PROCEEDINGS OF THE
AMERICAN MATHEMATICAL SOCIETY
Volume 126, Number 8, August 1998, Pages 2205–2211
S 0002-9939(98)04308-1

LINEAR OPERATORS THAT PRESERVE MAXIMAL COLUMN RANKS OF NONNEGATIVE INTEGER MATRICES

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(Communicated by Lance W. Small)

Abstract. The maximal column rank of an m by n matrix over a semiring is the maximal number of the columns of A which are linearly independent. We characterize the linear operators which preserve the maximal column ranks of nonnegative integer matrices.

1. Introduction and preliminaries

A semiring is essentially a ring in which only the zero is required to have an additive inverse. Thus all rings are semirings. The nonnegative integers (with the usual arithmetic), Z^+, and the Boolean algebra of two elements are combinatorially interesting examples of semirings. Algebraic operations on matrices over a semiring and such notions as *linearity* and *invertibility* are also defined as if the underlying scalars were in a field.

The set of $m \times n$ matrices with entries in Z^+ is denoted by $\mathbb{M}_{m,n}(Z^+)$.

A set of vectors ($m \times 1$ matrices) is a *semimodule* [1] if it is closed under addition and scalar multiplication. A subset $\mathcal{W}$ of a semimodule $\mathcal{V}$ is a *spanning set* if each vector in $\mathcal{V}$ can be written as a sum of scalar multiples (i.e. a linear combination) of elements of $\mathcal{W}$.

The $m \times n$ matrix all of whose entries are zero except its (i, j)th, which is 1, is denoted E_{ij}. We call E_{ij} a *cell*. The set of cells spans $\mathbb{M}_{m,n}(Z^+)$. Let $\mathbf{e}_i$ be the $n \times 1$ matrix with a "1" in the ith position and zero elsewhere. We say that A is a *column matrix* if $A = \mathbf{a}\mathbf{e}_i^t$ for some $1 \leq i \leq n$ and some $\mathbf{a} \in \mathbb{M}_{m,1}(Z^+)$.

The *column space* of a matrix $A \in \mathbb{M}_{m,n}(Z^+)$ is the semimodule spanned by the columns of A over Z^+. Since the column space is spanned by a finite set of vectors, it contains a spanning set of minimum cardinality; that cardinality is the *column rank* [2] of $A, \chi(A)$.

A set G of vectors over Z^+ is *linearly dependent* [2] if for some $g \in G$, g is a linear combination of elements in $G - \{g\}$. Otherwise G is *linearly independent*.

The *maximal column rank* [5], $\psi(A)$, of an $m \times n$ matrix $A \in \mathbb{M}_{m,n}(Z^+)$ is the maximal number of the columns of A which are linearly independent over Z^+.

Received by the editors June 4, 1996 and, in revised form, January 6, 1997.
1991 *Mathematics Subject Classification.* Primary 15A36, 15A03, 15A04.
Key words and phrases. Maximal column rank, linear operator.
The author wishes to acknowledge the financial support of the Korea Research Foundation made in the program year of 1997.

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It follows that

$$0 \le \chi(A) \le \psi(A) \le n \tag{1.1}$$

for all $m \times n$ matrices A over Z^+.

The inequality in (1.1) may be strict over Z^+. For example, we consider the matrix

$$A = [1, 2, 3]$$

over Z^+. Then the column rank of A is one, while the maximal column rank of it is two since the last two columns of A are linearly independent over Z^+.

Recently Hwang, Kim and Song [5] characterized the linear operators that preserve the maximal column rank of matrices over the binary Boolean algebra. They also compared the column rank and the maximal column rank for matrices over certain semirings, and found that, except for small values of m and n, the two ranks did not agree in general. In particular, they obtained the following relations between column rank and maximal column rank over $\mathbb{M}_{m,n}(Z^+)$.

