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A CONTINUATION TYPE RESULT FOR RANDOM OPERATORS

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Abstract. Fixed point results of continuation type are presented for random operators. Some applications to stochastic integral equations of Volterra type are also given.

1. Introduction and preliminaries

In this paper we present some random fixed point theorems for random operators. These results rely on classical continuation methods; in particular on the idea of an essential map. Our fixed point theory will then be applied to obtain a general existence principle for stochastic integral equations of Volterra type. This principle will then be used to establish the existence of sample solutions to a class of stochastic integral equations.

We now introduce some concepts which will be used throughout this paper. Let $(\Omega, \mathcal{A})$ denote a measurable space. For a metric space (X, d) we denote by $CD(X)$ all nonempty closed subsets of X, by $CB(X)$ all nonempty closed bounded subsets of X, and by $K(X)$ all nonempty compact subsets of X. A multivalued mapping $F : \Omega \to X$ is called measurable if for every open subset B of X, $F^{-1}(B) = \{w \in \Omega : F(w) \cap B \neq \emptyset\} \in \mathcal{A}$ (this type of measurability is usually called weakly measurable in the literature [4]). Notice that when $F(w) \in K(X)$ for all $w \in \Omega$, then F is measurable iff $F^{-1}(C) \in \mathcal{A}$ for every closed set C of X [4]. A measurable mapping $\xi : \Omega \to X$ is called a measurable selector of a measurable mapping $F : \Omega \to CD(X)$ if $\xi(w) \in F(w)$ for each $w \in \Omega$. Let Z be a nonempty closed subset of X. Then a mapping $F : \Omega \times Z \to X$ is called a *random operator* if for every $x \in Z$, the map $F(\,.\,, x) : \Omega \to X$ is measurable. A measurable map $\xi : \Omega \to X$ is called a *random fixed point* of a random operator $F : \Omega \times Z \to X$ if for every $w \in \Omega$ we have $F(w, \xi(w)) = \xi(w)$.

A single valued mapping $F : Z \subseteq X \to X$ is called a compact map if $F(Z)$ is precompact. We call F an α–Lipschitzian map if there is a constant $k \geq 0$ with $\alpha(F(Y)) \leq k\,\alpha(Y)$ for all bounded sets $Y \subseteq Z$; here $\alpha(Y)$ is the measure of noncompactness of Y, i.e.

$$\alpha(Y) = \inf\{\epsilon > 0 : Y \text{ can be covered by a finite number of sets of diameter } \leq \epsilon\}.$$

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F is called a condensing map if F is α–Lipschitzian with $k = 1$ and $\alpha(F(Y)) < \alpha(Y)$ for all bounded sets $Y \subseteq Z$ with $\alpha(Y) \neq 0$.

A random operator $F : \Omega \times Z \to X$ is called continuous (condensing, etc.) if for each $w \in \Omega$, $F(w, .)$ is continuous (condensing, etc.). If Z is any subset of a Banach space, then let $CK(Z)$ be the family of all nonempty compact convex subsets of Z.

Next we state the topological transversality theorem of Granas [3]. Let E be a Banach space, C a closed subset of E and U an open subset of C. We call $N : \overline{U} \times [0,1] \to C$ a condensing map if $\alpha(N(W)) \leq \alpha(\pi W)$ for all bounded sets W of $\overline{U} \times [0,1]$ and $\alpha(N(A)) < \alpha(\pi A)$ for all bounded nonprecompact subsets A of $\overline{U} \times [0,1]$; here $\pi : \overline{U} \times [0,1] \to \overline{U}$ is the natural projection. $K_{\partial U}(\overline{U}, C)$ denotes the set of all continuous, condensing maps $H : \overline{U} \to C$ with $H(\overline{U})$ a subset of a bounded set in C and with H fixed point free on ∂U; here $\overline{U}$ and ∂U denote the closure and boundary of U in C respectively. A mapping $F \in K_{\partial U}(\overline{U}, C)$ is essential (in the sense of Granas) if for every $H \in K_{\partial U}(\overline{U}, C)$ which agrees with F on ∂U we have that H has a fixed point in U. Two maps $F, G \in K_{\partial U}(\overline{U}, C)$ are homotopic in $K_{\partial U}(\overline{U}, C)$ written $F \cong G$ in $K_{\partial U}(\overline{U}, C)$ if there is a continuous, condensing map $N : \overline{U} \times [0,1] \to C$ with $N(\overline{U} \times [0,1])$ a subset of a bounded set in C and with $N_t(u) = N(u,t) : \overline{U} \to C$ belonging to $K_{\partial U}(\overline{U}, C)$ for each $t \in [0,1]$ and $N_0 = F$, $N_1 = G$. We now state two results of Granas [3], [6], [10], the first of which is called the topological transversality theorem in the literature.

