A SPLITTING THEOREM FOR EQUIFOCAL SUBMANIFOLDS IN SIMPLY CONNECTED COMPACT SYMMETRIC SPACES

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ABSTRACT. A submanifold in a symmetric space is called equifocal if it has a globally flat abelian normal bundle and its focal data is invariant under normal parallel transportation. This is a generalization of the notion of isoparametric submanifolds in Euclidean spaces. To each equifocal submanifold, we can associate a Coxeter group, which is determined by the focal data at one point. In this paper we prove that an equifocal submanifold in a simply connected compact symmetric space is a non-trivial product of two such submanifolds if and only if its associated Coxeter group is decomposable. As a consequence, we get a similar splitting result for hyperpolar group actions on compact symmetric spaces. These results are an application of a splitting theorem for isoparametric submanifolds in Hilbert spaces by Heintze and Liu.

1. INTRODUCTION

Let \( N = G/K \) be a Riemannian symmetric space, \( M \) an immersed submanifold of \( N \), and \( \nu(M) \) the normal bundle of \( M \). The end point map or normal exponential map \( \eta: \nu(M) \to N \) is the restriction of the exponential map \( \exp \) on \( TN \) to \( \nu(M) \). If \( \nu(x)(M) \) is a singular point of \( \eta \) and the dimension of the kernel of \( d_x \eta \) is \( k \), then \( \nu(x)(M) \) is called a multiplicity \( k \) focal normal and \( \exp(\nu(x)(M)) \) is called a multiplicity \( k \) focal point of \( M \) with respect to \( x \). The normal bundle of \( M \) is called abelian if \( \exp(\nu_x(M)) \) is contained in some flat of \( N \) for each \( x \in M \). It is called globally flat if the induced normal connection is flat and has trivial holonomy. Assume that \( M \) is connected and properly immersed in \( N \). \( M \) is called equifocal if \( \nu(M) \) is abelian and globally flat and the focal data is invariant under normal parallel transportation, i.e. if \( \nu(x)(M) \) is a parallel normal field on \( M \) such that \( \eta_{\nu(x)}(x_0) \) is a multiplicity \( k \) focal point of \( M \) with respect to \( x_0 \), then \( \eta_{\nu(x)}(x) \) is a multiplicity \( k \) focal point of \( M \) with respect to \( x \) for all \( x \in M \). (This definition was first given in [TT].) If the ambient space \( N \) is \( \mathbb{R}^n \), equifocal submanifolds are the same as isoparametric submanifolds, which have been studied since the 1930’s. (See [PT] and [Te3] for further reference.) Henceforth, we will assume that \( N = G/K \) is a simply connected symmetric space of compact type. Suppose \( M \) is a codimension \( r \) equifocal submanifold of \( N \). Then \( M \) is actually embedded in \( N \), and for all \( x \in M \), \( T_x := \eta(\nu_x(M)) \) is an \( r \)-dimensional flat torus in \( N \). We can associate to \( M \) an affine Weyl group \( W \), which acts isometrically on \( \nu_x(M) \), \( x \in M \). \( W \) can be obtained as a subgroup

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of $\text{Isom}(\nu_x(M))$, generated by reflections, where the set of reflection hyperplanes is exactly the set of focal normals in $\nu_x(M)$. Furthermore, we have $M \cap T_x = \exp_x(W \cdot 0)$. (Cf. [TT], Theorem 1.8.)

It is a well-known fact (cf. [Bou]) that a Coxeter group (hence in particular an affine Weyl group) decomposes as a product of two or more Coxeter groups if and only if its Coxeter diagram is not connected. The property of a Coxeter group associated to a submanifold to be decomposable actually gives an information about the submanifold itself. Let us review this for the case of the ambient space being a vector space. If $M$ is an isoparametric submanifold in a Hilbert space $V$, then we can associate to $M$ a Coxeter group $W$ in the same way as described above (cf. [Te1], [Te2]). $W$ is a (finite) Weyl group if $V$ is finite dimensional, and an affine Weyl group (i.e. infinite) if $V$ is infinite dimensional and $M$ not a product of a finite dimensional isoparametric submanifold with an affine subspace of $V$. $M$ is called decomposable if there exist two proper closed affine subspaces $V_1, V_2$ of $V$ and isoparametric submanifolds $M_i$ in $V_i$, $i = 1, 2$, such that $V = V_1 \oplus V_2$ and $M = M_1 \times M_2$. We will call such a decomposition trivial if one of the components $M_i$ has a trivial Coxeter group, i.e. if $M_i$ is an affine subspace of $V_i$. It is known that $M$ decomposes non-trivially if and only if its Coxeter group is decomposable. This has been proved by Terng [Te1] for the finite dimensional case, and by Heintze and Liu [HL] for the infinite dimensional case.

