

## CONVERGENCE OF THE POINCARÉ SERIES FOR SOME CLASSICAL SCHOTTKY GROUPS

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ABSTRACT. The Poincaré  $\theta_2$ -series for a multiply connected circular region can be either convergent or divergent absolutely. In this paper we prove a uniform convergence result for such a region.

### 1. INTRODUCTION

Let us consider mutually disjoint disks  $D_k := \{z \in \mathbb{C} : |z - a_k| < r_k\}$  ( $k = 1, 2, \dots, n$ ) on the complex plane  $\mathbb{C}$ . We assume that  $\overline{D_k} \cap \overline{D_j} = \emptyset$  for all  $k \neq j$ , where  $\overline{D_k}$  is the closure of  $D_k$ . Let

$$[k, z] := r_k^2 \overline{(z - a_k)} + a_k$$

be an inversion with respect to the circumference  $\partial D_k := \{t \in \mathbb{C} : |t - a_k| = r_k\}$ . Let us generate the Möbius transformations:

$$(1) \quad [k_1, k_2; z] := [k_1; [k_2; z]], \text{ where } k_1 = 1, 2, \dots, n; k_2 = 1, 2, \dots, n; k_2 \neq k_1,$$

$$[k_1, \dots, k_m; z] := [k_1, \dots, k_{m-1}; [k_m; z]], \text{ where } k_m = 1, 2, \dots, n; k_{m-1} = 1, 2, \dots, n;$$

$$k_m \neq k_{m-1}; \dots; k_2 = 1, 2, \dots, n, k_2 \neq k_1; k_1 = 1, 2, \dots, n.$$

Let  $[k, U]$  denote a set which is symmetric to  $U \subset \widehat{\mathbb{C}} := \mathbb{C} \cup \{\infty\}$  with respect to  $|t - a_k| = r_k$ ,  $[k_1, \dots, k_m; U] := [k_1, \dots, k_{m-1}; [k_m, U]]$ . We shall call  $m$  the level of the transformation (1). The functions (1) with even level can be rewritten in the form  $\gamma_j(z) = (\alpha_j z + b_j) / (c_j z + d_j)$ ,  $j = 0, 1, \dots$ , with  $\alpha_j d_j - c_j b_j = 1$ . Here  $\gamma_0(z) := z$ ,  $\gamma_1(z) := [1, 2; z]$ ,  $\gamma_2(z) := [1, 3; z]$ ,  $\dots$ ,  $\gamma_{n-1}(z) := [1, n; z]$ ,  $\gamma_n(z) := [2, 1; z]$  and so on. The numeration of the transformations is fixed in order of increasing level. The functions  $\gamma_j(z)$  generate the Schottky group  $\Gamma [1]-[3]$ .

Let  $H(z)$  be a meromorphic function in the extended complex plane  $\widehat{\mathbb{C}}$ . The Poincaré  $\theta_{2q}$ -series [1], [2]

$$(2) \quad \theta_{2q}(z) := \sum_{j=0}^{\infty} H(\gamma_j(z))(c_j z + d_j)^{-2q} \quad (q \in \mathbb{Z}/2)$$

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is associated with the group  $\Gamma$ . Here  $z \in B := \widehat{\mathbb{C}} - (B_1 \cup \Lambda(\Gamma))$ ,  $B_1$  is the set of poles of all  $H(\gamma_j(z))$  and  $\gamma_j(z)$ , and  $\Lambda(\Gamma)$  is the limit set of  $\Gamma$ . When  $q > 1$  the series (2) converges absolutely and uniformly in every compact subset of  $B$  [1], [2]. When  $q = 1$  the series (2) can be either absolutely convergent or absolutely divergent. It depends on the properties of  $\Gamma$ . Necessary and sufficient conditions for absolute and uniform convergence of the series have been found in [4], [5] in terms of the Hausdorff dimension of the limit set of  $\Gamma$ . Let us note that absolute and uniform convergencies was not studied separately in the previous papers.

**Definition.** A point  $z$  is called a regular point of  $\Gamma$  if there exist numbers  $k_1, k_2, \dots, k_m$  such that  $[k_1, k_2, \dots, k_m; z]$  belongs to  $\overline{D}$ , where  $D := \widehat{\mathbb{C}} - \bigcup_{k=1}^n \overline{D_k}$  and the closure  $\overline{D}$  is taken in  $\widehat{\mathbb{C}}$ .