Theorem 1.1. *Let U, C and E be as above. Suppose F and G are two maps in $K_{\partial U}(\overline{U}, C)$ such that $F \cong G$ in $K_{\partial U}(\overline{U}, C)$. Then F is essential iff G is essential.*

Theorem 1.2. *Let U, C and E be as above and let $u_0 \in U$. Define $F_0 : \overline{U} \to C$ by $F_0(u) = u_0$. Then the constant map $F_0 \in K_{\partial U}(\overline{U}, C)$ is essential.*

2. Fixed point theory

This section presents some general continuation type theorems for random operators. These generalize many well known random fixed point results in the literature [1], [2], [5], [8], [11], [12].

Theorem 2.1. *Let E be a Banach space, C a closed convex subset of E and U an open subset of C. Let $\overline{U}$ be a separable subset of C and $F : \Omega \times \overline{U} \to C$, $G : \Omega \times \overline{U} \to C$ be random operators. Assume for each $w \in \Omega$ that $F(w, .) \in K_{\partial U}(\overline{U}, C)$ and $G(w, .) \in K_{\partial U}(\overline{U}, C)$ with*

$$F(w, .) \cong G(w, .) \quad in \quad K_{\partial U}(\overline{U}, C).$$

If for each $w \in \Omega$ the map $G(w, .) \in K_{\partial U}(\overline{U}, C)$ is essential, then the random operator $F : \Omega \times \overline{U} \to C$ has a random fixed point.

Remark. In random fixed point theory in the literature $\overline{U}$ is a convex set.

Proof. Fix w and look at the set

$$H(w) = \{x \in \overline{U} : x = F(w,x)\}.$$

Now since we assumed $G(w, .)$ is essential the topological transversality theorem (theorem 1.1) implies that $F(w, .)$ is essential (in particular has a fixed point) and so $H(w) \neq 0$. Also $H(w)$ is compact valued (to see this notice that $H(w) \subseteq$

$F(w,H(w))$, so if $\alpha(H(w)) \neq 0$ then $\alpha(H(w)) \leq \alpha(F(w,H(w))) < \alpha(H(w))$, a contradiction). The mapping H is a multivalued mapping from Ω to $\overline{U}$. We claim H is measurable. Since H is compact valued, it suffices to show [4] that $H^{-1}(A)$ is measurable for any closed subset A of $\overline{U}$. Take a countable dense subset $\{x_n\}$ of $\overline{U}$ and look at

$$L(A) = \bigcap_{n=1}^{\infty} \bigcup_{x_i \in A_n} \left\{ w \in \Omega : \, |F(w,x_i) - x_i| < \frac{2}{n} \right\},$$

where

$$A_n = \left\{ x \in \overline{U} : \, d(x,A) < \frac{1}{n} \right\} \quad \text{and} \quad d(x,A) = \inf\{|x-y| : \, y \in A\}.$$

A standard argument [5, p. 263] yields $H^{-1}(A) = L(A)$ so H is measurable.

Now since the multivalued map $H : \Omega \to \overline{U}$ is measurable we may apply the selection theorem of Kuratowski and Ryll–Nardzewski [7] to $H(\,.\,)$ to deduce that there is a measurable selector $\phi : \Omega \to \overline{U}$ of H. In addition, $\phi(w) = F(w,\phi(w))$ for each $w \in \Omega$. □

Remark. It is possible, using the ideas in theorem 2.1 and in [5, p. 266], to obtain a multivalued version of theorem 2.1 for random operators (multivalued) $F : \Omega \times \overline{U} \to CK(C)$ which are continuous, bounded and condensing; here $CK(C)$ denotes the family of all nonempty compact convex subsets of C.