The purpose of this paper is to prove a similar result for equifocal submanifolds in symmetric spaces, as an application of the splitting theorem by Heintze and Liu. Let $N$ be a symmetric space, $M$ an equifocal submanifold in $N$. $M$ is called decomposable if there exist symmetric spaces $N_1, N_2$ and equifocal submanifolds $M_i$ in $N_i$, $i = 1, 2$, such that $N = N_1 \times N_2$ and $M = M_1 \times M_2$. Again, we will call such a decomposition trivial if one of the components $M_i$ has a trivial Coxeter group. Note that if we assume $N$ to be of compact type, the same holds for the $N_i$’s, and so the only possibility for $M_i \subset N_i$ to have a trivial Coxeter group is that $M_i = N_i$. 

**Theorem 1.** An equifocal submanifold $M$ in a simply connected symmetric space $N$ of compact type decomposes non-trivially if and only if its associated affine Weyl group is decomposable. In particular, if $N$ is irreducible, then $M$ and its affine Weyl group are indecomposable.

Due to this result, we can restrict any further investigation of equifocal submanifolds to the case of indecomposable ones. It is easy to see that if an equifocal submanifold decomposes non-trivially, then its affine Weyl group is decomposable. So we only have to prove the converse statement.

**2. The parallel transport map**

We will recall some facts about the method of studying submanifolds in symmetric spaces by lifting them to a Hilbert space, which was introduced in [TT]. Let $G$ be a compact, semisimple Lie group, $\mathfrak{g}$ its Lie algebra, equipped with the inner product $(.,.)$ given by the negative of the Killing form. Let $V := H^0([0, 1]; \mathfrak{g})$ denote the Hilbert space of $L^2$-integrable paths $u : [0, 1] \to \mathfrak{g}$, where the $L^2$-inner product is defined by

$$\langle u, v \rangle_0 := \int_0^1 \langle u(t), v(t) \rangle dt.$$
We can view $V$ as the space of connections of the trivial principal bundle $[0, 1] \times G$ over $[0, 1]$, by identifying the path $u(t)$ with the connection $u(t)dt$. For every $u \in V$, let $E_u(t)$ be the parallel translation corresponding to the connection defined by $u$, i.e. $E_u : [0, 1] \to G$ is the unique solution to the initial value problem
\[
\begin{cases}
E^{-1}E' = u, \\
E(0) = e.
\end{cases}
\]
Define the parallel transport map $\Phi : V \to G$ by
\[\Phi(u) := E_u(1).\]
The Hilbert Lie group $H^1([0, 1]; G)$ of $H^1$-paths from $[0, 1]$ to $G$ acts on $V$ isometrically via gauge transformations:
\[g \cdot u := gug^{-1} - g'g^{-1}.\]
For each subgroup $H$ of $G \times G$, we denote by $P(G, H)$ the following subgroup of $H^1([0, 1]; G)$:
\[P(G, H) := \{g \in H^1([0, 1]; G) \mid (g(0), g(1)) \in H\}.
\]
Define
\[\Omega_e(G) := P(G, e \times e).
\]
Then $\Phi : V \to G$ is a principal $\Omega_e(G)$-bundle via the gauge action ([TT], corollary 4.4).

Now let $\pi : G \to G/K$ be the natural Riemannian fibration of the symmetric space $N = G/K$, $M$ a submanifold in $N$ with globally flat abelian normal bundle, and $M^*$ a connected component of $\pi^{-1}(M)$. Then $M^*$ has a globally flat abelian normal bundle, and $M$ is equifocal in $N$ if and only if $M^*$ is equifocal in $G$.

Let $M^*$ be a submanifold in $G$ with globally flat abelian normal bundle, and $\hat{M}$ a connected component of $\Phi^{-1}(M^*)$. Then $\hat{M}$ has a globally flat normal bundle, and $M^*$ is equifocal in $G$ if and only if $\hat{M}$ is isoparametric in $V$.

These last two statements are part of theorems 1.9, 1.10 of [TT]. Note that if we assume $G$ to be simply connected and $K$ to be connected, then $\pi^{-1}(M)$ and $\Phi^{-1}(M^*)$ will be connected.

