : I = [0, 1] \rightarrow I$ was studied by many authors, including Smítal, Misiurewicz, Bruckner and Hu, and Xiong and Yang. In this note, first we introduce the notion of “$*$-chaos” which is related to chaos in the sense of Li-Yorke, and we prove a general theorem which is an improvement of a theorem of Kuratowski on independent sets. Second, we apply the result to scrambled sets of higher dimensional cases. In particular, we show that if a map $f : I_k \rightarrow I_k$ of the unit $k$-cube $I_k$ is $*$-chaotic on $I_k$, then for any $\epsilon > 0$ there is a map $g : I_k \rightarrow I_k$ such that $f$ and $g$ are topologically conjugate, $d(f, g) < \epsilon$ and $g$ has a scrambled set which has Lebesgue measure 1, and hence if $k \geq 2$, then there is a homeomorphism $f : I_k \rightarrow I_k$ with a scrambled set $S$ satisfying that $S$ is an $F_\sigma$-set in $I_k$ and $S$ has Lebesgue measure 1.

1. Introduction

All spaces considered in this note are assumed to be separable and complete metric spaces. Maps are continuous functions. By a compactum we mean a compact metric space.

Let $f : X \rightarrow X$ be a map of a compactum $X$ with metric $d$. A subset $S$ of $X$ is a scrambled set of $f$ if there is a positive number $\tau > 0$ such that for any $x, y \in S$ with $x \neq y$,

1. $\limsup_{n \to \infty} d(f^n(x), f^n(y)) > \tau$,
2. $\liminf_{n \to \infty} d(f^n(x), f^n(y)) = 0$.

If there is an uncountable scrambled set $S$ of $f$, then we say that $f$ is chaotic in the sense of Li-Yorke. In the original paper [6] of Li and Yorke, there was the following one more condition: for any $x \in S$ and any periodic point $p \in X$, $\limsup_{n \to \infty} d(f^n(x), f^n(p)) > 0$. But it is known that this condition is unnecessary.

In [6], it was proved that if a map $f : I = [0, 1] \rightarrow I$ has a periodic point with period 3, then $f$ is chaotic in the sense of Li-Yorke. The measure of scrambled sets of interval self-maps was studied by many authors, including Smítal [11], [12], Misiurewicz [7], Bruckner and Hu [1], and Xiong and Yang [13].

In this note, we prove a general theorem which is an improvement of a theorem [4] of Kuratowski on independent sets, and we apply this result to scrambled sets of higher dimensional cases.
Let $A$ be a closed subset of a compactum $X$. A map $f : X \to X$ is \textit{*}-chaotic on $A$ (in the sense of Li-Yorke) if the following conditions are satisfied:

1. there is $\tau > 0$ such that if $U$ and $V$ are any nonempty open subsets of $A$ with $U \cap V = \emptyset$ and $N$ is any natural number, then there is a natural number $n \geq N$ such that $d(f^n(x), f^n(y)) > \tau$ for some $x \in U$, $y \in V$, and

2. for any nonempty open subsets $U, V$ of $A$ and any $\epsilon > 0$ there is a natural number $n \geq 0$ such that $d(f^n(x), f^n(y)) < \epsilon$ for some $x \in U$, $y \in V$.

If $f : X \to X$ is \textit{*}-chaotic on the total space $X$, we say that $f : X \to X$ is everywhere \textit{*}-chaotic.

Note that if $S$ is a scrambled set of $f$, then $f$ is \textit{*}-chaotic on $\text{Cl}(S)$. A map $f : X \to X$ of a compactum $X$ has sensitive dependence on initial conditions on a closed subset $A$ of $X$ if there is $\tau > 0$ such that if $x \in A$ and $U$ is any neighborhood of $x$ in $A$, then there is $y \in U$ such that $d(f^n(x), f^n(y)) > \tau$ for some $n$. Suppose that $A$ is a closed subset of $X$ and $A$ has no isolated point ($= A$ is perfect). Then $f$ is \textit{*}-chaotic on $A$ if and only if $f$ has sensitive dependence on initial conditions on $A$ and the above condition 2 is satisfied.

