CONVERGENCE OF THE POINCARÉ SERIES
FOR SOME CLASSICAL SCHOTTKY GROUPS

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Abstract. The Poincaré $\theta_2$-series for a multiply connected circular region can be either convergent or divergent absolutely. In this paper we prove a uniform convergence result for such a region.

1. Introduction

Let us consider mutually disjoint disks $D_k := \{z \in \mathbb{C} : |z - a_k| < r_k\}$ ($k = 1, 2, ..., n$) on the complex plane $\mathbb{C}$. We assume that $\overline{D_k} \cap \overline{D_j} = \emptyset$ for all $k \neq j$, where $\overline{D_k}$ is the closure of $D_k$. Let

$$[k, z] := r_k^2 |z - a_k| + a_k$$

be an inversion with respect to the circumference $\partial D_k := \{t \in \mathbb{C} : |t - a_k| = r_k\}$. Let us generate the Möbius transformations:

(1) $[k_1, k_2; z] := [k_1; [k_2; z]]$, where $k_1 = 1, 2, ..., n$; $k_2 = 1, 2, ..., n$; $k_2 \neq k_1$,

$$[k_1, ..., k_m; z] := [k_1, ..., k_{m-1}; [k_m; z]]$$

where $k_m = 1, 2, ..., n$; $k_{m-1} = 1, 2, ..., n$;

$$k_m \neq k_{m-1};...; k_2 = 1, 2, ..., n, k_2 \neq k_1; k_1 = 1, 2, ..., n.$$

Let $[k, U]$ denote a set which is symmetric to $U \subset \hat{\mathbb{C}} := \mathbb{C} \cup \{\infty\}$ with respect to $|t - a_k| = r_k$, $[k_1, ..., k_m; U] := [k_1, ..., k_{m-1}; [k_m, U]]$. We shall call $m$ the level of the transformation (1). The functions (1) with even level can be rewritten in the form $\gamma_j(z) = (\alpha_j z + b_j) / (c_j z + d_j), j = 0, 1, ..., \alpha_j d_j - c_j b_j = 1$. Here $\gamma_0(z) := z, \gamma_1(z) := [1, 2; z], \gamma_2(z) := [1, 3; z], ..., \gamma_{n-1}(z) := [1, n; z], \gamma_n(z) := [2, 1; z]$ and so on. The numeration of the transformations is fixed in order of increasing level. The functions $\gamma_j(z)$ generate the Schottky group $\Gamma [1]-[3]$.

Let $H(z)$ be a meromorphic function in the extended complex plane $\hat{\mathbb{C}}$. The Poincaré $\theta_{2q}$-series [1], [2]

$$\theta_{2q}(z) := \sum_{j=0}^{\infty} H(\gamma_j(z))(c_j z + d_j)^{-2q} \ (q \in \mathbb{Z}/2)$$

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is associated with the group $\Gamma$. Here $z \in B := \hat{\mathbb{C}} - (B_1 \cup \Lambda(\Gamma))$, $B_1$ is the set of poles of all $H(\gamma_j(z))$ and $\gamma_j(z)$, and $\Lambda(\Gamma)$ is the limit set of $\Gamma$. When $q > 1$ the series (2) converges absolutely and uniformly in every compact subset of $B$ [1], [2]. When $q = 1$ the series (2) can be either absolutely convergent or absolutely divergent. It depends on the properties of $\Gamma$. Necessary and sufficient conditions for absolute and uniform convergence of the series have been found in [4], [5] in terms of the Hausdorff dimension of the limit set of $\Gamma$. Let us note that absolute and uniform convergencies was not studied separately in the previous papers.

**Definition.** A point $z$ is called a regular point of $\Gamma$ if there exist numbers $k_1, k_2, \ldots, k_m$ such that $[k_1, k_2, \ldots, k_m; z]$ belongs to $\overline{D}$, where $D := \hat{\mathbb{C}} - \bigcup_{k=1}^n \overline{D_k}$ and the closure $\overline{D}$ is taken in $\hat{\mathbb{C}}$.