3. Proof of the theorem in the case $N = G$

As a first step, we will prove the splitting theorem for the case of an equifocal submanifold $M$ in a simply connected compact semisimple Lie group $G$. Without loss of generality, assume $e \in M$. Let $\mathfrak{a} := \mathfrak{v}_e(M)$, hence $\mathfrak{a}$ is an abelian subalgebra of $\mathfrak{g}$. Let $\hat{M} := \Phi^{-1}(M)$. For each $x \in \mathfrak{g}$, we denote by $\hat{x} : [0, 1] \to \mathfrak{g}$ the constant map of value $x$. Let $\hat{\mathfrak{a}} := \{\hat{a} \mid a \in \mathfrak{a}\}$, hence $\hat{\mathfrak{a}}$ is the normal space of $\hat{M}$ at $0$. First we deal with the possibility of $\hat{M}$ having a trivial factor. Define
\[V' := \text{span}\{\hat{\nu}(u) \mid u \in \hat{M}, \text{ and } \hat{\nu}(u) \in \nu_e(M)\},\]
\[V_0 := (V')^\perp.
\]

**Lemma 3.1.** If the dimension of $V_0$ is bigger than 0, then there exists a submanifold $\hat{M}' \subset V'$ such that $\hat{M} = \hat{M}' \times V_0$ and $\hat{M}'$ is an isoparametric submanifold of $V'$. In fact $\hat{M}' = \hat{M} \cap V'$. 


This is lemma 3.1 of [HL]. According to corollary 4.6 of [TT], we have
\[ V' = \text{span}\{gvx^{-1}g^{-1} | g \in P(G, e \times G), \ x \in M, \ g^{-1}(1) = x, \ v \in \nu_x(M)\}. \]
The following lemma is easily checked.

**Lemma 3.2.** Let \( G \) be a connected Lie group, \( \mathfrak{g} \) its Lie algebra, \( A \subset \mathfrak{g} \) a subset. Let \( \mathfrak{h} \) be the ideal of \( \mathfrak{g} \) generated by \( A \). Then \( \mathfrak{h} = \text{span}\{gag^{-1} | a \in A, \ g \in G\} \).

Now define \( \mathfrak{h} := \text{span}\{gvx^{-1}g^{-1} | x \in M, \ v \in \nu_x(M), \ g \in G\} \).

According to lemma 3.2, \( \mathfrak{h} \) is the ideal of \( \mathfrak{g} \) generated by all elements \( vx^{-1} \), where \( x \in M, \ v \in \nu_x(M) \).

**Lemma 3.3.** We have \( V' = H^0([0, 1]; \mathfrak{h}) \).

**Proof.** Clearly \( V' \subseteq H^0([0, 1]; \mathfrak{h}) \). Since \( V = H^0([0, 1]; \mathfrak{h}) \oplus H^0([0, 1]; \mathfrak{h}^+) \), and also \( V' = V' \oplus V_0 \), it is sufficient to show that \( V_0 \subseteq H^0([0, 1]; \mathfrak{h}^+) \). Let \( \alpha \in V_0 \), i.e. \( \alpha : [0, 1] \to \mathfrak{g} \) such that for all \( x \in M, \ v \in \nu_x(M) \), \( \tilde{g} \in P(G, e \times G) \) with \( \tilde{g}^{-1}(1) = x \),
\[ \langle \alpha, \tilde{g}vx^{-1}\tilde{g}^{-1} \rangle_0 = \int_0^1 \langle \alpha(t), \tilde{g}(t)vx^{-1}\tilde{g}^{-1}(t) \rangle dt = 0. \]

Now assume that there is a \( g \in G, \ x \in M, \ v \in \nu_x(M) \), and a \( t \in [0, 1] \) such that \( \langle \alpha(t), gvx^{-1}g^{-1} \rangle \neq 0 \).

Choose a \( \tilde{g} \in P(G, e \times G) \) with \( \Phi(\tilde{g} \ast 0) = \tilde{g}^{-1}(1) = x \) and \( \tilde{g}(\frac{1}{2}) = g \). Define \( \varphi(t) := \langle \alpha(t), \tilde{g}(t)vx^{-1}\tilde{g}^{-1}(t) \rangle \).

\( \varphi : [0, 1] \to \mathbb{R} \) is continuous, \( \varphi(\frac{1}{2}) \neq 0 \), and
\[ \int_0^1 \varphi(t) dt = 0. \]

Hence there exists \( \varepsilon \in (0, \frac{1}{2}) \) such that
\[ \int_{t_1}^{t_2} \varphi(t) dt \neq 0, \]
where \( t_1 = \frac{1}{2} - \varepsilon, \ t_2 = \frac{1}{2} + \varepsilon \). Now define \( \lambda : [0, 1] \to [0, 1] \) by
\[ \lambda(t) := \begin{cases} 3t_2t, & \text{for } 0 \leq t \leq \frac{1}{3}, \\ 2t_2 - t_1 - 3(t_2 - t_1)t, & \text{for } \frac{1}{3} \leq t \leq \frac{2}{3}, \\ t_1 - 2t_2 + 3t_2t, & \text{for } \frac{2}{3} \leq t \leq 1. \end{cases} \]

