SOME EXTREMAL PROBLEMS IN $L^p(w)$

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(Communicated by J. Marshall Ash)

Abstract. Fix a positive integer $n$ and $1 < p < \infty$. We provide expressions for the weighted $L^p$ distance
\[
\inf\int_0^{2\pi} |1 - f|^p w \, d\lambda,
\]
where $d\lambda$ is normalized Lebesgue measure on the unit circle, $w$ is a nonnegative integrable function, and $f$ ranges over the trigonometric polynomials with frequencies in
\[
S_1 = \{\ldots, -3, -2, -1\} \cup \{1, 2, 3, \ldots, n\},
\]
or
\[
S_2 = \{\ldots, -3, -2, -1\} \setminus \{-n\},
\]
or
\[
S_3 = \{\ldots, -3, -2, -1\} \cup \{n\}.
\]
These distances are related to other extremal problems, and are shown to be positive if and only if $\log w$ is integrable. In some cases they are expressed in terms of the series coefficients of the outer functions associated with $w$.

Let $w$ be a nonnegative integrable function on the unit circle in the complex plane, and consider the Banach space $L^p(w)$ for $1 < p < \infty$. A natural subspace of $L^p(w)$ is associated with each subset $S$ of the integers $\mathbb{Z}$, namely
\[
\mathcal{M}(S) = \bigvee\{e^{ik\theta} : k \in S\}.
\]
Writing $d\lambda$ for normalized Lebesgue measure on the unit circle, we denote the distance from the constant function $1$ to the subspace $\mathcal{M}(S)$ by
\[
\sigma_p(w, S) = \inf \left\{ \left( \int |1 - f|^p w \, d\lambda \right)^{1/p} : f \in \mathcal{M}(S) \right\}.
\]
This notion has been of considerable interest in the theory of stationary processes and harmonic analysis (see [4, 6, 7, 9]). This paper is concerned with evaluating $\sigma_p(w, S)$, and exploring its relationship to other constructions. It is a sequel to [6], to which the reader is referred for further history and background material.

In particular, for a fixed integer $n \geq 1$, we are interested in examining $\sigma_p(w, S)$ for
\[
S_1 = \{\ldots, -3, -2, -1\} \cup \{1, 2, 3, \ldots, n\},
\]

Received by the editors March 27, 1996 and, in revised form, January 13, 1997.

1991 Mathematics Subject Classification. Primary 42A10, 60G25.

Key words and phrases. Prediction error, outer function, dual extremal problem, stationary sequences.

The second author’s research was supported by Office of Naval Research Grant No. N00014-89-J-1824 and U.S. Army Grant No. DAARO-96-1-0027.

The third author’s research was supported in part by NSA Grant No. MDA904-97-1-0013.

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w σ only the null function, without affecting the value of the supremum. This expresses σ

S_3 = \{ \ldots, -3, -2, -1 \} \cup \{ n \}.

All of these frequency sets are natural departures from the classical case of the halfline, S_0 = \{ \ldots, -3, -2, -1 \}.

Let Z_0 = \mathbb{Z} \setminus \{ 0 \}, and for any subset S \subseteq Z_0, let S^c = Z_0 \setminus S be the complement of S in Z_0. The following result provides a way to calculate \( \sigma_p(w, S) \) when the prediction problem for the complementary frequency set \( S^c \) is understood. It is a special case of Theorem 3.1 in [6] with a shorter and more direct proof. In what follows, we fix the parameter p with 1 < p < \infty, we define q by

\[
\frac{1}{q} + \frac{1}{p} = 1,
\]

and we put

\[
s = -\frac{1}{p - 1}.
\]

**Theorem 1.** Suppose that w is a nonnegative integrable function on the circle, and \( w^s \) is integrable. For any subset S \subseteq Z_0, we have

\[
\sigma_p(w, S) = \sigma_q(w^s, S^c)^{-1},
\]

provided that \( \sigma_p(w, S) \) or \( \sigma_q(w^s, S^c) \) is positive.