In this paper we prove

**Theorem 1.** Let a rational function $H(z)$ have poles only at regular points of $\Gamma$. Then the Poincaré $\vartheta_2$-series converges uniformly in every compact subset of each region $[k_1, k_2, \ldots, k_m; D] \cap B$. The order of summation depends on the region $[k_1, k_2, \ldots, k_m; D]$.

Let us consider the Banach space $C$ consisting of the functions $\Phi(t) := \Phi_k(t)$, where $|t - a_k| = r_k$ $(k = 1, 2, \ldots, n)$, which are continuous on $\bigcup_{k=1}^n \partial D_k$ with the norm $\|\Phi\| := \max_{1 \leq k \leq n} \max_{\partial D_k} |\Phi_k(t)|$. And let us consider the closed subspace $C^+ \in C$, for which the functions $\Phi_k(t)$ have analytic continuations to $D_k$.

The proof of Theorem 1 is based on the following

**Main Lemma.** The system of functional equations

$$
\Phi_k(z) = - \sum_{m=1}^n \left( \frac{1}{m, z} \right)^k \Phi_m([m, z]) + g_k(z), \ |z - a_k| \leq r_k, \ k = 1, 2, \ldots, n
$$

has a unique solution in $C^+$, where $g(z) := g_k(z), |z - a_k| \leq r_k$ $(k = 1, 2, \ldots, n)$, is in $C^+$. That solution can be found by the method of successive approximations. The approximations converge in $C^+$.

Let the function $g_k(z)$ is meromorphic in $|z - a_k| \leq r_k$ $(k = 1, 2, \ldots, n)$, i.e. $g_k(z) = s_k(z) + p_k(z)$, where $s_k(z)$ is analytic in $|z - a_k| < r_k$ and $p_k(z)$ is the principle part of $g_k(z)$. Let us consider the Banach space $C^+(p) := \{ \Phi : \Phi - p_k \in C^+, k = 1, 2, \ldots, n \}$ with the norm $\|\Phi\|_{C^+(p)} := \|\Phi - p_k\|_{C^+}$. Following [11], we can consider (3) with $g \in C^+(p)$ for $\Phi \in C^+(q)$, where the rational functions $q_k(z)$ are constructed by $p_k(z)$. We show how $q_k(z)$ is constructed by $p_k(z)$ in Sec. 3

2. **Functional equations**

Before proving the main results let us prove some auxiliary facts about functional equations. For brevity we shall write (3) in the form $\Phi = A \Phi + g$ in $C^+$.

Let us define the shift operator $S_m \Phi_m(z) = \Phi_m([m, z])$ on $|t - a_k| = r_k$ as the integral operator

$$
S_m \Phi_m(z) = \frac{1}{2\pi i} \int_{\partial D_m} \Phi_m(\tau) \frac{d\tau}{\tau - [m, t]}, \ |t - a_k| = r_k, \ m = 1, 2, \ldots, n; m \neq k.
$$
Let us consider the operator

$$A_C \Phi(z) = - \sum_{m=1 \atop m \neq k}^{n} \left( \frac{m}{z} \right)' \phi_m(z), \quad |t - a_k| = r_k, \quad k = 1, 2, \ldots, n.$$  

It is easily seen that if $\Phi$ satisfies $\Phi = A\Phi + g$ in $C^+$, then $\Phi$ satisfies $\Phi = A_C \Phi + g$ in $C$. Conversely, if $g \in C^+$, then from the properties of $A_C$ for $\Phi \in C$ we obtain that $\Phi$ is a solution of $\Phi = A\Phi + g$ in $C^+$. Thus the equations $\Phi = A\Phi + g$ in $C$ and $\Phi = A_C \Phi + g$ in $C$ are equivalent for each $g \in C^+$.