We now give some applications of our general random fixed point result (theorem 2.1).

Theorem 2.2. *Let E be a Banach space, C a closed convex subset of E and U an open subset of C. Let $\overline{U}$ be a separable subset of C, $p : \Omega \to C$ measurable with $p(w) \in U$ for each $w \in \Omega$, and $F : \Omega \times \overline{U} \to C$ a random operator. For each $w \in \Omega$ assume $F(w,\,.\,)$ is continuous and condensing with $F(w,\overline{U})$ bounded. In addition suppose*

$$(2.1) \quad \begin{cases} \text{for any } y : \Omega \to C \text{ measurable and any } w \in \Omega \text{ with } y(w) \in \partial U \\ \text{we have } y(w) \neq \lambda F(w,y(w)) + (1-\lambda)p(w) \\ \quad \text{for every } \lambda \in (0,1]. \end{cases}$$

Then F has a random fixed point (i.e. there exists a measurable $\phi : \Omega \to C$ such that $\phi(w) \in \overline{U}$ and $F(w,\phi(w)) = \phi(w)$ on Ω).

Proof. First we claim that $F(w,\,.\,)|_{\partial U}$ is fixed point free for each $w \in \Omega$. If this is not true then there exists $w_1 \in \Omega$ with

$$F(w_1,u_1) = u_1 \quad \text{for some} \quad u_1 \in \partial U.$$

Let $y(w) = u_1$ for each $w \in \Omega$. Then $y : \Omega \to C$ is measurable with $F(w_1,y(w_1)) = y(w_1)$ and $y(w_1) \in \partial U$. This contradicts (2.1), so our claim is true. Let $G : \Omega \times \overline{U} \to C$ be the random operator defined by $G(w,p(w)) = p(w)$. Notice that for each $w \in \Omega$ the map $G(w,\,.\,) \in K_{\partial U}(\overline{U},C)$ is essential by theorem 1.2. For each $w \in \Omega$, consider the homotopy $N_w : \overline{U} \times [0,1] \to C$ joining $G(w,\,.\,)$ and $F(w,\,.\,)$, given by

$$N_w(u,t) = tF(w,u) + (1-t)p(w).$$

It is clear that for each fixed $w \in \Omega$, $N_w : \overline{U} \times [0,1] \to C$ is continuous, bounded and condensing (to see this notice that if A is a bounded nonprecompact subset of $\overline{U} \times [0,1]$, then since $N_w(A) \subseteq co\,(F(w, \pi\, A) \cup \{p(w)\})$ we have $\alpha(N_w(A)) \leq \alpha(F(w, \pi\, A)) < \alpha(\pi\, A)$; here $\pi : \overline{U} \times [0,1] \to \overline{U}$ is the natural projection). Next we claim for each $w \in \Omega$ that $N_w(\,.\,, t)|_{\partial U}$ is fixed point free for each $t \in [0,1]$. If this is not true then there exist $w_2 \in \Omega$ and $t \in [0,1]$ with

$$tF(w_2, u_2) + (1-t)p(w_2) = u_2 \quad \text{for some} \quad u_2 \in \partial U.$$

Let $v(w) = u_2$ for each $w \in \Omega$. Then $v : \Omega \to C$ is measurable with $v(w_2) = tF(w_2, v(w_2)) + (1-t)p(w_2)$, $v(w_2) \in \partial U$ and $t \in [0,1]$. If $t \in (0,1]$ this contradicts (2.1), whereas if $t = 0$ we also have a contradiction since $p(w_2) \in U$. Thus $N_w(\,.\,, t)|_{\partial U}$ is fixed point free for each $t \in [0,1]$. Hence

$$F(w,\,.\,) \cong G(w,\,.\,) \quad \text{in} \quad K_{\partial U}(\overline{U}, C) \quad \text{for each} \quad w \in \Omega.$$

Now theorem 2.1 implies that the random operator $F : \Omega \times \overline{U} \to C$ has a random fixed point. □

Remark. We could have stated our Leray–Schauder condition (2.1) as follows: for each $w \in \Omega$, $u \neq \lambda F(w, u) + (1-\lambda)p(w)$ for all $u \in \partial U$ and $\lambda \in (0,1]$.