Theorem 1.1 ([5])**.** *Let $\alpha(Z^+, m, n)$ be the largest k such that for all $m \times n$ matrices A over Z^+, $\chi(A) = \psi(A)$ if $\chi(A) \le k$ and there is at least one $m \times n$ matrix A over Z^+ with $\chi(A) = k$. Then for $m \ge 1$,*

$$\alpha(Z^+, m, n) = \begin{cases} 1 & \text{if} \quad n = 1, \\ 2 & \text{if} \quad n = 2, \\ 0 & \text{if} \quad n \ge 3. \end{cases}$$

If A is a matrix over Z^+ and $A = \mathbf{u}\mathbf{a}^t$, then $\mathbf{a}$ and $\mathbf{u}$ are called right *and* left factors *of A respectively.*

Lemma 1.2. *For $A \in \mathbb{M}_{m,n}(Z^+), \psi(A) = 1$ if and only if A can be factored as $\mathbf{u}\mathbf{a}^t$ for some nonzero $\mathbf{u} \in \mathbb{M}_{m,1}(Z^+)$ and $\mathbf{a} \in \mathbb{M}_{m,1}(Z^+)$ with $\psi(\mathbf{a}^t) = 1$.*

Proof. If $\psi(A) = 1$, then there exists one column $\mathbf{a}_k$ of A such that all the other columns $\mathbf{a}_i$ are linearly dependent on each other, and hence all $\mathbf{a}_i$ are expressed as scalar multiples of $\mathbf{a}_k$, that is, $\mathbf{a}_i = \alpha_i \mathbf{a}_k$ for some $\alpha_i \in Z^+$. Therefore $A = \mathbf{a}_k[\alpha_1, \cdots, \alpha_n]$.

Let $\mathbf{u} = \mathbf{a}_k, \mathbf{a}^t = [\alpha_1, \cdots, \alpha_n]$. Then the fact that $\psi(\mathbf{a}^t) = 1$ follows from $\psi(A) = 1$.

The converse is clear. □

2. Linear operators that preserve maximal column ranks over $\mathbb{M}_{m,n}(Z^+)$

A function T mapping $\mathbb{M}_{m,n}(Z^+)$ into itself is called an *operator* on $\mathbb{M}_{m,n}(Z^+)$. The operator T

1. is *linear* if $T(\alpha A + \beta B) = \alpha T(A) + \beta T(B)$ for all $\alpha, \beta \in Z^+$ and all $A, B \in \mathbb{M}_{m,n}(Z^+)$;
2. *preserves maximal column rank* if $\psi(A) = \psi(T(A))$ for all $A \in \mathbb{M}_{m,n}(Z^+)$;
3. *strongly preserves maximal column rank 1* provided that $\psi(T(A)) = 1$ if and only if $\psi(A) = 1$ for all $A \in \mathbb{M}_{m,n}(Z^+)$.

In this section we obtain characterizations of the linear operators which preserve maximal column ranks of matrices over nonnegative integers.

Lemma 2.1. *Let* $A = \mathbf{h}(\mathbf{e}_1)^t, B = \mathbf{h}(\mathbf{e}_2)^t$ *be matrices in* $\mathbb{M}_{m,n}$ (Z^+) *with* $\mathbf{h} \in \mathbb{M}_{m,1}(Z^+)$. *Suppose* T *is a linear operator from* $\mathbb{M}_{m,n}(Z^+)$ *into itself which strongly preserves maximal column rank 1. If* $T(A) = \mathbf{u}\mathbf{a}^t, T(B) = \mathbf{v}\mathbf{b}^t$, *where* $\mathbf{a}^t = [a_1, \cdots, a_n], \mathbf{b}^t = [b_1, \cdots, b_n]$ *and* $c\mathbf{u} = d\mathbf{v}$ *for some nonzero* $c, d \in Z^+$, *then* $a_i b_i = 0$ *for some* i.