In other words, \( \lambda \) is running from 0 to \( t_2 \), back to \( t_1 \), and then on to 1. We have \( \tilde{g} \circ \lambda \in P(G, e \times G) \) with \( \Phi((\tilde{g} \circ \lambda) \ast 0) = \tilde{g}^{-1}(1) = x \), so
\[ \int_0^1 \varphi(\lambda(t)) dt = 0, \]
by our assumption on \( \alpha \). On the other hand,
\[ \int_0^1 \varphi(\lambda(t)) dt = \frac{1}{3t_2} \int_0^{t_1} \varphi(t) dt + \frac{2t_2 - t_1}{3t_2(t_2 - t_1)} \int_{t_1}^{t_2} \varphi(t) dt \neq 0, \]
which is a contradiction. Hence for all \( g \in G, \ t \in [0, 1], \ x \in M, \ v \in \nu_x(M) \), we have \( \langle \alpha(t), gvx^{-1}g^{-1} \rangle = 0 \), and so \( \alpha \) is a path in \( \mathfrak{h}^+ \). \( \qed \)
Assume $V' \neq V$. Then
\[ V_0 = H^0([0,1]; t), \]
where $t = h^+$, which is also an ideal in $g$. According to lemma 3.1, we have
\[ \tilde{M} = \tilde{M}' \times V_0, \]
where $\tilde{M}' = \tilde{M} \cap V'$ is an isoparametric submanifold in $V'$. Let $H, K$ be the connected Lie subgroups of $G$ whose Lie algebras are $\mathfrak{h}, \mathfrak{t}$, respectively. Since $G$ is simply connected, $G = H \times K$. Let $\Phi_1 := \Phi|V', \Phi_2 := \Phi|V_0$. Then $V_0 = \Phi_2^{-1}(K)$, and $M' = \Phi_1^{-1}(M')$, where $M' \times e = M \cap (H \times e)$. $M'$ is the image of the saturated submanifold $\tilde{M}'$ under the Riemannian submersion $\Phi_1$, therefore it is a submanifold of $H$. $M'$ inherits a globally flat abelian normal bundle from $\tilde{M}$, and so it is equifocal, since $\tilde{M}'$ is isoparametric. So we have $M = M' \times K$, i.e. $M$ has a trivial factor. Hence we see that the following is true.

**Lemma 3.4.** The following statements are equivalent:

(i) $M$ contains a trivial factor;
(ii) there is an ideal $\mathfrak{t}$ in $g$ such that for all $x \in M$, $\mathfrak{t} \perp \nu_x(M)x^{-1}$ or, equivalently, $\mathfrak{t}x \subset T_xM$,
(iii) $\tilde{M}$ contains a Euclidean factor.

Note that if $M$ is homogeneous, then lemma 3.4 yields that $M$ contains a trivial factor if and only if $T_eM$ contains an ideal of $g$, so we can recognize the existence of a trivial factor by looking at the tangent space of $M$ at a single point.

The following statement about Coxeter groups is easily checked (cf. [Bou], V.3.8).

**Lemma 3.5.** Let $P$ be a Euclidean space, $\{l_i \mid i \in I\}$ a set of affine hyperplanes, such that the reflections along the $l_i$’s generate a Coxeter group $W \subset \text{Isom}(P)$. For each $i \in I$ let $n_i$ be a normal vector to $l_i$. Then $W$ is decomposable if and only if there exist two proper linear subspaces $P_1$ and $P_2$ of $P$ such that $P_1 \perp P_2$ and $P_1 \cup P_2$ contains all the $n_i$’s.

The affine Weyl group $W$ associated to $M$ is the same as the one associated to the isoparametric submanifold $\tilde{M}$ in $V$ (cf. [TT], 1.8, 1.10, 6.17). Hence we can obtain $W$ as follows. The normal bundle of $\tilde{M}$ is flat, hence the tangent bundle splits as
\[ T\tilde{M} = \bigoplus \{E_i \mid i \in I\} \]
into the direct sum of the simultaneous eigenspaces $E_i$ of the shape operators $\tilde{A}$, where $I$ is a countable index set. Let $\{n_i \mid i \in I\}$ be the corresponding curvature normals of $\tilde{M}$, i.e. the parallel normal vector fields on $\tilde{M}$ such that for all $u \in \tilde{M}$, $\tilde{v} \in \nu_u(\tilde{M})$,
\[ \tilde{A}_\tilde{v}|E_i(u) = \langle \tilde{v}, n_i(u) \rangle \cdot \text{id}_{E_i(u)}. \]
Let
\[ l_i(u) := \{ \tilde{v} \in \nu_u(\tilde{M}) \mid \langle \tilde{v}, n_i(u) \rangle = 1 \}. \]
Then the union of all the affine hyperplanes $l_i(u)$ is exactly the set of focal normals in $\nu_u(\tilde{M})$, and $W$ is isomorphic to the subgroup of $\text{Isom}(\nu_u(\tilde{M}))$ generated by the reflections along the $l_i$’s, for all $u \in \tilde{M}$.
Lemma 3.6. Assume $V' = V$. Then $\tilde{M}$ is full, i.e. the curvature normals $n_i(u)$ span $\nu_u(\tilde{M})$ for all $u \in \tilde{M}$.