Essentially the same reasoning as in theorem 2.2 establishes the following more general version of theorem 2.2.

Theorem 2.3. *Let E be a Banach space, C a closed convex subset of E and U an open subset of C. Let $\overline{U}$ be a separable subset of C and $F : \Omega \times \overline{U} \to C$, $G : \Omega \times \overline{U} \to C$ random operators. Assume for each $w \in \Omega$ that the map $G(w,\,.\,) \in K_{\partial U}(\overline{U}, C)$ is essential. For each $w \in \Omega$ assume $F(w,\,.\,)$ is continuous and condensing, with $F(w, \overline{U})$ bounded. In addition suppose*

$$(2.2) \qquad \begin{cases} \text{for any } y : \Omega \to C \text{ measurable and any } w \in \Omega \\ \text{with } y(w) \in \partial U \text{ we have} \\ y(w) \neq \lambda\, F(w, y(w)) + (1-\lambda)G(w, y(w)) \\ \text{for every } \lambda \in (0,1]. \end{cases}$$

Then F has a random fixed point.

From an application point of view (see section 3) it is of interest to allow our set U to vary with w.

Theorem 2.4. *Let E be a Banach space, C a closed convex subset of E and $p : \Omega \to C$ measurable. Also assume $r : \Omega \to \mathbf{R}$ is measurable with $r(w) > 0$ for each $w \in \Omega$, and $Q_{r(w)} = \{x \in E : |x - p(w)| \leq r(w)\}$. Now suppose C_0 is a closed separable subset of C with $\overline{int\, C_0} = C_0$ and with $Q_{r(w)} \subseteq C_0$ for each $w \in \Omega$ and suppose the random operator $F : \Omega \times C_0 \to C$ is such that $F(w,\,.\,) : Q_{r(w)} \to C$ is a continuous, bounded, condensing map for every $w \in \Omega$. In addition assume*

$$(2.3) \qquad \begin{cases} \text{for any } y : \Omega \to C \text{ measurable and any } w \in \Omega \\ \text{with } y(w) \in \partial Q_{r(w)} \text{ we have} \\ (w) \neq \lambda\, F(w, y(w)) + (1-\lambda)p(w) \text{ for every } \lambda \in (0,1]. \end{cases}$$

Then F has a random fixed point (i.e. there exists a measurable $\phi : \Omega \to C$ such that $\phi(w) \in Q_{r(w)}$ and $F(w, \phi(w)) = \phi(w)$ on Ω).

Remark. In applications we usually have $C_0 = C = E$.

Proof. Without loss of generality assume $p(w) = 0$ for each $w \in \Omega$. Fix $w \in \Omega$. Let $R_w : E \to Q_{r(w)}$ be the continuous retraction given by

$$R_w(x) = \begin{cases} x, & x \in Q_{r(w)}, \\ r(w)\,\frac{x}{|x|}, & |x| > r(w). \end{cases}$$

Let $J(w,x) = F(w, R_w(x))$. Note that $J(w,\,.\,) : C_0 \to C$ is a continuous, bounded, condensing map (to see this note that if Z is a bounded subset of C_0 with $\alpha(Z) > 0$, then if $\alpha(R_w(Z)) > 0$ we have $\alpha(J(w,Z)) = \alpha(F(w,R_w(Z))) < \alpha(R_w(Z)) \le \alpha(Z)$ since $R_w(Z) \subseteq co\,(Z \cup \{0\})$, whereas if $\alpha(R_w(Z)) = 0$ then $\alpha(J(w,Z)) \le \alpha(R_w(Z)) < \alpha(Z)$). Next note that $J(\,.\,,x)$ is measurable for each $x \in C_0$. To see this note that for any $B \in \mathcal{L}$ (Borel σ–algebra of C) we have

$$\begin{aligned}\{w:\ J(w,x) \in B\} = [\,(F(\,.\,,x))^{-1}(B) \cap r^{-1}[|x|,\infty)\,] \\ \cup\,[\,\{w:\ F(w, r(w)\,\frac{x}{|x|}) \in B\} \cap r^{-1}(0,|x|)\,].\end{aligned}$$

Since $r(\,.\,)$ is a pointwise limit of step functions we have that $J(\,.\,,x)$ is measurable for each $x \in C_0$, and so $J : \Omega \times C_0 \to C$ is a random operator.