Proof. Suppose to the contrary that $a_i b_i \neq 0$ for all i. Since $c\mathbf{u} = d\mathbf{v}$, we have

$$\begin{aligned} dT(\alpha A + \beta B) &= \alpha d\mathbf{u}\mathbf{a}^t + \beta d\mathbf{v}\mathbf{b}^t = \alpha d\mathbf{u}\mathbf{a}^t + \beta c\mathbf{u}\mathbf{b}^t \\ &= \mathbf{u}(\alpha d\mathbf{a} + \beta c\mathbf{b})^t. \end{aligned} \tag{2.1}$$

By permuting columns, if necessary, we may assume that $b_1 \leq b_i$ for all i and $a_1 \leq a_j$ for all j such that $b_j = b_1$. Since c and d are nonzero, we have, for all sufficiently large β, that $da_1 + \beta b_1 c \leq da_i + \beta b_i c$ for all i. Since $dT(A + \beta B)$ has maximal column rank 1 for all β, any two columns are linearly dependent over Z^+. Thus, for any two columns, one is a scalar multiple of the other. Since $\psi(A + \beta B) = 1$, from (2.1) we have

$$(da_1 + \beta b_1 c) | (da_i + \beta b_i c) \tag{2.2}$$

for sufficiently large β, and all i. Choose k large and let $\beta = dka_1$. Then

$$(da_1 + dka_1 b_1 c) | (da_i + dka_1 b_i c)$$

for all i by (2.2). So $a_1 | a_i$ for all i, and so we may assume that $a_1 = 1$. Since $a_1 = 1$ and $b_1 \leq b_i, \alpha d + b_1 c \leq \alpha da_i + b_i c$ for all α and all i. Thus $(\alpha d + b_1 c) | (\alpha da_i + b_i c)$ for all α and all i, since $dT(\alpha A + B)$ has maximal column rank 1. Letting $\alpha = b_1 c$, we have $(b_1 cd + b_1 c) | (b_1 cda_i + b_i c)$ for all i. It now follows that $b_1 | b_i$ for all i, so we also may assume that $b_1 = 1$. Therefore

$$(d + \beta c) | (da_i + \beta b_i c) \tag{2.3}$$

for all β and all i from (2.2).

Suppose that $a_i \neq b_i$ for some i. Say $a_i < b_i$. Letting $\beta = b_i$ and $\beta = b_i + 1$, from (2.3) we have that for some $r, s \in Z^+$

$$da_i + b_i^2 c = r(d + b_i c), \tag{2.4}$$

$$da_i + (b_i^2 + b_i)c = s(d + (b_i + 1)c), \tag{2.5}$$

respectively. Subtracting (2.4) from (2.5), we have

$$b_i c = (s - r)(d + b_i c) + sc. \tag{2.6}$$

If $s = r$, then from (2.6) we have $b_i = s$. So (2.4) gives

$$da_i + s^2 c = s(d + sc) = sd + s^2 c,$$

that is, $a_i = s = b_i$, a contradiction since $a_i < b_i$. If $s > r$, then (2.6) gives

$$b_i c = (s - r)(d + b_i c) + sc > b_i c,$$

a contradiction. If $s < r$, then (2.6) gives

$$b_i c = (s - r)(d + b_i c) + sc < sc.$$

So $b_i < s < r$. From (2.4) we have

$$da_i + b_i^2 c = r(d + b_i c) > s(d + b_i c) > sd + b_i^2 c,$$

that is, $da_i > sd$. Thus we have $a_i > s > b_i$, which contradicts $a_i < b_i$.

For the case $b_i < a_i$, we also get contradictions by symmetric arguments.