In the definition of chaos in the sense of Li-Yorke, we only assume the condition that there exists an uncountable scrambled set $S$ of $f : X \to X$, and in general, $S$ is an arbitrary subset of $X$. Note that if $S$ is an uncountable set, then $\text{Cl}(S)$ contains a Cantor set by the Cantor-Bendixon theorem (e.g., see [5, p. 253]).

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2. Independent sets and scrambled sets

In this section, first, we prove the following general theorem which is an improvement of a theorem of Kuratowski on independent sets (see [4]). Let $X$ be a space and $R$ be any subset of $X^m$ ($m \geq 2$). A subset $F \subset X$ is said to be independent in $R$ if for any different $m$ points $x_1, \ldots, x_m$ of $F$ (i.e., $x_i \neq x_j$ for $i \neq j$), we have $(x_1, x_2, \ldots, x_m) \in X^m - R$. For a subset $R \subset X^m$, consider the following property:

(*) If $U_1, \ldots, U_m$ are mutually disjoint nonempty sets of $X$, there are nonempty open sets $V_1, \ldots, V_m$ of $X$ such that $V_i \subset U_i$ for each $i$ and $V_1 \times \cdots \times V_m \subset X^m - R$.

Note that if $X$ has no isolated point, $R \subset X^m$ has the property (*) if and only if $R$ is nowhere dense (see Lemma 2.2).

A countable union of nowhere dense sets is called a set of the first category.

**Theorem 2.1.** Suppose that $X$ is a separable and completely metrizable space and $m \geq 2$ is a fixed natural number. Let $R \subset X^m$.

1. If $R = \bigcup_{n=1}^{\infty} R_n$ and each $R_n$ has the property (*), then there is an $F_\sigma$-set $S$ of $X$ such that $S$ is independent in $R$, and $\text{Cl}(S) = X$.

2. If $X$ has no isolated point and $R$ is of the first category, then there is a subset $S$ of $X$ such that $S = \bigcup_{n=1}^{\infty} C_n$, where $C_n$ are Cantor sets in $X$, $S$ is independent in $R$, and $\text{Cl}(S) = X$.

To prove this theorem, we need the following lemma by Kuratowski [4, Lemma, p. 66].

**Lemma 2.2.** Let $X$ be an arbitrary space and let $G$ be an open dense set in the space $X^m$ ($m \geq 1$). Let $H_1, \ldots, H_n$, where $n \geq m$, be open non-empty sets in $X$. 
Then there exists a system of nonempty open sets $U_1, \cdots, U_n$ such that $U_i \subset H_i$ for $1 \leq i \leq n$, and

$$U_{i_1} \times \cdots \times U_{i_m} \subset G,$$

whenever $i_1, \cdots, i_m$ are different $m$ elements of $\{1, 2, \cdots, n\}$.

**Proof of Theorem 2.1.** We prove the second case. The first case can be similarly proved.

We assume that $X$ has no isolated points. Also, we may assume that $R = \bigcup_{k=1}^{\infty} R_k$, where $R_k$ is nowhere dense and a closed subset of $X$ for each $k$. Let $\{U_{i,j}\}_{i=1}^{\infty}$ be a countable open base of $X$. Let $i = 1$. By the above lemma, we can choose a family

$$\mathcal{V}_1 = \{V_{1,1}, V_{1,2}, \cdots, V_{1,n_1}\}$$

of mutually disjoint nonempty closed subsets of $X$ such that $\text{Int}(V_{1,j}) \neq \emptyset$ for each $j$, $(\bigcup \mathcal{V}_1) \cap U_1 \neq \emptyset$, and for any different $m$ elements $V_{1,i_1}, \cdots, V_{1,i_m}$ of $\mathcal{V}_1$,

$$(V_{1,i_1} \times \cdots \times V_{1,i_m}) \cap (R_1) = \emptyset,$$

where $\bigcup \mathcal{V}_1 = \bigcup \{V | V \in \mathcal{V}_1\}$.