We now claim that $J(w,\,.\,)|_{\partial C_0}$ is fixed point free for each $w \in \Omega$ (note that $\partial C_0 = \partial(int\ C_0)$ since $\overline{int\ C_0} = C_0$ and C_0 is closed in C). If this is not true then there exists $w_1 \in \Omega$ with $J(w_1,u_1) = u_1$ for some $u_1 \in \partial C_0$. Let $y(w) = u_1$ for each $w \in \Omega$. Then $y : \Omega \to C$ is measurable with $F(w_1, R_{w_1}(y(w_1))) = y(w_1)$. Suppose $|y(w_1)| > r(w_1)$. Let $v(w) = R_{w_1} \circ y(w)$. Then $v : \Omega \to C$ is measurable with

$$v(w_1) = \lambda F(w_1, v(w_1)),\ \ v(w_1) = \frac{r(w_1)y(w_1)}{|y(w_1)|} \in \partial Q_{r(w_1)}$$

and

$$\lambda = \frac{r(w_1)}{|y(w_1)|} \in (0,1).$$

This contradicts (2.3). Thus $y(w_1) \in Q_{r(w_1)}$, so $y : \Omega \to C$ is measurable with $y(w_1) = F(w_1, y(w_1))$. This contradicts (2.3), so $J(w,\,.\,)|_{\partial C_0}$ is fixed point free for each $w \in \Omega$. Let $G : \Omega \times C_0 \to C$ be the random operator defined by $G(w,u) = 0$. Notice that for each $w \in \Omega$, the map $G(w,\,.\,) \in K_{\partial(int\,C_0)}(C_0,C)$ is essential by theorem 1.2. Now for each $w \in \Omega$, consider the homotopy $N_w : C_0 \times [0,1] \to C$ joining $G(w,\,.\,)$ and $J(w,\,.\,)$, given by

$$N_w(u,t) = t\,J(w,u).$$

It is clear that for each fixed $w \in \Omega$, $N_w : C_0 \times [0,1] \to C$ is continuous, bounded and condensing. Next we claim for each $w \in \Omega$ that $N_w(\,.\,,t)|_{\partial C_0}$ is fixed point free for each $t \in [0,1]$. If this is not true, then there exist $w_2 \in \Omega$ and $t \in [0,1]$ with $t\,J(w_2,u_2) = u_2$ for some $u_2 \in \partial C_0$. Let $\eta(w) = u_2$ for each $w \in \Omega$. Then $\eta : \Omega \to C$ is measurable with $t\,F(w_2, R_{w_2}(\eta(w_2))) = \eta(w_2)$. Suppose $|\eta(w_2)| > r(w_2)$. Let $\tau(w) = R_{w_2} \circ \eta(w)$. Then $\tau : \Omega \to C$ is measurable with

$$\tau(w_2) = \lambda F(w_2, \tau(w_2)),\ \ \tau(w_2) \in \partial Q_{r(w_2)}\ \ \text{and}\ \ \lambda = \frac{t\,r(w_2)}{|\eta(w_2)|} \in [0,1).$$

This contradicts (2.3) if $\lambda \in (0,1)$, whereas if $\lambda = 0$ we have a contradiction since $0 \in int\,Q_{r(w_2)}$. Thus $\eta(w_2) \in Q_{r(w_2)}$; so $\eta : \Omega \to C$ is measurable with

$\eta(w_2) = t\,F(w_2, \eta(w_2))$. This is a contradiction. Thus for each $w \in \Omega$ we have that $N_w(\,.\,, t)|_{\partial C_0}$ is fixed point free for each $t \in [0,1]$. Hence

$$J(w,\,.\,) \cong G(w,\,.\,) \quad \text{in} \quad K_{\partial(int\, C_0)}(C_0, C)$$

for each $w \in \Omega$. Now theorem 2.1 implies that the random operator $J : \Omega \times C_0 \to C$ has a random fixed point ϕ. Hence $\phi : \Omega \to C$ is such that $\phi(w) = F(w, R_w(\phi(w)))$ for each $w \in \Omega$. We now show that $\phi(w) \in Q_{r(w)}$ for each $w \in \Omega$. If this is not true, then there exists $w_3 \in \Omega$ with $|\phi(w_3)| > r(w_3)$. Let $\delta(w) = R_{w_3} \circ \phi(w)$. Now $\delta : \Omega \to C$ is measurable and

$$\delta(w_3) = \lambda F(w_3, \delta(w_3)) \quad \text{with} \quad \delta(w_3) \in \partial Q_{r(w_3)} \quad \text{and} \quad \lambda = \frac{r(w_3)}{|\phi(w_3)|} \in (0,1).$$