**Lemma 1.** The homogeneous equation $\Phi = A\Phi$ in $C^+$ has only the trivial solution.

**Proof.** If $\Phi_m(z)$ is a solution of the system

$$\phi_k(z) = - \sum_{m=1 \atop m \neq k}^{n} \phi_m([m, z]) + \gamma_k, \quad |z - a_k| \leq r_k,$$

then the $\phi_k(z)$ are analytic in $|z - a_k| \leq r_k$. Let $\phi_k'(z) = \phi_k(z)$. Then integrating (4), we have

$$\psi(z) = - \sum_{m=1}^{n} \phi_m([m, z])$$

analytic in $\overline{D}$. From (5) we obtain

$$\psi(t) = \phi_k(t) - \overline{\phi_k(t)} - \gamma_k, \quad |t - a_k| = r_k.$$  

We shall rewrite the last relations in the form

$$\text{Re} \psi(t) = - \text{Re} \gamma_k,$$

$$2 \text{Im} \phi_k(t) = \text{Im} \psi(t) + \text{Im} \gamma_k, \quad |t - a_k| = r_k.$$  

According to [9] the problem

$$\text{Re} \psi(t) = f(t) + c_k, \quad |t - a_k| = r_k, \quad k = 1, 2, \ldots, n,$$

is called the modified Dirichlet problem. Here, the required analytic function $\psi(z)$ is single-valued in $D$, $f(t)$ is a given Hölder continuous function, and the $c_k$ are undetermined constants. If one of the constants $c_k$ is fixed, for example $c_1$, then the problem (8) has a unique solution up to a purely imaginary additive constant $i\alpha$ [9]. If $f(t) \equiv 0$, then $\psi(z) = c_1 + i\alpha$, $c_k = c_1$ ($k = 2, 3, \ldots, n$). This means that the problem (6) has only constant solutions. Hence, the right side of (7) is constant and the problem (7) with respect to the function $\phi_k(z)$ analytic in $|z - a_k| < r_k$ and continuous in $|z - a_k| \leq r_k$ has only constant solutions. Therefore, $\Phi_k(z) = \phi_k'(z) \equiv 0$.

The lemma is proved.

**Lemma 2.** The equation $\Phi = A_C \Phi + g$ is a Fredholm equation in $C$ and has a unique solution.
Proof. The shift operator $S_m \Phi_m$ is compact in $C(\partial D_m)$. The operator of complex conjugation is bounded in $C$. Therefore the operator $A_C$ is compact in $C$, and the equation $\Phi = A_C \Phi + g$ is a Fredholm equation in $C$. If $g \in C^+$, then $\Phi \in C^+$. In particular, the zero function belongs to $C^+$. Hence by Lemma 1 the homogeneous equation in $C$ has only the trivial solution. Then, according to the Fredholm theorem, the nonhomogeneous equation has a unique solution.

The lemma is proved.

Proof of the main lemma. By the properties of Fredholm equations [6] the spectrum of $A$ consists of eigenvalues only, and the condition $\rho(A) < 1$ expresses the convergence of the method of successive approximations. Here $\rho(A)$ is the spectral radius of $A$. To prove the lemma it is sufficient to demonstrate that the equation $\Phi = \lambda A \Phi$ when $|\lambda| \leq 1$ has only the trivial solution. Let us note that the case $\lambda = 1$ has been investigated in Lemma 1. Integrating the relations

$$\Phi_k(z) = -\lambda \sum_{m=1}^{n} \left( [m, z] \right) \Phi_m([m, z]), \quad |z - a_k| \leq r_k,$$

we obtain

$$\phi_k(z) = -\lambda \sum_{m=1}^{n} \frac{\phi_m([m, z])}{\rho_m([m, z])} + c_k, \quad |z - a_k| \leq r_k.$$

Here $\phi_k'(z) = \Phi_k(z)$, and the $c_k$ are arbitrary constants. Let us introduce the function

$$\psi(z) = -\lambda \sum_{m=1}^{n} \frac{\phi_m([m, z])}{\rho_m([m, z])}$$

analytic in $\partial D$. From the above equalities we obtain the relations

$$\psi(t) = \phi_k(t) - \lambda \pi_k(t) - c_k, \quad k = 1, 2, ..., n.$$  \hspace{1cm} (9)

According to [10] the problem (9) is called the boundary $\mathbb{R}$-linear problem. Here, the unknown functions $\psi(z)$ and $\phi_k(z)$ are analytic in $D$ and $D_k$, respectively, and continuous in $\partial D$ and $D_k$. The function $\psi(z)$ is single-valued in $D$.