Thus $a_i = b_i$ for all i, that is, $\mathbf{a} = \mathbf{b}$. Then $\alpha A + \beta B$ has maximal column rank 2 for relatively prime positive integers α and β, since the first two columns of $\alpha A + \beta B$ are linearly independent. But

$$\begin{aligned} T(\alpha A + \beta B) &= \alpha T(A) + \beta T(B) \\ &= \alpha \mathbf{u}\mathbf{a}^t + \beta \mathbf{v}\mathbf{b}^t \\ &= (\alpha \mathbf{u} + \beta \mathbf{v})\mathbf{a}^t \end{aligned}$$

has maximal column rank 1, since $\mathbf{a}^t$ has maximal column rank 1 from the construction. Hence we have a contradiction to the condition that T strongly preserves maximal column rank 1. □

Lemma 2.2. *Let T be a linear operator from $\mathbb{M}_{m,n}(Z^+)$ into itself. If T strongly preserves maximal column rank 1, then T maps column matrices to column matrices.*

Proof. Suppose to the contrary that T maps a column matrix to a matrix which is not a column matrix. Say $X_1 = \mathbf{x}(\mathbf{e}_1)^t$ and $T(X_1)$ has more than one nonzero column. For each $1 \le i \le n$, let $X_i = \mathbf{x}(\mathbf{e}_i)^t$. Let $S = \{1, 2, \cdots, n\}$ and let $S_1 = \{j :$ the jth column of $T(X_i)$ is zero for all $1 \le i \le n\}$. Then for each $i \in S - S_1$, there is a $j(i)$ such that the ith column of $T(X_{j(i)})$ is not zero. Now $T(X_1)$ has at least two nonzero columns, say columns k_1 and k_2. Let $S_2 = S - S_1 - \{k_1, k_2\}$, and let $A = X_1 + \sum_{i \in S_2} X_{j(i)}$. Note that for any $k \in S - S_1$, the kth column of $T(A)$ is nonzero. Further, since A consists of at most $n-1$ distinct summands, each of which is a column matrix, there is at least one zero column in A, say the ith. Let $B = X_i$. Since $T(A)$ has zero columns only corresponding to indices in S_1 (where $T(B)$ also must have a zero column), we can restrict our attention to those columns in $T(A)$ that are nonzero; hence we lose no generality in assuming that $T(A)$ has no zero column. Thus, since A, and hence $T(A)$, has maximal column rank 1, $T(A) = \mathbf{u}\mathbf{a}^t$, where $\mathbf{a}^t = [a_1, \cdots, a_n]$ has all nonzero entries which are linearly dependent, and some $u_j \neq 0$. Let $T(B) = \mathbf{v}\mathbf{b}^t$ with $\mathbf{b}^t = [b_1, \cdots, b_n]$. Now we consider two cases:

Case 1. Assume that $c\mathbf{u} \neq d\mathbf{v}$ for all nonzero c, d in Z^+. Since $\alpha A + B$ has maximal column rank 1 for any positive integer α,

$$T(\alpha A + B) = [\alpha a_1 \mathbf{u} + b_1 \mathbf{v} | \alpha a_2 \mathbf{u} + b_2 \mathbf{v} | \cdots | \alpha a_n \mathbf{u} + b_n \mathbf{v}]$$

also has maximal column rank 1. Thus we have, for some fixed j,

$$\alpha a_k \mathbf{u} + b_k \mathbf{v} = \mu_k (\alpha a_j \mathbf{u} + b_j \mathbf{v})$$

for some positive integer, $\mu_k, k = 1, \cdots, n$. If $a_k \neq \mu_k a_j$ for some k, then

$$\alpha |a_k - \mu_k a_j| \mathbf{u} = |\mu_k b_j - b_k| \mathbf{v},$$

which is a contradiction to the condition that $c\mathbf{u} \neq d\mathbf{v}$ for all nonzero c, d in Z^+. Thus $a_k = \mu_k a_j$ and $b_k = \mu_k b_j, k = 1, \cdots, n$. That is, $\mathbf{a} = a_j \mathbf{w}$ and $\mathbf{b} = b_j \mathbf{w}$, where $\mathbf{w}^t = [\mu_1, \cdots, \mu_n]$ with $\psi(\mathbf{w}^t) = \psi(\mathbf{a}^t) = 1$. Then $\psi(T(\alpha A + \beta B)) = \psi((\alpha a_j \mathbf{u} + \beta b_j \mathbf{v})\mathbf{w}^t) = 1$ for arbitrary α, β in Z^+. This contradicts the condition that T strongly preserves maximal column rank 1, since $\alpha A + \beta B$ has maximal column rank 2 for relatively prime α and β in Z^+.