**Proof.** By logical symmetry it suffices to assume that \( \sigma_p(w, S) \) is positive. Indeed, this is because p and q are conjugate indices to each other, \( (S^c)^c = S \), and

\[
(w^s)^{-1/(q-1)} = w^{1/(p-1)(q-1)} = w.
\]

Let us write \( H(S) \) for the collection of finite trigonometric sums with frequencies in S. Then

\[
\sigma_p(w, S)p = \inf \left\{ \int |1 - \phi|^p w \, d\lambda : \phi \in H(S) \right\}.
\]

Replace each \( 1 - \phi \) with a member of \( H(S \cup \{ 0 \}) \), divided by its constant term. Then we get

\[
\sigma_p(w, S) = \inf \left\{ \int |\hat{\phi}|^p w \, d\lambda : \phi \in H(S \cup \{ 0 \}), \hat{\phi}_0 \neq 0 \right\}.
\]

Note that we used \( \hat{\phi}_0 = \int \phi \, d\lambda \). Now

\[
\sigma_p(w, S) = \left[ \sup \left\{ \frac{\| \phi \|^p}{\| \phi w \|^p} : \phi \in H(S \cup \{ 0 \}), \phi_0 \neq 0 \right\} \right]^{-1}
\]

Note that \( s - sp = 1 \), so the denominator is indeed unchanged.

In the last expression, we may allow \( \phi \) to range over all of \( H(S \cup \{ 0 \}) \), excluding only the null function, without affecting the value of the supremum. This expresses \( \sigma_p(w, S) \) as the reciprocal of the norm of 1, viewed as a bounded linear functional on the span of \{ \phi w^{-s} : \phi \in H(S \cup \{ 0 \}) \} in \( L^p(w^s) \). But then this is just the distance from 1 to the annihilator of \{ \phi w^{-s} : \phi \in H(S \cup \{ 0 \}) \} in \( L^q(w^s) \). That annihilator consists exactly of those \( f \) in \( L^q(w^s) \) such that \( \int f \cdot \phi w^{-s} \cdot w^s \, d\lambda = 0 \) for
all \( \phi \in H(S \cup \{0\}) \). The collection of such \( f \), in turn, is spanned by \( H(\mathbb{Z} \setminus [S \cup \{0\}]) = H(S^c) \). Hence

\[
\sigma_p(w, S) = \left[ \inf \left\{ \left|1 - f\right|^q w^s \, d\lambda : f \in H(S^c) \right\} \right]^{-1} = [\sigma(q(w^s, S^c))]^{-1}.
\]

The next proposition asserts that if the index set \( S \) is a halfline with finitely many points of \( \mathbb{Z} \) added or deleted, then \( \sigma_p(w, S) \) is positive exactly when \( \log w \in L^1 = L^1(\lambda) \). In the classical case \( S = S_0 \) and \( p > 0 \), the result is well-known [3, p. 136]. Here and henceforth, the exponential function \( e^{ik\theta}, k \in \mathbb{Z} \), is denoted by \( e_k \).

**Theorem 2.** Suppose that \( w \) is a nonnegative integrable function on the circle. Let 

\[ S = (\{\ldots, -3, -2, -1\} \cup \{J_1, J_2, \ldots, J_M\}) \setminus \{K_1, K_2, \ldots, K_N\}, \]

where

\[ 0 < J_1 < J_2 < \ldots < J_M, \]

and

\[ 0 > K_1 > K_2 > \ldots > K_N. \]

Then \( \sigma_p(w, S) \) is positive if and only if \( \log w \in L^1 \).

**Proof.** If \( \log w \) is not integrable, then \( \sigma_p(w, T) = 0 \) for \( T = \{\ldots, K_N - 3, K_N - 2, K_N - 1\} \). Since \( T \subseteq S \), we have

\[ \sigma_p(w, T) \geq \sigma_p(w, S) \geq 0. \]

It follows that \( \sigma_p(w, S) = 0 \).

Conversely, suppose that \( \sigma_p(w, S) = 0 \), so that \( e_0 \) belongs to the subspace \( M(S) \).

Then \( e_0 \) certainly belongs to \( M(U) \), where \( U \) is an index set of the form

\[ U = \{\ldots, -3, -2, -1\} \cup \{1, 2, \ldots, r\}. \]

Indeed, the inclusion \( S \subseteq U \) holds when \( r = J_M \). But let us choose \( r \) to be the smallest positive integer for which \( \sigma_p(w, U) = 0 \).

There exist coefficients \( c^{(j)}_r \) and \( V_j \in M(S_0 \cup \{1, 2, \ldots, r - 1\}) \) such that

\[ e_0 = \lim_{j \to \infty} (c^{(j)}_r e_r + V_j) \]

in the norm of \( L^p(w) \). The sequence \( c^{(1)}_r, c^{(2)}_r, c^{(3)}_r, \ldots \) must be bounded away from zero by some positive distance \( \rho \). If not, then \( c^{(jm)}_r \to 0 \) for some subsequence. This would imply that

\[ e_0 = \lim_{m \to \infty} c^{(jm)}_r e_r + \lim_{m \to \infty} V_{jm}. \]

The first limit is zero, and the resulting equation violates the minimality of \( r \).