Inductively, by the above lemma we can choose a sequence

$$\mathcal{V}_i = \{V_{i,1}, \cdots, V_{i,n_i}\} \ (i \geq 1)$$

of families of mutually disjoint nonempty closed subsets of $X$ such that

1. $\text{Int}(V_{i,j}) \neq \emptyset$ and $\text{mesh}(\mathcal{V}_i) < 1/i$,
2. for each $V_{i,j}$, there are at least two elements $V_{i+1,p}, V_{i+1,q}$ ($p \neq q$) of $\mathcal{V}_{i+1}$ such that $V_{i+1,p} \cup V_{i+1,q} \subset \text{Int}(V_{i,j})$,
3. $U_i \cap (\bigcup \mathcal{V}_i) \neq \emptyset$, and
4. for any different $m$ elements $V_{i,j_1}, \cdots, V_{i,j_m}$ of $\mathcal{V}_i$,

$$(V_{i,j_1} \times \cdots \times V_{i,j_m}) \cap \left( \bigcup_{k=1}^{i} R_k \right) = \emptyset.$$

For each $i$, by induction we can obtain the sequence $\{i,j \ (j = i, i+1, i+2, \cdots)\}$ of subfamilies of $\mathcal{V}_j$ as follows:

$\mathcal{F}_{i,i} = \mathcal{V}_i, \quad \mathcal{F}_{i,j+1} = \{V \in \mathcal{V}_{j+1} | V \text{ is contained in some element of } \mathcal{F}_{i,j}\} \ (j \geq i)$.

Put $C_i = \bigcap_{j=i}^{\infty} (\bigcup \mathcal{F}_{i,j})$. Then we see that $C_i$ is a Cantor set and $\{C_i\}_{i=1}^{\infty}$ is increasing. Then by the condition 4, $S = \bigcup_{i=1}^{\infty} C_i$ is independent in $R = \bigcup_{k=1}^{\infty} R_k$ and by the condition 3, $S$ is dense in $X$. This completes the proof.

Let $I = [0, 1]$ be the unit interval and $I^k = [0, 1]^k$ the unit $k$-cube. A space homeomorphic to $I^k$ is a $k$-cell. A $k$-cell $B$ in $I^k - \partial I^k$ is flat if there is a homeomorphism $h : I^k \to I^k$ such that $h(B) = J^k$, where $J = [1/3, 2/3] \subset I$. A 0-dimensional compactum $D$ in $I^k - \partial I^k$ is flat in $I^k$ if for any neighborhood $V$ of $D$ in $I^k$, there is a neighborhood $U$ of $D$ in $I^k$ such that $U \subset V$ and $U = B_1 \cup \cdots \cup B_p$, where $B_i$ ($i = 1, 2, \cdots, p$) are mutually disjoint $k$-cells. Then we may assume that $B_i$ is flat in $I^k$ for each $i$, because we can choose a $k$-cell $B'$ in $\text{Int}(B_i)$ such that $\partial B'$ is locally flat and hence $B'$ is flat by Generalized Schoenflies Theorem (e.g. see [10, p. 48]). Note that if $C$ and $C'$ are flat Cantor sets in $I^k$ and $k \geq 2$, then any homeomorphism $f : C \cup \partial I^k \to C' \cup \partial I^k$ can be extended to a homeomorphism $F : I^k \to I^k$ (e.g., see the proof of [8, p. 93, Theorem 7]). Also, note that any
closed subset of a flat 0-dimensional compactum is also flat. It is well known that if \( k \leq 2 \), any Cantor set in \( I^k - \partial I^k \) is flat, but if \( k \geq 3 \), there are Cantor sets in \( I^k - \partial I^k \) which are not flat (see [8, p. 127]).