If $\lambda = \exp(2i\Theta)$, i.e. $|\lambda| = 1$, then the $\mathbb{R}$-linear problem (9) reduces to the modified Dirichlet problems

$$\text{Re} \exp(-i\Theta) \psi(t) = -\text{Re} \exp(-i\Theta) c_k, \quad |t - a_k| = r_k, \quad k = 1, 2, ..., n,$$

$$2 \text{Im} \exp(-i\Theta) \phi_k(t) = \text{Im} \exp(-i\Theta) (\psi(t) + c_k), \quad |t - a_k| = r_k.$$

Like the problems (6) and (7), the last problems have only constant solutions.

If $|\lambda| < 1$, then according to [7], [8] the homogeneous problem (9) (all $c_k = 0$) has only one linearly independent solution. In this case we can write this solution as $\psi^0(z) = \beta, \phi_k^0(z) = (\beta + \lambda \beta) / (1 - |\lambda|^2)$, where $\beta$ is an arbitrary constant.

According to [7], [8] the nonhomogeneous problem (9) (all $c_k$ are fixed) has the general solution $\psi(z) = \psi^0(z) + \psi^1(z), \phi_k(z) = \phi_k^0(z) + \phi_k^1(z)$, where $\psi^0(z), \phi_k^0(z)$ is the general solution of the homogeneous problem; $\psi^1(z), \phi_k^1(z)$ is a particular solution of the nonhomogeneous problem. In this case we can write this solution as $\psi^1(z) = 0; \phi_k^1(z) = (c_k + \lambda \beta c_k) / (1 - |\lambda|^2)$. 
Therefore, for $|\lambda| \leq 1$ the problem has only constant solutions. Hence, $\Phi_k(z) = \phi_k^* (z) \equiv 0$.

The lemma is proved.

Applying the main lemma to $g_k/i$ instead of $g_k$, and setting $\Omega = i\Phi$, one obtains Lemma 3.

**Lemma 3.** The system of functional equations

$$
\Omega_k(z) = \sum_{m=1}^{n} (m, z) \phi_m([m, z]) + \gamma_k(z), \quad |z - a_k| \leq r_k, \quad k = 1, 2, \ldots, n,
$$

has a unique solution in $C^+$, where $g(z) := g_k(z)$, $|z - a_k| \leq r_k$ ($k = 1, 2, \ldots, n$), is in $C^+$. That solution can be found by the method of successive approximations. The approximations converge in $C^+$.

### 3. The Poincaré $\theta_2$-series in $D$

Let us consider the systems (3) and (10) when $g_k(z) = -H(z)$. At first we assume that $H(z)$ has poles only in the region $D$. Let $\Phi_k(z)$ be a solution of (3) obtained by the method of successive approximation:

$$
\Phi_k(z) = -H(z) + \sum_{k_1=1}^{n} (k_1, z) \phi_{k_1}([k_1, z])
$$

$$
- \sum_{k_1=1}^{n} \sum_{k_2=1}^{n} (k_2, k_1, z) H([k_2, k_1, z]),
$$

$|z - a_k| \leq r_k.$

The solution of (10) has the form

$$
\Omega_k(z) = -H(z) - \sum_{k_1=1}^{n} (k_1, z) H([k_1, z])
$$

$$
- \sum_{k_1=1}^{n} \sum_{k_2=1}^{n} (k_2, k_1, z) H([k_2, k_1, z]),
$$

$|z - a_k| \leq r_k.$

These series converge in $C^+$, i.e. uniformly in $|z - a_k| \leq r_k$. Let us construct the functions

$$
\phi(z) := \sum_{k=1}^{n} (k, z) \phi_k([k, z]), \quad \omega(z) := \sum_{k=1}^{n} (k, z) \Omega_k([k, z]).
$$

Let us consider the generating elements of $\Gamma [m, k, z]$ ($k, m = 1, 2, \ldots, n; k \neq m$).