Case 2. Assume that $c\mathbf{u} = d\mathbf{v}$ for some nonzero c, d in Z^+. Since $T(A) = \mathbf{u}\mathbf{a}^t$ has maximal column rank 1, all the columns of $\mathbf{a}^t$ are linearly dependent. So, without loss of generality, we can assume that $a_1 = 1$. For $T(B) = \mathbf{v}\mathbf{b}^t$, we shall show that $b_i \neq 0$ for all i.

Suppose $b_i = 0$ for some i. Choose j such that $b_j = 0$ and $a_j \leq a_h$ for all h such that $b_h = 0$. Since

$$\begin{aligned} dT(A+\beta B) &= d(\mathbf{u}\mathbf{a}^t + \beta \mathbf{v}\mathbf{b}^t) \\ &= \mathbf{u}[da_1 + \beta cb_1, da_2 + \beta cb_2, \cdots, da_n + \beta cb_n] \end{aligned}$$

has maximal column rank 1 for all β, choose β such that $da_k + \beta cb_k > da_j$ for all k with $b_k \neq 0$. Thus there exist distinct integers γ, δ such that, for fixed k with $b_k \neq 0$,

$$da_k + \beta cb_k = \gamma da_j \tag{2.7}$$

and

$$da_k + (\beta + 1)cb_k = \delta da_j. \tag{2.8}$$

Subtracting (2.7) from (2.8), we have

$$cb_k = (\delta - \gamma) da_j. \tag{2.9}$$

Further, if $j \neq 1$, then $b_1 \neq 0$. For, if $b_1 = 0$, then $a_1 = 1$ is the minimal entry in $\mathbf{a}$, and hence $j = 1$ from the construction of a_j. So, for $k = 1$, $da_1 + \beta cb_1 = \gamma_1 da_j$ from (2.6). Since $da_j | cb_1$ from (2.9) for $k = 1$, it follows that $da_j | da_1$, that is, $a_j | a_1 (= 1)$. Thus $a_j = 1$. So we may assume that $j = 1$.

Now,

$$\begin{aligned} cT(\alpha A + B) &= [c\alpha \mathbf{u} | c\alpha \mathbf{u} a_2 + c\mathbf{v} b_2 | \cdots | c\alpha \mathbf{u} a_n + c\mathbf{v} b_n] \\ &= \mathbf{v}[\alpha d, \alpha d a_2 + cb_2, \cdots, \alpha d a_n + cb_n] \end{aligned}$$

must have maximal column rank 1 for all α. Thus there are γ_i such that $\alpha da_i + cb_i = \gamma_i(\alpha d)$ for all i.

It follows that $\alpha d | cb_i$ for all α, and all $i = 1, \cdots, n$, a contradiction since $b_i \neq 0$ for at least one i.

We now have shown that $b_i \neq 0$ for all i. Now, letting $A = X_i$ and $B = X_j, j = 1, \cdots, n$ and $j \neq i$, the above argument implies that $T(A)$ and $T(B)$ have no zero columns. This contradicts Lemma 2.1.