This contradicts (2.3). Consequently $\phi(w) \in Q_{r(w)}$ for each $w \in \Omega$, and so $\phi(w) = F(w, \phi(w))$ for each $w \in \Omega$. □

3. Stochasic integral equations of Volterra type

In this section we use theorem 2.4 to establish the existence of sample solutions to the stochasic Volterra problem

$$x(t,w) = \int_0^t k(t,s,w) f(s, x(s,w), w)\, ds, \quad t \in [0,T]. \tag{3.1}$$

Throughout, $(\Omega, \mathcal{A}, P)$ is a probability measure space (in fact the results hold for any measure space which is σ–finite). By a sample solution to (3.1) we mean a stochastic process $x : [0,T] \times \Omega \to \mathbf{R}^n$ (i.e., x is measurable in w for each $t \in [0,T]$) such that $x(\,.\,, w)$ is continuous on $[0,T]$ for each $w \in \Omega$ and (3.1) is satisfied on $[0,T] \times \Omega$.

We first prove a very general existence principle for the stochastic integral equation (3.1). This existence principle can then be used to establish existence results for (3.1).

Theorem 3.1. *Let $\alpha > 1$ be a constant and β the conjugate to α. Suppose the following conditions are satisfied:*

$$f(t,\,.\,,w) \quad \textit{is continuous for all} \quad (t,w), \tag{3.2}$$

$$f(\,.\,, x,\,.\,) \quad \textit{is measurable for all} \quad x, \tag{3.3}$$

$$\begin{cases} \textit{for each } r_0 > 0 \textit{ there exists a measurable } h_{r_0}(t,w) \\ \textit{with } h_{r_0} \in L^\beta[0,T] \textit{ for each } w \in \Omega, \\ \textit{and } |f(t,x,w)| \le h_{r_0}(t,w) \\ \textit{for a.e. } t \in [0,T] \textit{ and each } |x| \le r_0,\, w \in \Omega\,; \\ \textit{also assume } \int_\Omega \int_0^T h_{r_0}(t,w)\, dt\, dw < \infty, \end{cases} \tag{3.4}$$

$$k(t,\,.\,,\,.\,) \quad \textit{is measurable for all} \quad t, \tag{3.5}$$

$$\textit{for each } w \in \Omega,\ k(t,s,w) \in L^\alpha[0,T] \textit{ for each } t \in [0,T], \tag{3.6}$$

and

$$\begin{cases} \textit{for each } w \in \Omega, \quad \textit{the map } t \mapsto k(t,s,w) \\ \textit{is continuous from } [0,T] \textit{ to } L^\alpha[0,T]. \end{cases} \tag{3.7}$$

In addition assume there is a measurable $r : \Omega \to \mathbf{R}$*, with* $r(w) > 0$ *for each* $w \in \Omega$*, and for each fixed* $w \in \Omega$ *suppose* $|y(w)|_0 = \sup_{t\in[0,T]} |y(t,w)| \neq r(w)$ *for any solution* $y(t,w)$ *(*w *fixed) to*

$$(3.8)_\lambda \qquad y(t,w) = \lambda \int_0^t k(t,s,w)\, f(s, y(s,w), w)\, ds$$

for each $\lambda \in (0,1]$*; also assume* $r(w)$ *is independent of* λ*. Then* (3.1) *has a solution (as described above).*

Remark. For each fixed $w \in \Omega$, by a solution to $(3.8)_\lambda$ we mean a function $y(t)\,(= y(t,w)) \in C[0,T]$ with $y(t) = \lambda \int_0^t k(t,s,w)\, f(s,y(s),w)\, ds$ for $t \in [0,T]$.

