ON JB*-TRIPLES WHICH ARE M-IDEALS IN THEIR BIDUALS

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(Communicated by Palle E. T. Jorgensen)

Abstract. The object of this paper is to investigate JB*-triples which are M-ideals in their biduals.

M-ideals in their biduals is a subject of current interest (see [9], [10], [11], [12], [14]), c0 and the compact operators on a Hilbert space being representative examples.

Let us recall that a Banach space $X$ is an M-ideal in its bidual (in short, M-ideal) if
\[ \|\varphi\| = \|\pi(\varphi)\| + \|\varphi - \pi(\varphi)\|, \]
for all $\varphi \in X^{**}$, where $\pi$ is the canonical projection of $X$.

M-ideals can also be characterized by intersection properties of balls. For further information we refer to [2]. In particular, A. Lima proved (see e.g. [12, p. 43]) that a Banach space $X$ is M-ideal in its bidual if, and only if, $X$ has the 2-ball property in its bidual.

On the other hand, it is known (see [14, Theorem 2.6]) that Banach spaces which are M-ideals in their biduals are Asplund spaces (see [5, Chapter VII, Section 5] for a definition).

A JB*-triple is a complex Banach space $J$ together with a continuous triple product $\{.,.,.\} : J \times J \times J \to J$, which satisfies:
1. $\{x, y, z\}$ is bilinear and symmetric in $x$ and $z$ and conjugate-linear in $y$.
2. The Jordan identity
\[ \{a, b, \{x, y, z\}\} = \{\{a, b, x\}, y, z\} - \{x, \{b, a, y\}, z\} + \{x, y, \{a, b, z\}\}. \]
3. For each $x \in J$, the operator $x\Delta x : J \to J$ defined by $x\Delta x(y) = \{x, x, y\}$ is a hermitian operator with non-negative spectrum.
4. $\|x\Delta x\| = \|x\|^2$, for each $x \in J$.

As an example, any C*-algebra with the triple product $\{x, y, z\} = \frac{1}{2}(xy^*z + zy^*x)$ is a JB*-triple.

We recall that the bidual $J^{**}$ of a JB*-triple $J$ is in a natural way a JB*-triple containing $J$ as a subtriple (see [6]).

An (algebraic) ideal in a JB*-triple $J$ is a complex subspace $F$ satisfying $\{x, y, F\} \subseteq F$ and $\{x, F, y\} \subseteq F$, for all $x, y \in J$. Observe that it is enough to take $x = y$ in this definition.
For the statement of our main result, let us fix some notation.

Given a closed subspace \( X \) of a Banach space \( Y \), and \( y \in Y \), we write \( P_X(y) \) for the set of best approximants of \( y \) in \( X \):

\[
P_X(y) = \{ x \in X : \| x - y \| = \| y + X \| \}.
\]

Also for \( x \in X \) and \( \epsilon > 0 \), \( B^X_\epsilon(x, \epsilon) \) will mean the open ball in \( X \) with center at \( x \) and radius \( \epsilon \).

In what follows, we identify, if there is no ambiguity, a Banach space \( Z \) with \( j_Z(Z) \) and \( Z^o \) with \( j_Z(Z)^\circ \), where \( j_Z \) denotes the natural injection of \( Z \) into \( Z^\star \).

We say that the natural projection \( Z^\star \star \rightarrow Z^* \) is of best approximation if, for every \( \varphi \in Z^\star \star \), we have that \( \pi(\varphi) \in P_{Z^*}(\varphi) \).

Now, we state and prove our main result.

**Theorem 1.** Let \( J \) be a JB*-triple. The following assertions are equivalent:

1. \( J \) is an (algebraic) ideal in \( J^\star \).
2. \( J \) is an \( M \)-ideal in \( J^\star \).
3. There exists \( t > 1 \) such that
   \[
   B^J_\epsilon(0, t\| F + J \|) \subseteq P_J(F) - P_J(F)
   \]
   for all \( F \in J^\star \).
4. There exists \( \epsilon > 0 \) such that
   \[
   \| w \| + \epsilon \| f \| \leq \| w + f \|
   \]
   for all \( w \in J^o \) and \( f \in J^* \).
5. \( J \) is an Asplund space and the natural projection \( J^\star \star \rightarrow J^* \) is of best approximation.

The following results are crucial for the proof of Theorem 1.