In this paper we prove

**Theorem 1.** *Let a rational function  $H(z)$  have poles only at regular points of  $\Gamma$ . Then the Poincaré  $\theta_2$ -series converges uniformly in every compact subset of each region  $[k_1, k_2, \dots, k_m; D] \cap B$ . The order of summation depends on the region  $[k_1, k_2, \dots, k_m; D]$ .*

Let us consider the Banach space  $C$  consisting of the functions  $\Phi(t) := \Phi_k(t)$ , where  $|t - a_k| = r_k$  ( $k = 1, 2, \dots, n$ ), which are continuous on  $\bigcup_{k=1}^n \partial D_k$  with the norm  $\|\Phi\| := \max_{1 \leq k \leq n} \max_{\partial D_k} |\Phi_k(t)|$ . And let us consider the closed subspace  $C^+ \in C$ , for which the functions  $\Phi_k(t)$  have analytic continuations to  $D_k$ .

The proof of Theorem 1 is based on the following

**Main Lemma.** *The system of functional equations*

$$(3) \quad \Phi_k(z) = - \sum_{m=1, m \neq k}^n \left( \overline{[m, z]} \right)' \overline{\Phi_m([m, z])} + g_k(z), \quad |z - a_k| \leq r_k, \quad k = 1, 2, \dots, n$$

has a unique solution in  $C^+$ , where  $g(z) := g_k(z), |z - a_k| \leq r_k$  ( $k = 1, 2, \dots, n$ ), is in  $C^+$ . That solution can be found by the method of successive approximations. The approximations converge in  $C^+$ .

Let the function  $g_k(z)$  is meromorphic in  $|z - a_k| \leq r_k$  ( $k = 1, 2, \dots, n$ ), i.e.  $g_k(z) = s_k(z) + p_k(z)$ , where  $s_k(z)$  is analytic in  $|z - a_k| < r_k$  and  $p_k(z)$  is the principle part of  $g_k(z)$ . Let us consider the Banach space  $C^+(p) := \{\Phi : \Phi - p_k \in C^+, k = 1, 2, \dots, n\}$  with the norm  $\|\Phi\|_{C^+(p)} := \|\Phi - p_k\|_{C^+}$ . Following [11], we can consider (3) with  $g \in C^+(p)$  for  $\Phi \in C^+(q)$ , where the rational functions  $q_k(z)$  are constructed by  $p_k(z)$ . We show how  $q_k(z)$  is constructed by  $p_k(z)$  in Sec. 3

## 2. FUNCTIONAL EQUATIONS

Before proving the main results let us prove some auxiliary facts about functional equations. For brevity we shall write (3) in the form  $\Phi = A\Phi + g$  in  $C^+$ .

Let us define the shift operator  $S_m \Phi_m(z) = \Phi_m([m, z])$  on  $|t - a_k| = r_k$  as the integral operator

$$S_m \Phi_m(z) = \frac{1}{2\pi i} \int_{\partial D_m} \Phi_m(\tau) \frac{d\tau}{\tau - [m, t]}, \quad |t - a_k| = r_k, \quad m = 1, 2, \dots, n; m \neq k.$$

Let us consider the operator

$$A_C\Phi(z) = - \sum_{m=1}^n \sum_{m \neq k} \left( \overline{[m, t]} \right)' S_m \Phi_m(t), \quad |t - a_k| = r_k, \quad k = 1, 2, \dots, n.$$

It is easily seen that if  $\Phi$  satisfies  $\Phi = A\Phi + g$  in  $C^+$ , then  $\Phi$  satisfies  $\Phi = A_C\Phi + g$  in  $C$ . Conversely, if  $g \in C^+$ , then from the properties of  $A_C$  for  $\Phi \in C$  we obtain that  $\Phi$  is a solution of  $\Phi = A\Phi + g$  in  $C^+$ . Thus the equations  $\Phi = A\Phi + g$  in  $C$  and  $\Phi = A_C\Phi + g$  in  $C$  are equivalent for each  $g \in C^+$ .