**Proposition 2.3.** Suppose that \( X \) is the unit \( k \)-cube \( I^k \) \( (k \geq 1) \). Let \( m \geq 2 \) be a fixed natural number and let \( R \subset X^m \) be of the first category in \( X^m \). Then there is a subset \( S \) of \( X \) such that \( S = \bigcup_{n=1}^{\infty} C_n \), where \( C_n \) are flat Cantor sets in the unit \( k \)-cube \( X \), \( S \) is independent in \( R \), and \( Cl(S) = X \).

**Proof.** In the proof of the above theorem, we may assume that each \( V_{i,j} \) is a \( k \)-cell (see Lemma 2.2).

**Theorem 2.4.** If \( f : X \to X \) is a map of a compactum \( X \) and \( f \) is \(*\)-chaotic on a closed set \( A \), then there is an \( F_\sigma \)-set \( S \subset A \) such that \( S \) is a scrambled set of \( f \) and \( Cl(S) = A \). If \( A \) has no isolated points, we can choose \( S \) such that \( S \) is a countable union of Cantor sets \( C_n \). Moreover if \( X = I^k \), then the Cantor sets \( C_n \) can be chosen as flat Cantor sets in \( X \).

**Proof.** Suppose that \( \tau \) is a positive number as in the definition of scrambled set \( S \). Consider the following sets:

\[
R_1 = \{(x, y) \in A^2 | \limsup_{i \to \infty} d(f^i(x), f^i(y)) < \tau \},
\]

\[
R_2 = \{(x, y) \in A^2 | \liminf_{i \to \infty} d(f^i(x), f^i(y)) > 0 \}.
\]

Let \( \epsilon_n = 1/n \) \((n = 1, 2, \ldots)\). Then \( R_1 = \bigcup_{n=1}^{\infty} R_n \), where

\[
T_n = \{(x, y) \in A^2 | d(f^i(x), f^i(y)) \leq \tau - \epsilon_n \text{ for every } i \geq n \}.
\]

Also, \( R_2 = \bigcup_{n=1}^{\infty} W_n \), where

\[
W_n = \{(x, y) \in A^2 | d(f^i(x), f^i(y)) \geq \epsilon_n \text{ for every } i \geq n \}.
\]

Note that \( T_n, W_n \subset A^2 \) are closed. Since \( f \) is \(*\)-chaotic on \( A \), they have the property (*). By Theorem 2.1 and Proposition 2.3, we obtain a desired scrambled set \( S \).

**Corollary 2.5.** Let \( f : X \to X \) be a map of a compactum \( X \) and \( S \) an uncountable scrambled set of \( f \). Then there is an \( F_\sigma \)-set \( S' \) of \( X \) such that \( S' \) is a scrambled set of \( f \) and \( Cl(S') = Cl(S) \), and \( S' \) contains a Cantor set. Moreover, if \( Cl(S) \) has no isolated point, \( S' \) can be chosen so that \( S' \) is a countable union of Cantor sets \( C_n \). Moreover if \( X = I^k \), then the Cantor sets \( C_n \) can be chosen as flat Cantor sets in \( I^k \).

Let \( \mu \) be the Lebesgue measure on \( I^k \). Note that there are subsets \( E \) of \( I^k - \partial I^k \) with \( \mu(E) = 1 \) which are countable union of flat Cantor sets in \( I^k \).