Proof. Let $\tilde{P}$ be the subspace of $\nu_0(\tilde{M})$ spanned by all $n_i(0)$. Let $P := \Phi_*(\tilde{P}) \subset a$, let $P^\perp$ be the orthogonal complement of $P$ in $a$. If $z \in P^\perp$, then there are no focal normals on the line $\mathbb{R}z$ in $a = \nu_\circ(\tilde{M})$. Since $G$ has only non-negative sectional curvature, this implies that the curvature operator $R_z$ is zero, i.e. for all $v \in \mathfrak{g}$, $R_z(v) = -\frac{1}{4} \text{ad}_z^2(v) = 0$. Now $\text{ad}_z$ is skew-symmetric, hence it follows that $[z, v] = 0$ for all $v \in \mathfrak{g}$. So $P^\perp$ is an abelian ideal in $\mathfrak{g}$. Since $\mathfrak{g}$ is semisimple, $P^\perp$ must be zero, which means that $\tilde{M}$ is full.

Henceforth, we will assume that $V' = V$. This implies that $\tilde{M}$ is full, which is one of the assumptions in the proof of the splitting theorem for isoparametric submanifolds in Hilbert spaces in [HL], chap. 3.

If we now assume that $W$ is decomposable, then there exist two proper linear subspaces $\mathfrak{a}_1$ and $\mathfrak{a}_2$ of $a$, such that $\mathfrak{a}_1 \perp \mathfrak{a}_2$ and $\mathfrak{a}_1 \cup \mathfrak{a}_2$ contains all the curvature normals $n_i(0)$ of $\tilde{M}$ at 0.

Let $P_i$ be the subbundle of $\nu(M)$ obtained by parallel transport of $\hat{a}_i$ in the normal bundle. Let $\hat{P}_i$ be the subbundle of $\nu(\tilde{M})$ obtained by parallel transport of $\hat{a}_i$ in the normal bundle, i.e. $\hat{P}_i$ is the horizontal lift of $P_i$.

Lemma 3.7. For all $g_1, g_2 \in G$, $x_1, x_2 \in M$, $v_i \in P_i(x_i)$, $i = 1, 2$, we have

$$\langle g_1 v_1 x_1^{-1} g_1^{-1}, g_2 v_2 x_2^{-1} g_2^{-1} \rangle = 0.$$ 

Proof. From lemma 3.5 of [HL], we know that for all $u, w \in \tilde{M}$,

$$\hat{P}_1(u) \perp \hat{P}_2(w).$$

By corollary 4.6 of [TT] we know that for all $u \in \tilde{M}$, $i = 1, 2$,

$$\hat{P}_i(u) = \{ \hat{g} v x_i^{-1} \hat{g}^{-1} | \hat{g} \in P(G, e \times G), \hat{g} \ast \hat{0} = u, \Phi(u) = x, v \in P_i(x) \}.$$ 

So for all $\hat{g}_i \in P(G, e \times G)$, $x_i = \Phi(\hat{g}_i \ast \hat{0})$, $v_i \in P_i(x_i)$, $i = 1, 2$,

$$\langle \hat{g}_1 v_1 x_1^{-1} \hat{g}_1^{-1}, \hat{g}_2 v_2 x_2^{-1} \hat{g}_2^{-1} \rangle_0 = 0.$$ 

Now assume that there are $g_i \in G$, $x_i \in M$, $v_i \in P_i(x_i)$, $i = 1, 2$, such that

$$\langle g_1 v_1 x_1^{-1} g_1^{-1}, g_2 v_2 x_2^{-1} g_2^{-1} \rangle \neq 0.$$ 

Choose for $i = 1, 2$ a $\hat{g}_i \in P(G, e \times G)$ with $\Phi(\hat{g}_i \ast \hat{0}) = \hat{g}_i^{-1}(1) = x_i$ and $\hat{g}_i(\frac{1}{2}) = g_i$.

Define

$$\varphi(t) := \langle \hat{g}_1(t) v_1 x_1^{-1} \hat{g}_1^{-1}(t), \hat{g}_2(t) v_2 x_2^{-1} \hat{g}_2^{-1}(t) \rangle.$$ 

Now we can apply exactly the same argument already used in lemma 3.3 to get a contradiction.