Proof. Let $E = C[0,T]$ and $Q_{r(w)} = \{x \in C[0,T] : |x|_0 \le r(w)\}$. Define $F : \Omega \times C[0,T] \to C[0,T]$ by

$$F(w,x)(t) = \int_0^t k(t,s,w)\, f(s,x(s),w)\, ds.$$

We wish to apply theorem 2.4. Now (3.2), (3.3) (since $f(\,.\,,x,w)$ is measurable for all (x,w)), (3.4), (3.6) and (3.7) together with standard arguements [10] imply that $F(w,\,.\,) : Q_{r(w)} \to C[0,T]$ is a continuous, compact map for every $w \in \Omega$ (notice that $F(w,\,.\,) : C[0,T] \to C[0,T]$ is completely continuous for every $w \in \Omega$). Next we show that $F : \Omega \times C[0,T] \to C[0,T]$ is a random operator. Fix $x \in C[0,T]$. We need to show that $F(\,.\,,x)$ is measurable. Now for any $x \in C[0,T]$, (3.3) and (3.5) imply for fixed $t \in [0,T]$ that $k(t,s,w)\, f(s,x(s),w)$ is product measurable on $[0,t] \times \Omega$, and so $w \mapsto \int_0^t k(t,s,w)\, f(s,x(s),w)\, ds$ is measurable on Ω [13] (see Tonelli's and Fubini's theorem). Thus $F(\,.\,,x)(t)$ is measurable for each fixed $t \in [0,T]$. Then since $C[0,T]$ is separable, a standard argument [5] implies that $F(\,.\,,x)$ is measurable.

Remark. For completeness we provide the argument. It suffices to show, for any $r_1 > 0$, $x \in C[0,T]$ and rational $t \in [0,T]$, that $(F(\,.\,,x))^{-1}(B) \in \mathcal{A}$ if $B = \{v \in C[0,T] : |v(t) - x(t)| \le r_1\}$. Now

$$\begin{aligned}(F(\,.\,,x))^{-1}(B) &= \{w \in \Omega : F(w,x)(t) \in B\} \\ &= \{w \in \Omega : |F(w,x)(t) - x(t)| \le r_1\} \in \mathcal{A}\end{aligned}$$

since $F(\,.\,,x)(t)$ is measurable for each fixed $t \in [0,T]$.

Finally we show (2.3) is satisfied. If (2.3) is not true then there exists $y : \Omega \to C[0,T]$ measurable with $y(w) = \lambda F(w, y(w))$ for some $w \in \Omega$ and some $\lambda \in (0,1]$ with $y(w) \in \partial Q_{r(w)}$, i.e., for $t \in [0,T]$, $y(w)(t) = \lambda F(w, y(w)(t))$ for some $w \in \Omega$ and some $\lambda \in (0,1]$ with $r(w) = |y(w)|_0 = \sup_{[0,T]} |y(w)(t)|$. For the above w, $y(w)$ is a solution to $(3.8)_\lambda$ with $\sup_{[0,T]} |y(w)(t)| = r(w)$. This is a contradiction. Consequently (2.3) is true. Now theorem 2.4 implies that there exists a measurable $\phi : \Omega \to C[0,T]$ with $\phi(w) = F(w, \phi(w))$ on Ω (i.e. for $t \in [0,T]$, $\phi(w)(t) = F(w, \phi(w)(t))$). Define $\phi^\star : [0,T] \times \Omega \to C[0,T]$ by $\phi^\star(w,t) = \phi(w)(t)$. Now $\phi^\star(\,.\,,t)$ is measurable for every $t \in [0,T]$, since for t fixed we have $\phi^\star(w,t) = j_t \circ \phi(w)$, where $j_t : C[0,T] \to \mathbf{R}^n$ is the continuous mapping given by $j_t(u) = u(t)$. It is also clear that $\phi^\star(w,\,.\,)$ is continuous for every $w \in \Omega$. Also for each $w \in \Omega$ and each $t \in [0,T]$ we have

$$\phi^\star(w,t) = \phi(w)(t) = F(w, \phi(w)(t)) = F(w, \phi^\star(w,t)).$$

Thus $\phi^\star(w,t)$ is a solution of $(3.8)_1$ (i.e. (3.1)). □

Remark. One could also consider a general measure space $(\Omega, \mathcal{A}, P)$. Notice that in theorem 3.1 we only needed the measure space to be σ–finite and

$$\int_\Omega \int_0^T h_{r_0}(t,w)\,dt\,dw < \infty$$

in (3.4) so that Tonelli's and Fubini's theorem could be applied. In many cases, even in a general measure space $(\Omega, \mathcal{A}, P)$, one can show (without the aid of Tonelli's and Fubini's theorem) that $w \mapsto \int_0^t k(t,s,w)\,f(s,x(s),w)\,ds$ is measurable on Ω, for any $x \in C[0,T]$.