**Theorem 2.6.** Suppose that \( E \) is a countable union of flat Cantor sets of \( I^k \) \( (k \geq 1) \) such that \( \mu(E) = 1 \). If \( f : I^k \to I^k \) is everywhere \(*\)-chaotic, then for any \( \epsilon > 0 \) there is a map \( g : I^k \to I^k \) such that \( f \) and \( g \) are topologically conjugate, \( d(f, g) < \epsilon \) and \( E \) is a scrambled set of \( g \), where \( d(f, g) = \sup\{d(f(x), g(x)) | x \in X \} \). In particular, \( g \) has a scrambled set which has Lebesgue measure 1.
Proof. First, we assume \( k \geq 2 \). Note that the space \( H(X) \) of all homeomorphisms of a compactum \( X \) has a complete metric, i.e., \( \rho(f, g) = d(f, g) + d(f^{-1}, g^{-1}) \). By Theorem 2.4, we can choose a scrambled set \( S \) of \( f \) such that \( S \) is a countable union of Cantor sets which are flat in \( I^k \), and \( S \) is dense in \( I^k \). Let \( \epsilon_1 > \epsilon_2 > \cdots \) be a sequence of positive numbers with \( \sum_{n=1}^{\infty} \epsilon_n < \epsilon \). Since any closed subset of a flat Cantor set is also flat, we can choose mutually disjoint flat Cantor sets \( D_n \) \( (n = 1, 2, \cdots) \) such that \( E = \bigcup_{n=1}^{\infty} D_n \). For \( n = 1 \), we choose mutually disjoint \( k \)-cells \( B_j \) \( (j = 1, 2, \cdots, p) \) such that \( U = \bigcup_{j=1}^{p} B_j \) is a neighborhood of \( D_1 \) and \( \text{diam}(B_j) < \epsilon_1 \). Also, since any closed subset of a flat Cantor set is flat, we can choose a flat Cantor set \( C_1 \subset S \cap \text{Int}(U) \) such that \( C_1 \cap B_j \neq \emptyset \) for each \( j \). Then there is a homeomorphism \( h_1 : I^k \rightarrow I^k \) such that \( h_1(D_1) = C_1 \subset S, h_1|I^k - U = id \) (see the proof of [8, p. 93, Theorem 7]). Then \( \rho(id, h_1) < \epsilon_1 \). Put \( D' = h_1(D_2) \). Note that \( D' \) is flat. We can choose a neighborhood \( V \) of \( D' \) such that \( V \cap h_1(D_2) = \emptyset \).

By the above argument, we obtain a homeomorphism \( h' : I^k \rightarrow I^k \) such that \( h'(D') \subset S, h'|I^k - V = id, \rho(id, h') < \epsilon_2 \). Put \( h_2 = h' \cdot h_1 \).

If we continue this procedure, we obtain a sequence \( h_n : I^k \rightarrow I^k \) of homeomorphisms such that

1. \( \rho(h_1, 1) < \epsilon_1, \rho(h_n, h_{n+1}) < \epsilon_n \) for each \( n \),
2. \( h_{n+1}((\bigcup_{j=1}^{p} D_j)) = h_n((\bigcup_{j=1}^{p} D_j)), \) and \( h_n(D_n) \subset S \).

Then we obtain a homeomorphism \( h = \lim_{n \to \infty} h_n \), since \( H(X) \) is complete and the sequence \( \{h_n\}_{n=1}^{\infty} \) is a Cauchy sequence. By the condition 2, we see that \( h(E) \subset S \). Put \( g = h^{-1} : f \cdot h : I^k \rightarrow I^k \). Then \( g \) is a desired map. The case \( k = 1 \) is similarly proved by using Theorem 2.4 and [2]. This completes the proof.

Remark 2.7. For the proof of a weak version of Theorem 2.6, we can use the following theorem [9, Theorem 9] by Oxtoby and Ulam: for any subset \( B \) of any \( k \)-cube \( R \) \( (k \geq 1) \), there is a homeomorphism \( h : R \rightarrow R \) with \( h|\partial R = id \) such that \( \mu(h(B)) = 0 \) if and only if \( R - B \) contains a sequence of perfect sets whose union is dense in \( R \). In fact, for any \( \delta > 0 \), we decompose \( I^k \) into \( k \)-cubes \( R_1, \cdots, R_p \) such that \( \text{Int}(R_i) \cap \text{Int}(R_j) = \emptyset \) \( (i \neq j) \), \( \text{diam}(R_i) < \delta \), and \( I^k = R_1 \cup \cdots \cup R_p \). If \( S \) is a scrambled set of \( f \) such that \( S \) is dense in \( I^k \) and \( S \) is a countable union of Cantor sets, then for each \( R_i \) there is a homeomorphism \( h_i : R_i \rightarrow R_i \) such that \( h_i|\partial R_i = id, \mu(h_i(R_i - S)) = 0 \). Define a map \( h : I^k \rightarrow I^k \) by \( h|R_i = h_i \). Put \( S' = h(S) \) and \( g = h \cdot f \cdot h^{-1} \). Then \( \mu(S') = 1 \). If we choose a sufficiently small positive number \( \delta \), then \( h \) has a scrambled set \( S' \) which has Lebesgue measure 1, and \( d(f, g) < \epsilon \).