Now define for $i = 1, 2$,

$$\mathfrak{g}_i := \text{span}\{ g v x^{-1} g^{-1} | x \in M, v \in P_i(x), g \in G \}.$$ 

According to lemmas 3.2 and 3.7, $\mathfrak{g}_1$ and $\mathfrak{g}_2$ are ideals of $\mathfrak{g}$, and $\mathfrak{g}_1 \perp \mathfrak{g}_2$. Since we assume that $M$ has no trivial factors, lemma 3.4 implies that $\mathfrak{g} = \mathfrak{g}_1 \oplus \mathfrak{g}_2$. Let $G_i$, $i = 1, 2$, be the connected Lie subgroup of $G$ whose Lie algebra is $\mathfrak{g}_i$. Since $G$ is simply connected, $G = G_1 \times G_2$. This splitting of $G$ then yields a splitting of $M$. 
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To see this, we have to take a closer look at the proof of the splitting theorem for isoparametric submanifolds in Hilbert spaces, as it is given in [HL], chap. 3. Define for \( i = 1, 2, \)

\[
V_{\tilde{P}_i} := \text{span}\{ \tilde{v}(u) \mid u \in M, \, \tilde{v}(u) \in \tilde{P}_i(u) \} = \text{span}\{ gx^{-1}g^{-1} \mid g \in P(G, e \times G), \, x = \Phi(g \times 0) \in M, \, v \in P_i(x) \}.
\]

Clearly \( V_{\tilde{P}_i} \subseteq H^0([0, 1]; \mathfrak{g}_i). \) But according to corollary 3.6 of [HL], we have \( V = V_{\tilde{P}_1} \oplus V_{\tilde{P}_2}, \) so \( V_{\tilde{P}_i} = H^0([0, 1]; \mathfrak{g}_i), \ i = 1, 2. \) Furthermore \( \tilde{M} = M_1 \times M_2, \) where \( M_i = M \cap V_{\tilde{P}_i} \) is an isoparametric submanifold of \( V_{\tilde{P}_i}. \) If we define \( \Phi_i := \Phi|V_{\tilde{P}_i} \) and \( M_i := \Phi_i(M_i), \) we have \( M_i = \Phi_i^{-1}(M_i) \) and \( M = M_1 \times M_2. \) \( M_i \) is the image of the saturated submanifold \( \tilde{M}_i \) under the Riemannian submersion \( \Phi_i, \) therefore it is a submanifold of \( G_i. \) \( M_i \) inherits a globally flat and abelian normal bundle from \( M, \) its lift \( \tilde{M}_i \) is isoparametric in \( V_{\tilde{P}_i}, \) so it is equifocal in \( G_i, \ i = 1, 2. \)

4. PROOF OF THE THEOREM IN THE CASE \( N = G/K \)

Let \( N = G/K \) be a simply connected symmetric space of compact type, in particular \( G \) semisimple. We choose \( G \) to be simply connected, which implies that \( K \) is also simply connected. Let \( \pi : G \to G/K \) be the natural projection, \( M \subset N \) an equifocal submanifold, and \( M^* := \pi^{-1}(M) \). Without loss of generality assume \( eK \in M, \) i.e. \( e \in M^*. \) \( M \) and \( M^* \) have the same affine Weyl group \( W \) associated to them (cf. [TT], 1.8, 1.9). We have seen that if \( W \) is decomposable, then \( M^* \) is decomposable. So we only have to check that such a decomposition of \( M^* \subset G \) yields a decomposition of \( M \subset G/K. \)

Let \( s : G \to G \) be the involution of \( G \) with \( K \) as the identity component of the fixed group of \( s. \) Let \( \theta := ds : \mathfrak{g} \to \mathfrak{g} \) be the corresponding Cartan involution. Let \( G = G_1 \times G_2, \ M^*_i \subset G_i \) such that \( M^* = M^*_1 \times M^*_2. \) Let \( \mathfrak{g}_i \) be the Lie algebra of \( G_i, \) i.e. \( \mathfrak{g} = \mathfrak{g}_1 \oplus \mathfrak{g}_2. \)

Assume that \( \theta(\mathfrak{g}_i) = \mathfrak{g}_i \) for \( i = 1, 2. \) Then we can define \( \mathfrak{k}_i \) to be the 1-eigenspace of \( \theta|\mathfrak{g}_i, \) and \( \mathfrak{k}_i \) to be the connected Lie subgroup of \( G_i \) with Lie algebra \( \mathfrak{k}_i. \) \( K \) is simply connected, so \( K = \mathfrak{k}_1 \times \mathfrak{k}_2. \) If we define \( N_i := G_i/K_i, \) then \( N_i \) is a symmetric space, and \( N = N_1 \times N_2. \) Define \( \pi_i := \pi|G_i, \ M_i := \pi_i(M_i^*). \) Then \( M_i^* = \pi^{-1}_i(M_i), \) and \( M_i \) inherits a globally flat and abelian normal bundle from \( M, \) so \( M_i \) is equifocal in \( N_i, \) and \( M = M_1 \times M_2. \)