Now we have

\[
0 \leq \rho \left\| \frac{1}{c^{(j)}_r} e_0 - e_r - \frac{1}{c^{(j)}_r} V_j \right\|_p \\
\leq \|e_0 - c^{(j)}_r e_r - V_j\|_p \\
\to 0.
\]

This shows that \( e_r \in M(\{\ldots, r - 3, r - 2, r - 1\}) \), giving \( \log w \notin L^1 \). \[\square\]
Thus the condition \( w^* \in L^1 \) in Theorem 1 is not natural to the present applications, and we seek to replace it with the weaker condition \( \log w \in L^1 \). Unfortunately, the related quantity \( \sigma_q(w^*, S^c) \) might not be defined under this weaker condition. Thus we must bring in yet another dual extremal problem, one tied to the metric projection of \( L^p \) onto the Hardy space \( H^q \) of the unit circle.

**Theorem 3.** Suppose that \( w \) is nonnegative and integrable. If \( \log w \in L^1 \), then

\[
\sigma_p(w, S_1) = \text{dist}_{L^q}(\phi^{(n)}, e_{n+1} H^q)^{-1},
\]

where \( \phi \) is the outer function satisfying

\[
w^* = |\phi|^q,
\]

and \( \phi^{(n)} \) is the truncated series

\[
\phi^{(n)}(e^{i\theta}) = \sum_{k=0}^{n} \hat{\phi}_k e^{ik\theta}.
\]

**Proof.** First assume that \( w^* \in L^1 \). Then Theorem 1 applies, yielding

\[
\sigma_p(w, S_1) = \sigma_q(w^*, S^c_1)^{-1}
\]

\[
= \inf \left\{ \left( \int |1 - f|^q w^* \, d\lambda \right)^{1/q} : f \in H(S^c_1) \right\}^{-1}
\]

\[
= \inf \left\{ \left( \int |\phi - \phi f|^q \, d\lambda \right)^{1/q} : f \in H(S^c_1) \right\}^{-1}
\]

\[
= \inf \left\{ \left( \int |\phi^{(n)} - f|^q \, d\lambda \right)^{1/q} : f \in H(S^c_1) \right\}^{-1},
\]

where in the last step we used the fact that \( \phi \) is outer in \( H^q \). This confirms (1) when \( w^* \in L^1 \). More generally, for any positive integer \( m \) define \( w_m = \max\{w, 1/m\} \), and let \( \phi_m \) be the outer function satisfying

\[
w^*_m = |\phi_m|^q.
\]

Since \( \log w_m \in L^1 \), the preliminary result applies, and we get

\[
\sigma_p(w_m, S_1) = \text{dist}_{L^q}(\phi^{(m)}_m, e_{n+1} H^q)^{-1}.
\]

Next, we argue that \( \sigma_p(w_m, S_1) \to \sigma_p(w, S_1) \) as \( m \to \infty \). To see this, note that for any \( \epsilon > 0 \), there exists an \( f_0 \in H(S_1) \) such that

\[
\sigma_p(w, S_1)^p \leq \int |1 - f_0|^p w \, d\lambda < \sigma_p(w, S_1)^p + \frac{\epsilon}{2}. \tag{2}
\]

Since \( w_m \) is a decreasing sequence of functions converging to \( w \), the dominated convergence theorem (with dominating function \( w + 1 \)) provides that

\[
\int |1 - f_0|^p w_m \, d\lambda \to \int |1 - f_0|^p w \, d\lambda.
\]

Hence there exists an \( M > 0 \) such that for all \( m > M \) we have

\[
\int |1 - f_0|^p w \, d\lambda \leq \int |1 - f_0|^p w_m \, d\lambda \leq \int |1 - f_0|^p w \, d\lambda + \frac{\epsilon}{2}. \tag{3}
\]
Combining (2) and (3), we get
\[
\sigma_p(w, S_1)^p \leq \int |1 - f_0|^p w_m \, d\lambda \leq \sigma_p(w, S_1)^p + \epsilon,
\]
whenever \( m > M \). This implies that
\[
\sigma_p(w, S_1)^p \leq \sigma_p(w_m, S_1)^p \leq \sigma_p(w, S_1)^p + \epsilon,
\]
and hence
\[
\lim_{m \to \infty} \sigma_p(w_m, S_1) = \sigma_p(w, S_1).
\]