**Lemma 1.** *The homogeneous equation  $\Phi = A\Phi$  in  $C^+$  has only the trivial solution.*

*Proof.* If  $\Phi_m(z)$  is a solution of the system

$$(4) \quad \Phi_k(z) = - \sum_{m=1}^n \sum_{m \neq k} \left( \overline{[m, z]} \right)' \overline{\Phi_m([m, z])}, \quad |z - a_k| \leq r_k,$$

then the  $\Phi_k(z)$  are analytic in  $|z - a_k| \leq r_k$ . Let  $\phi'_k(z) = \Phi_k(z)$ . Then integrating (4), we have

$$(5) \quad \phi_k(z) = - \sum_{m=1}^n \sum_{m \neq k} \overline{\phi_m([m, z])} + \gamma_k, \quad |z - a_k| \leq r_k,$$

where  $\gamma_k$  are arbitrary constants of integration. Let us introduce the function

$$\psi(z) = - \sum_{m=1}^n \overline{\phi_m([m, z])}$$

analytic in  $\overline{D}$ . From (5) we obtain

$$\psi(t) = \phi_k(t) - \overline{\phi_k(t)} - \gamma_k, \quad |t - a_k| = r_k.$$

We shall rewrite the last relations in the form

$$(6) \quad \operatorname{Re} \psi(t) = - \operatorname{Re} \gamma_k,$$

$$(7) \quad 2 \operatorname{Im} \phi_k(t) = \operatorname{Im} \psi(t) + \operatorname{Im} \gamma_k, \quad |t - a_k| = r_k.$$

According to [9] the problem

$$(8) \quad \operatorname{Re} \psi(t) = f(t) + c_k, \quad |t - a_k| = r_k, \quad k = 1, 2, \dots, n,$$

is called the modified Dirichlet problem. Here, the required analytic function  $\psi(z)$  is single-valued in  $D$ ,  $f(t)$  is a given Hölder continuous function, and the  $c_k$  are undetermined constants. If one of the constants  $c_k$  is fixed, for example  $c_1$ , then the problem (8) has a unique solution up to a purely imaginary additive constant  $i\alpha$  [9]. If  $f(t) \equiv 0$ , then  $\psi(z) = c_1 + i\alpha$ ,  $c_k = c_1$  ( $k = 2, 3, \dots, n$ ). This means that the problem (6) has only constant solutions. Hence, the right side of (7) is constant and the problem (7) with respect to the function  $\phi_k(z)$  analytic in  $|z - a_k| < r_k$  and continuous in  $|z - a_k| \leq r_k$  has only constant solutions. Therefore,  $\Phi_k(z) = \phi'_k(z) \equiv 0$ .

The lemma is proved.

**Lemma 2.** *The equation  $\Phi = A_C\Phi + g$  is a Fredholm equation in  $C$  and has a unique solution.*

*Proof.* The shift operator  $S_m \Phi_m$  is compact in  $C(\partial D_m)$ . The operator of complex conjugation is bounded in  $C$ . Therefore the operator  $A_C$  is compact in  $C$ , and the equation  $\Phi = A_C \Phi + g$  is a Fredholm equation in  $C$ . If  $g \in C^+$ , then  $\Phi \in C^+$ . In particular, the zero function belongs to  $C^+$ . Hence by Lemma 1 the homogeneous equation in  $C$  has only the trivial solution. Then, according to the Fredholm theorem, the nonhomogeneous equation has a unique solution.

The lemma is proved.

*Proof of the main lemma.* By the properties of Fredholm equations [6] the spectrum of  $A$  consists of eigenvalues only, and the condition  $\rho(A) < 1$  expresses the convergence of the method of successive approximations. Here  $\rho(A)$  is the spectral radius of  $A$ . To prove the lemma it is sufficient to demonstrate that the equation  $\Phi = \lambda A \Phi$  when  $|\lambda| \leq 1$  has only the trivial solution. Let us note that the case  $\lambda = 1$  has been investigated in Lemma 1. Integrating the relations

$$\Phi_k(z) = -\lambda \sum_{m=1, m \neq k}^n \left( \overline{[m, z]} \right)' \overline{\Phi_m([m, z])}, \quad |z - a_k| \leq r_k,$$

we obtain

$$\phi_k(z) = -\lambda \sum_{m=1, m \neq k}^n \overline{\phi_m([m, z])} + c_k, \quad |z - a_k| \leq r_k.$$