To conclude the proof of the theorem, we have to deal with the possibility of \( \theta \) not respecting the decomposition \( \mathfrak{g} = \mathfrak{g}_1 \oplus \mathfrak{g}_2, \) i.e. \( \theta(\mathfrak{g}_i) \neq \mathfrak{g}_i \) for \( i = 1, 2. \) The semisimple Lie algebra \( \mathfrak{g} \) decomposes as a direct sum of its simple ideals

\[
\mathfrak{g} = \mathfrak{h}_1 \oplus \cdots \oplus \mathfrak{h}_r,
\]

(cf. [Hel], II.6.3), and the ideals \( \mathfrak{g}_1, \mathfrak{g}_2 \) are each a direct sum of certain \( \mathfrak{h}_\rho, \ 1 \leq \rho \leq r. \) On the other hand, the orthogonal symmetric Lie algebra \( (\mathfrak{g}, \theta) \) decomposes as a direct sum of irreducible orthogonal symmetric Lie algebras \( (\mathfrak{l}_\kappa, \theta_\kappa), \ 1 \leq \kappa \leq k, \) where the \( \mathfrak{l}_\kappa \)'s are ideals in \( \mathfrak{g} \) and \( \theta_\kappa = \theta|\mathfrak{l}_\kappa \) (cf. [Hel], VIII.5.2). Each \( \mathfrak{l}_\kappa \) is either a simple ideal or a sum of two simple ideals which are interchanged by \( \theta_\kappa \) (cf. [Hel], VIII.5.3).

If we assume that \( \theta(\mathfrak{g}_i) \neq \mathfrak{g}_i \) for \( i = 1, 2, \) then there must be a \( \kappa \in \{1, \ldots, k\} \) such that \( \mathfrak{l}_\kappa = \mathfrak{h}_\rho \oplus \mathfrak{h}_\sigma \) for some \( 1 \leq \rho, \sigma \leq r, \) and \( \theta_\kappa(\mathfrak{h}_\rho) = \mathfrak{h}_\sigma, \ \theta_\kappa(\mathfrak{h}_\sigma) = \mathfrak{h}_\rho, \)
Using the notation introduced in the paper, we define the following statements are equivalent:

1. The action of $\hat{H}_i$ on Riemannian manifolds $N_i$ for $i = 1, 2$ are called $\omega$-equivalent if there exists an isometry $f : N_1 \to N_2$ such that $f(H_1 \cdot x) = H_2 \cdot f(x)$ for all $x \in N_1$. An isometric action of $H$ on $N$ is called $\omega$-decomposable if there exist Riemannian $H_i$-manifolds $N_i$ for $i = 1, 2$ such that $N = N_1 \times N_2$ and the action of $H$ on $N$ is $\omega$-equivalent to the product action of $H_1 \times H_2$ on $N_1 \times N_2$. If one of the $H_i$-actions is transitive, then we say that this decomposition has a trivial factor. If $G$ is a compact semisimple Lie group and $H$ a closed subgroup of $G \times G$, $H$ has a natural isometric action on $G$ defined by $(h_1, h_2) \cdot g = h_1 g h_2^{-1}$ for $(h_1, h_2) \in H, g \in G$.

Assume that the action of $H$ on $G$ is hyperpolar. Let $M$ be a principal $H$-orbit. Then $M$ is equifocal ([TT], theorem 2.1). Let $A$ be a torus section through $e \in G$ and let $\mathfrak{a}$ denote its Lie algebra. Let $\hat{W}$ be the affine Weyl group associated to $M$. We can view $\hat{W}$ as a subgroup of $\text{Isom}(\mathfrak{a})$ generated by reflections. Let $\Lambda := \{ \lambda \in \mathfrak{a} \mid \exp(\lambda) = e \}$ be the unit lattice of $A$. If we identify $\lambda \in \Lambda$ with the reflection along the hyperplane $\frac{1}{2}\lambda + \lambda^\perp$ in $\mathfrak{a}$, then $\Lambda$ is a normal subgroup of $\hat{W}$, and $\hat{W}/\Lambda = W(A)$ (cf. [Te2], theorem 8.10, and [Te4], theorem 1.2).