Finally we confirm that \( \phi_m^{(n)} \to \phi^{(n)} \) uniformly on the circle, hence in \( L^q \) norm. Since
\[
|\log w_m| \leq |\log w|
\]
for all positive integers \( m \), it follows that for each \( z \) with \( |z| < 1 \) we have
\[
\left| \frac{e_1 + z}{e_1 - z} \log w_m \right| \leq g(|z|) |\log w_m| \leq g(|z|) |\log w|
\]
for some positive function \( g \). Applying the dominated convergence theorem to
\[
\phi_m = \exp \left[ \frac{s}{q} \int \frac{e_1 + z}{e_1 - z} \log w_m \, d\lambda \right],
\]
we get
\[
\lim_{m \to \infty} \phi_m(z) = \phi(z)
\]
for all \( |z| < 1 \). In fact, the convergence is uniform on any closed disc \( |z| \leq R < 1 \). Thus for the corresponding power series coefficients we have
\[
\lim_{m \to \infty} \hat{\phi}_{m,k} = \hat{\phi}_k,
\]
k = 0, 1, 2, \ldots. In particular, the truncated series \( \phi_m^{(n)} \) converges uniformly to \( \phi^{(n)} \). Now the continuity of the metric projection in \( L^q \) yields
\[
\lim_{m \to \infty} \text{dist}_{L^q}(\phi_m^{(n)}, e_{n+1} H^q) = \text{dist}_{L^q}(\phi^{(n)}, e_{n+1} H^q).
\]
The claim is proved.

This second formulation turns out to be a fruitful one, since it reduces the computation of \( \sigma_p(w, S_1) \) to the well established dual extremal problem of computing \( \text{dist}_{L^q}(\phi^{(n)}, e_{n+1} H^q) \) (see [2, pp. 136–146]).

With these preliminaries it is possible to solve the prediction problems for the case \( p = 2 \). Here we suppose that \( \log w \in L^1 \), so that \( w = |\psi|^2 \) for some outer function \( \psi \) in \( H^2 \). In this case \( s = -1 \), and the function \( w^s \) factors into \( |1/\psi|^2 \). We write the associated power series expansions
\[
\psi(z) = \sum_{k=0}^{\infty} c_k z^k,
\]
\[
\frac{1}{\psi(z)} = \sum_{k=0}^{\infty} d_k z^k,
\]
\(|z| < 1\).
Theorem 4. Suppose that $w$ is nonnegative and integrable. If $\log w \in L^1$, then

$$\sigma_2(w, S_1) = \left( \sum_{k=0}^{n} |d_k|^2 \right)^{-1/2}.$$

Otherwise, $\sigma_2(w, S_1) = 0$.

Proof. The quantity $\sigma_2(w, S_1)$ is nonzero if and only if $\log w \in L^1$, by Theorem 2. Now Theorem 3 shows that

$$\sigma_2(w, S_1) = \text{dist}_{L^2}(\{(\phi^{-1})_n, e_{n+1}H^2\}^{-1}).$$

The right side is easily seen to be

$$\left( \sum_{k=0}^{n} |d_k|^2 \right)^{-1/2}.$$

This improves upon the corresponding results of [6, 8], in which the stronger hypothesis $w^{-1} \in L^1$ is needed.

Next, an elementary argument settles the case $p = 2$ and $S = S_3$.

Theorem 5. Let $w$ be nonnegative and integrable. If $\log w \in L^1$, then

$$\sigma_2(w, S_3) = |c_0| \left( \sum_{k=0}^{n-1} |c_k|^2 \right)^{1/2} \left( \sum_{k=0}^{n} |c_k|^2 \right)^{-1/2}.$$

Otherwise, $\sigma_2(w, S_3) = 0$.

Proof. Again, Theorem 2 asserts that $\sigma_2(w, S_3)$ is nonzero precisely when $\log w$ is integrable. In that case, let $\hat{e}_n$ be the orthogonal projection of $e_n$ onto $\mathcal{M}(S_0)$. Since

$$S_3 = \mathcal{M}(S_0) \oplus \sqrt{\{e_n - \hat{e}_n\}},$$

the projection of $e_0$ onto $\mathcal{M}(S_3)$ is given by

$$P_{S_3}e_0 = \hat{e}_0 + a(e_n - \hat{e}_n),$$

where

$$a = \frac{\langle e_0, e_n - \hat{e}_n \rangle_{L^2(w)}}{\|e_n - \hat{e}_n\|^2_{L^2(w)}}.$$

Thus, using the orthogonality of $\hat{e}_0$ and $e_n - \hat{e}_n$ we get

$$\sigma_2(w, S_3)^2 = \|e_0 - P_{S_3}e_0\|^2_{L^2(w)}$$

$$= \|e_0 - \hat{e}_0\|^2_{L^2(w)} - \frac{\|e_0 - \hat{e}_0, e_n - \hat{e}_n\|^2_{L^2(w)}}{\|e_n - \hat{e}_n\|^2_{L^2(w)}}.$$

But it is straightforward to check that

$$e_n - \hat{e}_n = \left( \sum_{k=0}^{n} c_k e_{n-k} \right) \phi^{-1}.$$

The desired result now follows from substituting this in (4).