Here  $\phi'_k(z) = \Phi_k(z)$ , and the  $c_k$  are arbitrary constants. Let us introduce the function

$$\psi(z) = -\lambda \sum_{m=1}^n \overline{\phi_m([m, z])}$$

analytic in  $\overline{D}$ . From the above equalities we obtain the relations

$$(9) \quad \psi(t) = \phi_k(t) - \lambda \overline{\phi_k(t)} - c_k, \quad k = 1, 2, \dots, n.$$

According to [10] the problem (9) is called the boundary  $\mathbb{R}$ -linear problem. Here, the unknown functions  $\psi(z)$  and  $\phi_k(z)$  are analytic in  $D$  and  $D_k$ , respectively, and continuous in  $\overline{D}$  and  $\overline{D}_k$ . The function  $\psi(z)$  is single-valued in  $D$ .

If  $\lambda = \exp(2i\Theta)$ , i.e.  $|\lambda| = 1$ , then the  $\mathbb{R}$ -linear problem (9) reduces to the modified Dirichlet problems

$$\begin{aligned} \operatorname{Re} \exp(-i\Theta) \psi(t) &= -\operatorname{Re} \exp(-i\Theta) c_k, \quad |t - a_k| = r_k, \quad k = 1, 2, \dots, n, \\ 2 \operatorname{Im} \exp(-i\Theta) \phi_k(t) &= \operatorname{Im} \exp(-i\Theta) (\psi(t) + c_k), \quad |t - a_k| = r_k. \end{aligned}$$

Like the problems (6) and (7), the last problems have only constant solutions.

If  $|\lambda| < 1$ , then according to [7], [8] the homogeneous problem (9) (all  $c_k = 0$ ) has only one linearly independent solution. In this case we can write this solution as  $\psi^0(z) = \beta$ ,  $\phi_k^0(z) = (\beta + \lambda \overline{\beta}) / (1 - |\lambda|^2)$ , where  $\beta$  is an arbitrary constant.

According to [7], [8] the nonhomogeneous problem (9) (all  $c_k$  are fixed) has the general solution  $\psi(z) = \psi^0(z) + \psi^1(z)$ ,  $\phi_k(z) = \phi_k^0(z) + \phi_k^1(z)$ , where  $\psi^0(z), \phi_k^0(z)$  is the general solution of the homogeneous problem;  $\psi^1(z), \phi_k^1(z)$  is a particular solution of the nonhomogeneous problem. In this case we can write this solution as  $\psi^1(z) = 0; \phi_k^1(z) = (c_k + \lambda \overline{\beta c_k}) / (1 - |\lambda|^2)$ .

Therefore, for  $|\lambda| \leq 1$  the problem has only constant solutions. Hence,  $\Phi_k(z) = \phi'_k(z) \equiv 0$ .

The lemma is proved.

Applying the main lemma to  $g_k/i$  instead of  $g_k$ , and setting  $\Omega = i\Phi$ , one obtains Lemma 3.

**Lemma 3.** *The system of functional equations*

$$(10) \quad \Omega_k(z) = \sum_{m=1, m \neq k}^n \left( \overline{[m, z]} \right)' \overline{\Omega_m([m, z])} + g_k(z), \quad |z - a_k| \leq r_k, \quad k = 1, 2, \dots, n,$$

has a unique solution in  $C^+$ , where  $g(z) := g_k(z)$ ,  $|z - a_k| \leq r_k$  ( $k = 1, 2, \dots, n$ ), is in  $C^+$ . That solution can be found by the method of successive approximations. The approximations converge in  $C^+$ .