The equalities $\gamma_j^2(z) = (c_j z + d_j)^{-2}$, $\gamma_j \in \Gamma$, hold here [1], [2]. Hence, we have from (11)

$$
\frac{1}{2} [\phi(z) - \omega(z)] = \sum_{j=1}^{\infty} H(\gamma_j(z)) (c_j z + d_j)^{-2}.
$$
Remember that the order of summation is fixed in accordance with the level of $\gamma_j \in \Gamma$. Thus the Poincaré $\theta_2$-series (2) converges uniformly in $\bar{\mathcal{D}}$, since

$$
(12) \quad \theta_2(z) = -\frac{1}{2}[(\phi(z) + H(z)) - (\omega(z) - H(z))].
$$

Let us study system (3) for the functions $g_k(z) = -H(z)$ meromorphic in $\bar{\mathcal{D}}$. Let $H(z)$ have a pole at a regular point $w$ belonging to some disk $\bar{\mathcal{D}}_p$. Let us describe the process of removing this pole. Let $H(z) = H_0(z) + h(z)$, where $H_0(z)$ is analytic near $D_p$ and $h(z)$ is the principal part of $H(z)$ at $w$. Let us make the change of the $p$-th function $\Phi_p(z) = \Phi_0^p(z) - h(z)$. If $[p, w]$ doesn’t belong to $\bar{\mathcal{D}}_k$ for each $k \neq p$, then the process stops. If $[p, w]$ belongs to $\bar{\mathcal{D}}_q$, then we make the change of the $q$-th function $\Phi_q(z) = \Phi_q^0(z) + \left(\frac{[p, w]}{[w, z]}\right)^{\gamma} h([p, z])$. If $[p, q, w]$ belongs to $\bar{\mathcal{D}}_m$, then we make the next change, and so on. After a finite number of steps the process stops because $w$ is a regular point of $\Gamma$. Along similar lines we can make changes in accord with other poles of $H(z)$.

So we can construct a meromorphic function $q(z)$ such that the equation $\Phi = A\Psi - H$ for $H \in C^+(p)$ with respect to $\Phi \in C^+(p)$ is reduced to the equation $\Psi = A\Psi - G$ in $C^+$. Here $p(z)$ is the principle part of $-H(z)$ in $\bigcup_{k=1}^{p} \bar{\mathcal{D}}_k$, and the elements $\Psi = \Phi - q, G = H + q - Aq$ belong to $C^+$. By virtue of the main lemma the series $\Psi = -\sum_{k=0}^{+\infty} A^k G$ converges in $C^+$. This means that the series $\Phi = q + \Psi = q - \sum_{k=0}^{+\infty} A^k (H + q - Aq)$ converges uniformly in every compact subset of $\bigcup_{k=1}^{p} \bar{\mathcal{D}}_k \setminus \{\text{poles of } q\}$. Let us prove that the last series is $-\sum_{k=0}^{+\infty} A^k H$. The finite sum $q - \sum_{k=0}^{N} A^k (H + q - Aq) = -\sum_{k=0}^{N} A^k H + A^{N+1} q$. Since $q$ has poles only at regular points, there exists a number $M$ such that $A^{N+1} q \in C^+$. Therefore $A^{N+1} q \to 0$ as $N \to \infty$. The same argument is valid for the system (10) when the function $g_k(z) = -H(z)$ is meromorphic in $\bar{\mathcal{D}}_k$. Using relations (11) and (12), let us construct the function

$$
\theta_2(z) = \sum_{j=1}^{\infty} H \left(\gamma_j(z)\right) \left(c_j z + d_j\right)^{-2},
$$

meromorphic in $\bar{\mathcal{D}}$. This series converges uniformly in every compact subset of $\bar{\mathcal{D}} \cap B$.

4. THE POINCARÉ SERIES AS AN AUTOMORPHIC FUNCTION

Theorem 2. The Poincaré $\theta_2$-series (2) is an automorphic function:

$$
(13) \quad \theta_2(z) = \theta_2(\gamma_j(z)) (c_j z + d_j)^{-2} \text{ for each } \gamma_j \text{ from } \Gamma.
$$

The proof of the theorem in the case of absolute convergence is based on the change of the order of summation in (2) [1], [2]. In our case it is forbidden to change the order. Thus we present another proof of the theorem.