With this done, the case with $p = 2$ and $S = S_2$ is immediate.
Theorem 6. Suppose that \( w \) is nonnegative and integrable. If \( \log w \in L^1 \), then
\[
\sigma_2(w, S_2) = |c_0| \left( \sum_{k=0}^{n} |d_k|^2 \right)^{1/2} \left( \sum_{k=0}^{n-1} |d_k|^2 \right)^{-1/2}
\]
Otherwise, \( \sigma_2(w, S_2) = 0 \).

Proof. By Theorem 2 the quantity \( \sigma_2(w, S_2) \) is nonzero precisely when \( \log w \in L^1 \). When that holds, Theorem 1 gives
\[
\sigma_2(w, S_2) = \sigma_2(w^{-1}, S_2^c)^{-1}.
\]
But \( S_2^c = \{-n\} \cup \{1, 2, 3, \ldots\} \) is simply the reflection of \( S_3 \) about the origin. Now an application of Theorem 5 completes the proof.

It is interesting and instructive to compare these least-squares error formulas with the classical \( n \)-step prediction error variance, where \( S = \{\ldots, -n - 3, -n - 2, -n - 1\} \); for then
\[
\sigma_2(w, S) = \left( \sum_{k=0}^{n} |c_k|^2 \right)^{1/2}.
\]
In particular, we can explicitly observe the changes in \( \sigma_2(w, S) \) as indices are added to or deleted from the frequency set \( S \).

When \( p \neq 2 \) these Hilbert space techniques do not apply. However, in this situation the work of Rajput and Sundberg [10, Theorem 2 and Remark 1(b)] makes the following approach possible. As before, let \( w \) be nonnegative and integrable with \( \log w \in L^1 \), and let \( \phi \) be the outer function for which \( w^s = |\phi|^q \). Assume for the sake of argument that \( w^s \in L^1 \), so that \( \phi \in H^q \). Let \( Q(h) \) denote the metric projection of \( h \in L^q \) onto \( H^q \); by Theorem 3, we need to consider \( h = e_{-(n+1)} \phi^{(n)} \).

Note that \( \phi^{(n)}(0) = \hat{\phi}_0 > 0 \), and let
\[
[\phi^{(n)}(z)]q^{1/2} = \sum_{k=0}^{\infty} c_k z^k = P_n(z) + \sum_{k=n+1}^{\infty} c_k z^k
\]
be an analytic root of \( \phi^{(n)}(z) \) defined in a neighborhood of zero. Assuming that \( P_n(z) \neq 0 \) for all \( |z| < 1 \), we get
\[
(Q(h))(z) = z^{-(n+1)}|\phi^{(n)}(z) - P_n(z)|^{2/q},
\]
\( |z| < 1 \), where \( P_n^{2/q} \) is the analytic root of \( P_n \) which satisfies and is uniquely determined by the condition
\[
P_n^{2/q}(0) = \hat{\phi}_0.
\]
Now the desired distance is given by
\[
\sigma_p(w, S_1) = \text{dist}_{L^p}(h, H^q)^{-1}
\]
\[
= \left( \int |P_n|^2 d\lambda \right)^{-1/q}
\]
\[
= \left( \sum_{k=0}^{n} |c_k|^2 \right)^{-1/q} .
\]
For more general \( w \), we apply the above argument with \( w \) replaced by \( w_m = \max\{w, 1/m\} \), and extract limits as \( m \to \infty \), as in the proof of Theorem 3. Note
that if the polynomial $P_n$ has no roots in the closed disk $|z| \leq 1$, then its approximates also do not vanish in the open disk $|z| < 1$, for sufficiently large $m$. This yields the following.

**Theorem 7.** Suppose that $w$ is nonnegative and integrable. If $\log w \in L^1$ and $P_n(z)$ has no roots in the closed disk $|z| \leq 1$, then

$$
\sigma_p(w, S_1) = \left( \sum_{k=0}^{n} |c_k|^2 \right)^{-1/q},
$$

where $P_n$ and the coefficients $c_k$ are given in (5).

The explicit formula (6) and [6, Theorem 3.1] can be used to compute the function in $M(S_1)$ for which $\sigma_p(w, S_1)$ is attained.

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