Hence the two cases show that T maps column matrices to column matrices. □

Theorem 2.3. *Let $T : \mathbb{M}_{m,n}(Z^+) \to \mathbb{M}_{m,n}(Z^+)$ be a linear operator. Then T strongly preserves maximal column rank 1 if and only if there exist $Q \in \mathbb{M}_{m,m}(Z^+)$ which is nonsingular as a real matrix and a permutation matrix $P \in \mathbb{M}_{n,n}(Z^+)$ such that $T(A) = QAP$ for all $A \in \mathbb{M}_{m,n}(Z^+)$.*

Proof. Suppose there exist Q and P such that $T(A) = QAP$ for all $A \in \mathbb{M}_{m,n}(Z^+)$ and A has maximal column rank 1. Then $A = \mathbf{x}\mathbf{a}^t$ with $\psi(\mathbf{a}^t) = 1$. That is, all the columns in $\mathbf{a}^t$ are linearly dependent on each other. Let P^t correspond to a permutation $\pi \in S_n$. Then

$$QAP = Q\mathbf{x}(P^t \mathbf{a})^t$$

and the columns of $(P^t\mathbf{a})^t$ are linearly dependent on each other. Hence QAP has maximal column rank 1. Further, assume that QAP has maximal column rank 1. Since P is a permutation matrix in $\mathbb{M}_{n,n}(Z^+)$, multiplying QAP on the right by a permutation matrix P^{-1} does not change the maximal column rank of QAP. Hence QA has maximal column rank 1. Therefore all the columns $Q\mathbf{a}_i$ are linearly dependent, with $A = [\mathbf{a}_1, \cdots, \mathbf{a}_n]$. Thus for any two columns $Q\mathbf{a}_k$ and $Q\mathbf{a}_h$ of QA we have $Q\mathbf{a}_k = r_k Q\mathbf{a}_h$ or $Q\mathbf{a}_h = r_h Q\mathbf{a}_h$ with $r_k, r_h \in Z^+$. Hence $\mathbf{a}_k = r_k\mathbf{a}_h$ or $\mathbf{a}_h = r_h\mathbf{a}_k$ over the real field, and hence over Z^+ since Q is invertible as a real matrix. That is, $\psi(A) = 1$. Thus T strongly preserves maximal column rank 1.

Conversely, suppose T strongly preserves maximal column rank 1. Let $X_i = \mathbf{x}(\mathbf{e}_i)^t, i = 1, \cdots, n$, for some fixed $\mathbf{x} \in (Z^+)^m$. By Lemma 2.2, $T(X_i) = \mathbf{y}(\mathbf{e}_{\pi(i)})^t$, where $\pi : \{1, \cdots, n\} \to \{1, \cdots, n\}$. If π is not a permutation, then $\alpha T(X_i) + \beta T(X_j)$ has only one nonzero column for all $\alpha, \beta \in Z^+$. That is, $T(\alpha X_i + \beta X_j)$ has maximal column rank 1 for all α, β, a contradiction since $\alpha X_i + \beta X_j$ has maximal column rank 2 for relatively prime $\alpha, \beta \in Z^+$. Thus π is a permutation. So without loss of generality, we assume π is the identity permutation, so that $T(X_1) = \mathbf{u}(\mathbf{e}_1)^t$ and $T(X_2) = \mathbf{v}(\mathbf{e}_2)^t$. If $u_i \neq 0$ and $v_i = 0$, or vice versa, then $\mathbf{u}(\mathbf{e}_1)^t + \mathbf{v}(\mathbf{e}_2)^t$ has maximal column rank 2, contradicting the condition that T strongly preserves maximal column rank 1 since $X_1 + X_2$ has maximal column rank 1. Thus $u_i = 0$ if and only if $v_i = 0$. We assume without loss of generality that $0 \neq u_1 \leq v_1$. Since $X_1 + X_2$ has maximal column rank 1, $\mathbf{v} = r\mathbf{u}$ for some $r \in Z^+$. If $r \neq 1$, choose p relatively prime to r; then

$$\begin{aligned} T(pX_1 + X_2) &= [p\mathbf{u}|\mathbf{v}|\mathbf{0}|\cdots|\mathbf{0}] \\ &= [p\mathbf{u}|r\mathbf{u}|\mathbf{0}|\cdots|\mathbf{0}] \end{aligned}$$