### 3. THE POINCARÉ $\theta_2$ -SERIES IN D

Let us consider the systems (3) and (10) when  $g_k(z) = -H(z)$ . At first we assume that  $H(z)$  has poles only in the region  $D$ . Let  $\Phi_k(z)$  be a solution of (3) obtained by the method of successive approximation:

$$\begin{aligned} \Phi_k(z) &= -H(z) + \sum_{k_1=1, k_1 \neq k}^n \left( \overline{[k_1, z]} \right)' \overline{H([k_1, z])} \\ &\quad - \sum_{k_1=1, k_1 \neq k}^n \sum_{k_2=1, k_2 \neq k_1}^n \left( [k_2, k_1, z] \right)' H([k_2, k_1, z]) + \dots, \\ |z - a_k| &\leq r_k. \end{aligned}$$

The solution of (10) has the form

$$\begin{aligned} \Omega_k(z) &= -H(z) - \sum_{k_1=1, k_1 \neq k}^n \left( \overline{[k_1, z]} \right)' \overline{H([k_1, z])} \\ &\quad - \sum_{k_1=1, k_1 \neq k}^n \sum_{k_2=1, k_2 \neq k_1}^n \left( [k_2, k_1, z] \right)' H([k_2, k_1, z]) - \dots, \\ |z - a_k| &\leq r_k. \end{aligned}$$

These series converge in  $C^+$ , i.e. uniformly in  $|z - a_k| \leq r_k$ . Let us construct the functions

$$(11) \quad \phi(z) := \sum_{k=1}^n \left( \overline{[k, z]} \right)' \overline{\Phi_k([k, z])}, \quad \omega(z) := \sum_{k=1}^n \left( \overline{[k, z]} \right)' \overline{\Omega_k([k, z])}.$$

Let us consider the generating elements of  $\Gamma$   $[m, k, z]$  ( $k, m = 1, 2, \dots, n; k \neq m$ ). The equalities  $\gamma'_j(z) = (c_j z + d_j)^{-2}$ ,  $\gamma_j \in \Gamma$ , hold here [1], [2]. Hence, we have from (11)

$$\frac{1}{2} [\phi(z) - \omega(z)] = \sum_{j=1}^{\infty} H[\gamma_j(z)] (c_j z + d_j)^{-2}.$$

Remember that the order of summation is fixed in accordance with the level of  $\gamma_j \in \Gamma$ . Thus the Poincaré  $\theta_2$ -series (2) converges uniformly in  $\overline{D}$ , since

$$(12) \quad \theta_2(z) = -\frac{1}{2}[(\phi(z) + H(z)) - (\omega(z) - H(z))].$$

Let us study system (3) for the functions  $g_k(z) = -H(z)$  meromorphic in  $\overline{D}_k$ . Let  $H(z)$  have a pole at a regular point  $w$  belonging to some disk  $\overline{D}_p$ . Let us describe the process of removing this pole. Let  $H(z) = H_0(z) + h(z)$ , where  $H_0(z)$  is analytic near  $D_p$  and  $h(z)$  is the principal part of  $H(z)$  at  $w$ . Let us make the change of the  $p$ -th function  $\Phi_p(z) = \Phi_p^0(z) - h(z)$ . If  $[p, w]$  doesn't belong to  $\overline{D}_k$  for each  $k \neq p$ , then the process stops. If  $[p, w]$  belongs to  $\overline{D}_q$ , then we make the change of the  $q$ -th function  $\Phi_q(z) = \Phi_q^0(z) + \left(\overline{[p, z]}\right)' \overline{h([p, z])}$ . If  $[p, q, w]$  belongs to  $\overline{D}_m$ , then we make the next change, and so on. After a finite number of steps the process stops because  $w$  is a regular point of  $\Gamma$ . Along similar lines we can make changes in accord with other poles of  $H(z)$ .

So we can construct a meromorphic function  $q(z)$  such that the equation  $\Phi = A\Phi - H$  for  $H \in C^+(p)$  with respect to  $\Phi \in C^+(p)$  is reduced to the equation  $\Psi = A\Psi - G$  in  $C^+$ . Here  $p(z)$  is the principle part of  $-H(z)$  in  $\bigcup_{k=1}^n \overline{D}_k$ , and the elements  $\Psi = \Phi - q, G = H + q - Aq$  belong to  $C^+$ . By virtue of the main lemma the series  $\Psi = -\sum_{k=0}^{\infty} A^k G$  converges in  $C^+$ . This means that the series  $\Phi = q + \Psi = q - \sum_{k=0}^{\infty} A^k (H + q - Aq)$  converges uniformly in every compact subset of  $\bigcup_{k=1}^n \overline{D}_k \setminus \{\text{poles of } q\}$ . Let us prove that the last series is  $-\sum_{k=0}^{\infty} A^k H$ . The finite sum  $q - \sum_{k=0}^N A^k (H + q - Aq) = -\sum_{k=0}^N A^k H + A^{N+1}q$ . Since  $q$  has poles only at regular points, there exists a number  $M$  such that  $A^{N+1}q \in C^+$ . Therefore  $A^{N+1}q \rightarrow 0$  as  $N \rightarrow \infty$ . The same argument is valid for the system (10) when the function  $g_k(z) = -H(z)$  is meromorphic in  $\overline{D}_k$ . Using relations (11) and (12), let us construct the function