Theorem 3.2. *Let* $\alpha > 1$ *be a constant and* β *the conjugate to* α. *Assume* (3.2)–(3.7) *hold. In addition suppose*

$$(3.9)\qquad \begin{cases} \text{there exist a measurable } a:\Omega \to \mathbf{R}, \quad \text{with } a(w) \geq 0 \\ \text{for each } w \in \Omega, \text{ and a function } \mu:[0,T] \to [0,\infty), \\ \text{with } \int_0^T \mu(s)\,ds < \infty, \\ \text{such that } |k(t,s,w)\,f(s,x,w)| \leq \mu(s)\,a(w)\,[1+|x|] \\ \text{for } (t,s,w) \in [0,T]\times[0,t]\times\Omega \text{ and } x \in \mathbf{R}. \end{cases}$$

Then (3.1) *has a sample solution.*

Proof. Let

$$r(w) = \ln\left(1 + a(w)\int_0^T \mu(s)\,ds\right) + 1.$$

Clearly $r:\Omega \to \mathbf{R}$ is measurable. Let $w \in \Omega$ be fixed and suppose $y(t,w)$ (w fixed) is a solution of $(3.8)_\lambda$ for $0 < \lambda \leq 1$. Then for $t \in [0,T]$ we have

$$(3.10)\qquad |y(t,w)| \leq a(w)\int_0^t \mu(s)\,[1+|y(s,w)|]\,ds \equiv u(t,w).$$

Now

$$u'(t,w) = a(w)\,\mu(t)\,[1+|y(t,w)|] \leq a(w)\,\mu(t)\,[1+u(t,w)].$$

Integration from 0 to x, $x \in [0,T]$, yields

$$(3.11)\qquad \int_0^{u(x,w)} \frac{dv}{1+v} \leq a(w)\int_0^x \mu(s)\,ds \leq a(w)\int_0^T \mu(s)\,ds,$$

and so

$$u(x,w) \leq \ln\left(1 + a(w)\int_0^T \mu(s)\,ds\right).$$

This together with (3.10) yields

$$|y(x,w)| \leq \ln\left(1 + a(w)\int_0^T \mu(s)\,ds\right) \neq r(w) \quad \text{for each} \quad x \in [0,T].$$

Thus $|y(w)|_0 \neq r(w)$ for any solution $y(t,w)$ (w fixed) to $(3.8)_\lambda$ for $0 < \lambda \leq 1$. Now theorem 3.1 implies that (3.1) has a sample solution. □

Remark. Of course $1+|x|$ in assumption (3.9) can be replaced by $\psi(|x|)$, where $\psi:[0,\infty)\to(0,\infty)$ is a nondecreasing Borel function with $\frac{1}{\psi}\in L^1_{loc}[0,\infty)$. Then (3.1) has a solution if

$$a(w)\int_0^T \mu(s)\,ds < \int_0^\infty \frac{dx}{\psi(x)} \quad \text{for each} \quad w\in\Omega.$$

To see this, notice that (3.11) in this case becomes

$$\int_0^{u(x,w)} \frac{dv}{\psi(v)} \le a(w)\int_0^x \mu(s)\,ds \le a(w)\int_0^T \mu(s)\,ds.$$

Let $J:[0,\infty)\to[0,\infty)$ be the increasing function $J(z)=\int_0^z \frac{dx}{\psi(x)}$; then we have

$$|y(x,w)| \le u(x,w) \le J^{-1}\left(a(w)\int_0^x \mu(s)\,ds\right) \quad \text{for each} \quad x\in[0,T].$$

Now let $r(w)=J^{-1}\left(a(w)\int_0^x \mu(s)\,ds\right)+1$, and we are finished.

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