**Theorem 2.** Let $G$ be a simply connected compact semisimple Lie group. Then the following statements are equivalent:

1. $M$ is $\omega$-decomposable.
2. $M$ is $\omega$-equivalent to a principal $H$-orbit. 
3. $M$ is $\omega$-equivalent to a principal $H$-orbit.
(i) the $H$-action on $G$ is $\omega$-decomposable into two non-trivial factors,
(ii) the isometric action of $W(A)$ on $A$ is decomposable,
(iii) $\hat{W}$ is decomposable.

Proof. The equivalence of (i) and (ii) is theorem B of [HL]. Furthermore it is clear that (i) implies (iii). If $\hat{W}$ is decomposable, then we know by the previous results that the $\hat{W}$-action on $\mathfrak{a}$ is a product action. The $\hat{W}$-action on $\mathfrak{a}$ induces the $W(A)$-action on $A$, so the latter one is also a product action. Hence (iii) implies (ii).

More generally, let $G/K$ be a compact symmetric space and $H$ a closed subgroup of $G$ that acts hyperpolarly on $G/K$. Then the subgroup $H \times K$ of $G \times G$ acts hyperpolarly on $G$ (cf. [HPTT]), and the $H \times K$-orbits in $G$ are the lifts of the $H$-orbits in $G/K$ under $\pi : G \rightarrow G/K$. The principal $H$-orbits are equifocal in $G/K$ (cf. [TT]). Let $A$ be a torus section through $e \in G$ for the $H \times K$-action. Then $B := \pi(A)$ is a section through $eK \subset G/K$ for the $H$-action. The projection $\pi : H \times K \rightarrow H$ induces isomorphisms $N(A) \rightarrow N(B)$ and $Z(A) \rightarrow Z(B)$, hence $W(A) = W(B)$, and for all $a \in A$ and $w \in W(A) = W(B)$, $\pi(w \cdot a) = w \cdot \pi(a)$. Let $\hat{W}$ be the affine Weyl group associated to a principal $H$-orbit, $\Lambda$ the unit lattice of $A$, hence $W(B) = \hat{W}/\Lambda$.

**Theorem 3.** Let $G/K$ be a simply connected symmetric space of compact type. Then the following statements are equivalent:

(i) the $H$-action on $G/K$ is $\omega$-decomposable into two non-trivial factors,
(ii) the isometric action of $W(B)$ on $B$ is decomposable,
(iii) $\hat{W}$ is decomposable.

Proof. Clearly (i) implies (iii), and (iii) implies (ii). Let the action of $W(B)$ on $B$ be decomposable. Then the action of $W(A)$ on $A$ is also decomposable. By theorem 2 this implies that the $H \times K$-action on $G$ is $\omega$-decomposable. More precisely, let $G = G_1 \times G_2$ be the decomposition of $G$. In the proof of theorem 1 we have seen that we can choose this decomposition in a way that there are subgroups $K_i$ of $G_i$, $i = 1, 2$, such that $K = K_1 \times K_2$, and $G/K = (G_1/K_1) \times (G_2/K_2)$. $G \times G$ decomposes as a product of $G_1 \times G_1$ and $G_2 \times G_2$. Let $H_i$ be the projection of $H$ to $G_i$, $i = 1, 2$. Then $H_1 \times K_1$ is the projection of $H \times K$ to $G_1 \times G_1$, and the $(H_1 \times K_1) \times (H_2 \times K_2)$-action on $G_1 \times G_2$ is $\omega$-equivalent to the $H \times K$-action on $G$ (cf. [HL], proof of theorem B). This implies that the $H_1 \times H_2$-action on $(G_1/K_1) \times (G_2/H_2)$ is $\omega$-equivalent to the $H$-action on $G/K$. 

Note that a hyperpolar action can be $\omega$-decomposable without being a product action. For example, let $G_0 := SO(2n)$, $K_0 := U(n)$. Then $(G_0, K_0)$ is an irreducible symmetric pair of compact type (type D III in the list of [Hel], X.6). Let $G := G_0 \times G_0$, $K := K_0 \times K_0$. Then $(G, K)$ is also a symmetric pair of compact type. Let $N := G/K$. $K$ acts hyperpolarly on $N$ by $k \cdot (gK) = kgK$, for $k \in K$, $g \in G$. Let $H := S(U(n) \times U(n))$. $H$ is a subgroup of $K$. More precisely,

$$H = \{(k_1, k_2) \in K_0 \times K_0 \mid \det(k_1) \det(k_2) = 1\}.$$

Let $k = (k_1, k_2) \in K$, let $\lambda := \det(k_1) \det(k_2)$. Then $h = (h_1, h_2) := (k_1, \lambda^{-\frac{1}{2}} k_2)$ is an element of $H$, and for all $g \in G$, we have $hgK = kgK$. Hence the actions of $H$ and $K$ have the same orbits, i.e. they are $\omega$-equivalent. So the $H$-action is $\omega$-decomposable, but it is not a product action.
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