$$\theta_2(z) = \sum_{j=1}^{\infty} H[\gamma_j(z)] (c_j z + d_j)^{-2},$$

meromorphic in  $\overline{D}$ . This series converges uniformly in every compact subset of  $\overline{D} \cap B$ .

#### 4. THE POINCARÉ SERIES AS AN AUTOMORPHIC FUNCTION

**Theorem 2.** *The Poincaré  $\theta_2$ -series (2) is an automorphic function:*

$$(13) \quad \theta_2(z) = \theta_2(\gamma_j(z)) (c_j z + d_j)^{-2} \text{ for each } \gamma_j \text{ from } \Gamma.$$

The proof of the theorem in the case of absolute convergence is based on the change of the order of summation in (2) [1], [2]. In our case it is forbidden to change the order. Thus we present another proof of the theorem.

It follows from (11) that

$$\phi(t) = -\frac{\overline{t - a_k}}{t - a_k} \overline{\Phi_k(t)} + \sum_{m=1, m \neq k}^n \left(\overline{[m, t]}\right)' \overline{\Phi_m([m, t])}, |t - a_k| = r_k,$$

since  $(\overline{[k, t]})' = -\frac{\overline{t-a_k}}{t-a_k}$ ,  $|t - a_k| = r_k$ . Using (3), we calculate

$$\begin{aligned}
 & \operatorname{Im}(t - a_k) \phi(t) \\
 (14) \quad & = \operatorname{Re}(t - a_k) \left[ \Phi_k(t) + \sum_{m=1, m \neq k}^n (\overline{[m, t]})' \overline{\Phi_m([m, t])} \right] \\
 & = \operatorname{Im}(t - a_k) H(t), \quad |t - a_k| = r_k.
 \end{aligned}$$

The region  $[k; D]$  is symmetric to the region  $D$  with respect to  $|t - a_k| = r_k$ . The circumference  $[k; \partial D_m]$  is symmetric to the circumference  $|t - a_m| = r_m$  with respect to  $|t - a_k| = r_k$  ( $m \neq k$ ), and the region  $[k, m; D]$  is symmetric to the region  $[k; D]$  with respect to  $[k; \partial D_m]$ . Let us note that the numbers  $k$  and  $m$  are fixed in the definitions of these regions;  $D \cup \partial D_k \cup [k; D]$  and  $[k; D] \cup [k; \partial D_m] \cup [k, m; D]$  are the fundamental regions of  $\Gamma$ . The relation (14) implies meromorphic continuation of  $(\phi(z) + H(z))$  to  $[k; D]$  and  $[k, m; D]$ . Using the reflection principle we have

$$\begin{aligned}
 \phi(z) + H(z) &= -(\overline{[k, z]})' \left[ \overline{\phi([k, z])} + \overline{H([k, z])} \right], \quad z \in [k, D], \\
 \phi(z) + H(z) &= \gamma'_p(z) [\phi(\gamma_p(z)) + H(\gamma_p(z))], \quad z \in [k, m, D],
 \end{aligned}$$

where  $\gamma_p(z)$  is the composition of symmetries with respect to  $[k; \partial D_m]$  and  $\partial D_k$ . The transformation  $\gamma_p(z)$  is an element of the group  $\Gamma$ .