It follows from (11) that

$$
\phi(t) = -\frac{t - a_k}{t - a_k} \Phi_k(t) + \sum_{m=1}^{n} \Phi_m([m, t]), \quad |t - a_k| = r_k,
$$

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since \( (k, t) \) = \( \frac{-a_k}{r_k} \), \( |t - a_k| = r_k \). Using (3), we calculate
\[
\text{Im} (t - a_k) \phi (t)
\]
\[
(14) \quad \text{Re} (t - a_k) \left[ \Phi_k (t) + \sum_{m=1, m \neq k}^{n} \left( [m, t] \right)' \Phi_m ([m, t]) \right]
\]
\[
= \text{Im} (t - a_k) H (t), \ |t - a_k| = r_k.
\]

The region \([k; D]\) is symmetric to the region \(D\) with respect to \( |t - a_k| = r_k \). The circumference \([k; \partial D_m]\) is symmetric to the circumference \( |t - a_m| = r_m \) with respect to \( |t - a_k| = r_k \) \( (m \neq k) \), and the region \([k; m; D]\) is symmetric to the region \([k; D]\) with respect to \([k; \partial D_m]\). Let us note that the numbers \(k\) and \(m\) are fixed in the definitions of these regions; \(D \cup \partial D_k \cup [k; D]\) and \([k; D]\) are the fundamental regions of \(\Gamma\). The relation (14) implies meromorphic continuation of \((\phi (z) + H (z))\) to \([k; D]\) and \([k; m; D]\). Using the reflection principle we have
\[
\phi (z) + H (z) = - \left( [k, z] \right)' \left[ \phi ([k, z]) + H ([k, z]) \right], \ z \in [k, D],
\]
\[
\phi (z) + H (z) = \gamma_p (z) \left[ \phi (\gamma_p (z)) + H (\gamma_p (z)) \right], \ z \in [k, m, D],
\]
where \(\gamma_p (z)\) is the composition of symmetries with respect to \([k; \partial D_m]\) and \(\partial D_k\). The transformation \(\gamma_p (z)\) is an element of the group \(\Gamma\).

Along similar lines the relation
\[
\text{Re} (t - a_k) [\omega (t) - H (t)] = 0, \ |t - a_k| = r_k, k = 1, 2, ..., n,
\]
holds. Hence, the function \([\omega (z) - H (z)]\) can be meromorphically continued to \([k; D]\) and \([k; m; D]\), by
\[
\omega (z) - H (z) = \left( [k, t] \right)' \left[ \omega ([k, t]) - H ([k, t]) \right], \ z \in [k, D],
\]
\[
\omega (z) - H (z) = \gamma_p (z) \left[ \omega (\gamma_p (z)) - H (\gamma_p (z)) \right], \ z \in [k, m, D].
\]
It follows from (12) that the function \(\theta_2 (z)\) can be meromorphically continued to \([k; D]\) and \([k; m; D]\). Moreover, the values of \(\theta_2 (z)\) in \(D\) and \([k; m; D]\) are related by the equality
\[
\theta_2 (z) = \theta_2 (\gamma_p (z)) \gamma_p (z), \ z \in [k, m; D] \cap B = \gamma_p (D \cap B).
\]

Using the reflection principle, we can continue \(\theta_2 (z)\) through \([k; \partial D_m]\) to the next symmetric domain, and so on to the region of discontinuity of \(\Gamma\). Moreover the values of \(\theta_2 (z)\) in \(\hat{C}\)−{poles} are related by the equality (13).

Theorem 2 is proved.

It follows from the proof of Theorem 2 that \(\phi (z)\) and \(\omega (z)\) are represented as series in each \([k_1, ..., k_m; D]\). Substituting these series into (12), we obtain a \(\theta_2\)-series with another order of summation. Thus \(\theta_2 (z)\) can be represented as the series
\[
\sum_{j=1}^{\infty} H \left[ \gamma_{\sigma (j)} (z) \right] \gamma_{\sigma (j)} (z)
\]
in each image \([k_1, ..., k_m; D]\) of \(D\) under the mapping \(z \rightarrow [k_1, ..., k_m; z]\), where \(\sigma\) is the bijection of the set of all non-negative integers that corresponds canonically to \([k_1, ..., k_m; D]\).

This proves Theorem 1.
The function $\theta_2(z)$ can be applied to construct the Schwarz operator (Green’s function) for a circular multiply connected region in analytic form. This was done in the papers [12], [13].

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REFERENCES


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