has maximal column rank 2 while $pX_1 + X_2$ has maximal column rank 1, a contradiction. Thus $r = 1$. That is, $\mathbf{u} = \mathbf{v}$. It follows that $T(X_i) = \mathbf{u}(\mathbf{e}_i)^t$. In particular, when $X_i = E_{ji}$, there exists some vector $\mathbf{u}_j$ such that $T(E_{ji}) = \mathbf{u}_j(\mathbf{e}_i)^t$ for all i, j. Let Q be the matrix $[\mathbf{u}_1|\mathbf{u}_2|\cdots|\mathbf{u}_m]$. Then for an arbitrary $A \in \mathbb{M}_{m,n}(Z^+)$,

$$\begin{aligned} T(A) &= \sum_{j=1}^{m}\sum_{i=1}^{n} a_{ji}T(E_{ji}) \\ &= \sum_{j=i}^{m}\sum_{i=1}^{n} a_{ji}\mathbf{u}_j(\mathbf{e}_i)^t. \end{aligned}$$

So the (k, j) entry of $T(A)$ is $\sum_{i=1}^{m} a_{ij}u_{ki}$. The (k, j) entry of QA is $\sum_{i=1}^{m} u_{ki}a_{ij}$, which is the (k, j) entry of $T(A)$. Thus, $T(A) = QA$ for all $A \in \mathbb{M}_{m,n}(Z^+)$.

Finally, we show that Q is nonsingular as a real matrix. Suppose that $Q = (q_{ij})$ is singular. Say, $Q\mathbf{x} = \mathbf{0}$ for some nonzero real vector $\mathbf{x}$. Since $\mathbf{x}$ can be considered as a solution of the homogeneous system of linear equations with coefficients $q_{ij} \in Z^+$, we may assume, without loss of generality, that the entries of $\mathbf{x}$ are all integers. So let $\alpha = 1 + \max_{1 \leq i \leq m} |x_i|$ and $\mathbf{z} = \alpha\mathbf{j} + \mathbf{x}$, where $\mathbf{j}$ is the vector of all 1's. Then $\mathbf{z} \in \mathbb{M}_{m,1}(Z^+)$ and $Q\mathbf{z} = Q(\alpha\mathbf{j} + \mathbf{x}) = Q(\alpha\mathbf{j})$. Thus $T(\mathbf{z}\mathbf{e}_1^t + \alpha\mathbf{j}\mathbf{e}_2^t) = Q(\mathbf{z}\mathbf{e}_1^t) + Q(\alpha\mathbf{j}\mathbf{e}_2^t) = Q(\alpha\mathbf{j})\mathbf{e}_1^t + Q(\alpha\mathbf{j})\mathbf{e}_2^t = Q(\alpha\mathbf{j})(\mathbf{e}_1 + \mathbf{e}_2)^t$ has maximal column rank 1. Then $\mathbf{z} = k\alpha\mathbf{j}$, or $k\mathbf{z} = \alpha\mathbf{j}$, for some $k \in Z^+$. But then $Q\mathbf{z} = \mathbf{0}$ and hence $T(\mathbf{z}\mathbf{e}_1^t) = \mathbf{0}$, contradicting the condition that T strongly preserves maximal column rank 1. Thus Q is nonsingular as a real matrix. □

Corollary 2.4. *A linear operator* $T : \mathbb{M}_{m,n}(Z^+) \to \mathbb{M}_{m,n}(Z^+)$ *preserves maximal column rank if and only if there exist* $Q \in \mathbb{M}_{m,m}(Z^+)$ *which is nonsingular as a real matrix and a permutation matrix* $P \in \mathbb{M}_{n,n}(Z^+)$ *such that* $T(A) = QAP$ *for all* $A \in \mathbb{M}_{m,n}(Z^+)$.

Thus we have characterized the linear operators that preserve maximal column rank of nonnegative integer matrices.

Acknowledgement

The author would like to thank his parents (Pal-Bok Song, Yong-Bun Park) and his mother-in-law (Youn-Hee Seol) for their encouragements.

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