Along similar lines the relation

$$\operatorname{Re}(t - a_k)[\omega(t) - H(t)] = 0, \quad |t - a_k| = r_k, k = 1, 2, \dots, n,$$

holds. Hence, the function  $[\omega(z) - H(z)]$  can be meromorphically continued to  $[k; D]$  and  $[k, m; D]$ , by

$$\begin{aligned}
 \omega(z) - H(z) &= (\overline{[k, t]})' \left[ \overline{\omega([k, t])} - \overline{H([k, t])} \right], \quad z \in [k, D], \\
 \omega(z) - H(z) &= \gamma'_p(z) [\omega(\gamma_p(z)) - H(\gamma_p(z))], \quad z \in [k, m, D].
 \end{aligned}$$

It follows from (12) that the function  $\theta_2(z)$  can be meromorphically continued to  $[k; D]$  and  $[k, m; D]$ . Moreover, the values of  $\theta_2(z)$  in  $D$  and  $[k, m; D]$  are related by the equality

$$\theta_2(z) = \theta_2(\gamma_p(z))\gamma'_p(z), \quad z \in [k, m; D] \cap B = \gamma_p(D \cap B).$$

Using the reflection principle, we can continue  $\theta_2(z)$  through  $[k; \partial D_m]$  to the next symmetric domain, and so on to the region of discontinuity of  $\Gamma$ . Moreover the values of  $\theta_2(z)$  in  $\widehat{\mathbb{C}} - \{\text{poles}\}$  are related by the equality (13).

Theorem 2 is proved.

It follows from the proof of Theorem 2 that  $\phi(z)$  and  $\omega(z)$  are represented as series in each  $[k_1, \dots, k_m; D]$ . Substituting these series into (12), we obtain a  $\theta_2$ -series with another order of summation. Thus  $\theta_2(z)$  can be represented as the series

$$\sum_{j=1}^{\infty} H[\gamma_{\sigma(j)}(z)] \gamma'_{\sigma(j)}(z)$$

in each image  $[k_1, \dots, k_m; D]$  of  $D$  under the mapping  $z \rightarrow [k_1, \dots, k_m; z]$ , where  $\sigma$  is the bijection of the set of all non-negative integers that corresponds canonically to  $[k_1, \dots, k_m; D]$ .

This proves Theorem 1.

The function  $\theta_2(z)$  can be applied to construct the Schwarz operator (Green's function) for a circular multiply connected region in analytic form. This was done in the papers [12], [13].

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#### REFERENCES

- [1] H.Poincaré, Oeuvres, Gauthier-Villar, Paris t.2, 1916; t.4, 1950; t.9, 1954. MR **97c**:01053
- [2] L.Ford, Automorphic functions, New York, 1929.
- [3] A.F.Berdon, Geometry of discrete groups, Springer-Verlag, New York, 1986. MR **87h**:20014
- [4] T.Akaza, Singular sets of some Kleinian groups, Nagoya Math J **26** (1966), 127-143. MR **34**:7794
- [5] T.Akaza, K.Inoue, Limit sets of geometrically finite free Kleinian groups, Tohoku Math J **36** (1984), 1-16. MR **85b**:30058
- [6] M.A.Krasnoselski, et al., Approximate solution of operator equations, Nauka, Moscow, 1969; English transl., Wolters-Noordhoff, Groningen, 1974. MR **41**:4271; MR **52**:6515
- [7] B.Bojarski, On a boundary value problem of function theory, Soobschenia AN Gruz. SSR **25** (1960), 385-390. (Russian)
- [8] L.G.Mikhajlov, A new class of singular integral equations, Wolters-Noordhoff Publ., Groningen 1970. MR **41**:8812
- [9] F.D.Gakhov, Boundary value problems, Perg.Press, Oxford, 1966. MR **33**:6311
- [10] V.Mityushev, A method of functional equations for boundary value problems of continuous media, Reports Math Phys **33** (1993), 137-147. MR **95e**:30038
- [11] V.Mityushev, Solution of linear functional equation with shift into domain in class of analytic functions, Izvestia AN BSSR. Ser. Phys.-Math. 5 (1983) 117. (Russian)
- [12] V.Mityushev, Solution of the Hilbert boundary value problem for a multiply connected domain, Slupskie Prace Mat.-Przyr. 9a (1994), 37-69. MR **96d**:30050
- [13] V.Mityushev, Hilbert boundary value problem for multiply connected domains, Complex Variables [to appear]

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