By [3, Corollary 3.3], every compact connected \( k \)-manifold \( (k \geq 2) \) admits an everywhere \( \ast \)-chaotic homeomorphism. Hence we obtain

Corollary 2.8. There is a homeomorphism \( f : I^k \rightarrow I^k \) \((k \geq 2)\) such that there is a scrambled set \( S \) of \( f \) with \( \mu(S) = 1 \).

Remark 2.9. There is no homeomorphism \( f : I \rightarrow I \) with a nonempty scrambled set.

By the similar proof of Theorem 2.6, we obtain the following.

Theorem 2.10. Let \( f : I^k \rightarrow I^k \) be a map of the unit \( k \)-cube \( I^k \) and suppose that there is an uncountable scrambled set \( S \) of \( f \) with \( S \subset I^k - \partial I^k \). Then for any \( \epsilon > 0 \), there is a map \( g : I^k \rightarrow I^k \) such that \( d(f, g) < \epsilon \), \( g \) is topologically conjugate to

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f, and there is a scrambled set S′ of g such that S′ is a Cantor set and S′ has a positive Lebesgue measure.

Let f : X → X be a map of a compactum X. A point x ∈ X is recurrent if for any neighborhood U of x in X, there is a natural number N > 0 such that fN(x) ∈ U. A point x ∈ X is nonwandering if for any neighborhood U of x in X, there is a natural number N > 0 such that fN(U) ∩ U ≠ ∅. If there is a point of X whose orbit is dense, then f is called transitive. It is well known that the set Ω(f) of all nonwandering points of f is closed, and the set R(f) of all recurrent points of f is Gδ. Put T(f) = {x ∈ X | the orbit of x is dense in X}. Then T(f) is Gδ-dense if f is transitive. Note that if X = Ω(f), then R(f) is dense in X. Note that, in general, if a set F is of the first category in X, then (F × X) ∪ (X × F) is of the first category in X2. Hence by the same argument as above, we can add the following restriction to scrambled sets.

**Corollary 2.11.** Let f : Ik → Ik be an everywhere *-chaotic map.
1. If f is transitive, then for any ε > 0 there is a map g : Iκ → Iκ such that f and g are topologically conjugate, d(f, g) < ε and there is a scrambled set S ⊂ T(g) of g which has Lebesgue measure 1.
2. If Ω(f) = X, then for any ε > 0 there is a map g : Iκ → Iκ such that f and g are topologically conjugate, d(f, g) < ε and there is a scrambled set S ⊂ R(g) of g which has Lebesgue measure 1.

A map f : X → X is mixing if for nonempty open sets U, V of X, there is a natural number N such that if n > N, then fκ(U) ∩ V ≠ ∅.

**Corollary 2.12 (cf. [13]).** Let f : Iκ → Iκ be a mixing map. Then for any ε > 0 there is a map g : Iκ → Iκ such that f and g are topologically conjugate, d(f, g) < ε and there is a scrambled set S ⊂ T(g) of g which has Lebesgue measure 1.

**Remark 2.13 (cf. [4, Applications]).** 1. There is a Vitali set in the space E of reals containing a countable union S of Cantor sets such that S is dense in E.
2. If X is an indecomposable continuum, then X contains a subset S such that S is a countable union of Cantor sets, S is dense in X, and no two of points of S belong to the same composant of X.

**References**


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