**Lemma 2** ([3, Lemma 1]). Let \( X \) be a closed subspace of a Banach space \( Y \) and \( t \geq 0 \). If

\[
B^X_\epsilon(0, t\| y + X \|) \subseteq P_X(y) - P_X(y), \forall y \in Y,
\]

then

\[
\| w \| \leq \| h \| + (1 - t)\| h + X^o \|, \forall h \in Y^*, w \in P_X^\star(h).
\]

**Lemma 3** (see [4, Proposition 2.5 and Theorem 4.2]). Let \( X \) be a Banach space and \( \epsilon > 0 \) such that

\[
\| w \| + \epsilon \| f \| \leq \| w + f \|, \forall w \in X^o, f \in X^*.
\]

Then, the following statements are true:

1. For all \( \varphi \in X^\star \star \), \( P_{X^\star}(\varphi) = \{ \pi(\varphi) \} \).
2. \( X \) is an Asplund space.
3. If \( Z \) is a Banach space for which the natural projection \( \pi_Z \) is of best approximation and \( I \) is any isometric linear mapping from \( X^\star \star \) onto \( Z^\star \star \), then \( I \) is the bitranspose of an isometric linear mapping from \( X \) onto \( Z \).

**Proof of Lemma 3.** 1. Let \( \varphi \in X^\star \star \) and \( x^* \in X^* \) with \( \pi(\varphi) \neq x^* \). Then

\[
\| \varphi - x^* \| \geq \| (\varphi - x^*) - \pi(\varphi - x^*) \| + \epsilon \| \pi(\varphi - x^*) \| > \| \varphi - \pi(\varphi) \|.
\]

Therefore,

\[
P_{X^\star}(\varphi) = \{ \pi(\varphi) \}, \forall \varphi \in X^\star \star.
\]
2. If \( Z \) is a separable subspace of \( X \), again \( Z \) satisfies (cf. [12, p.111])
\[
\|\chi\| \geq \|\chi - \pi_Z(\chi)\| + \varepsilon\|\pi_Z(\chi)\|, \quad \forall \chi \in Z^{***},
\]
where \( \pi_Z \) is the canonical projection of \( Z \).

On the other hand, it is clear that, for every \( \chi \in Z^{***} \),
\[
\|\chi - 2\pi_Z(\chi)\| \leq \|\chi - \pi_Z(\chi)\| + \epsilon\|\pi_Z(\chi)\| + (1 - \epsilon)\|\pi_Z(\chi)\| \leq (2 - \epsilon)\|\chi\|,
\]
and so, by [8, Proposition 2.8], \( Z^* \) is separable, that is, \( X \) is an Asplund space.

3. Let \( I : X^{**} \to Z^{**} \) be an isometric isomorphism. Since \( X \) and \( Z \) contain no copy of \( l_1 \) (see [8, Proposition 2.6]), by [8, Lemma 5.6] and [7, Corollary 5.5], \( I \) is \( w^*-w^* \)-continuous. In particular, \( I^*(Z^*) = X^* \). It is clear that
\[
\|\chi + Z^*\| = \|I^*(\chi) + X^*\|,
\]
for all \( \chi \in Z^{***} \) (of course, \( \pi_Z(\chi) \in P_{Z^*}(\chi) \)), and so,
\[
I^*\pi_Z = \pi I^*.
\]
Hence,
\[
I^*(Z^*) = X^*.
\]

Therefore, by the Hahn-Banach theorem, \( I(X) = Z \). Now, we can define \( H : X \to Z \) by
\[
H(x) = j_Z^{-1}I_j\chi (x), \quad \forall x \in X.
\]

The operator \( H \) is continuous and \( H^{**} \) coincides with \( I \) on \( X \). Since both operators are \( w^*-w^* \)-continuous, \( H^{**} = I \).

Proof of Theorem 1. The equivalence 1) \( \Leftrightarrow \) 2) follows from the well-known fact that the closed ideals of a \( JB^* \)-triple \( J \) are precisely the M-ideals of \( J \) (see [1, Theorem 3.2]). The implications 2) \( \Rightarrow \) 3) \( \Rightarrow \) 4) \( \Rightarrow \) 5) have sense for any Banach space \( J \) and actually are true in this more general context. Indeed, 2) \( \Rightarrow \) 3) (with \( t = 2 \)) have been proved in [14, Theorem 1.2]. Next, we show that 3) \( \Rightarrow \) 4). Assume that \( \pi \) is the natural projection of \( J \). Note that
\[
\|\varphi - (\varphi - \pi(\varphi))\| = \|\pi(\varphi)\| = \|\pi(\varphi - w)\| \leq \|\varphi - w\|, \quad \forall \varphi \in J^{***}, w \in J^o
\]
so,
\[
\varphi - \pi(\varphi) \in P_{J^o}(\varphi) \quad \text{and} \quad \|\pi(\varphi)\| = \|\varphi + J^o\|.
\]
(In particular, if \( w \in J^o \), then \( w \in P_{J^o}(f + w), \forall f \in J^* \).) Therefore, by Lemma 2, taking \( \epsilon = t - 1 \), \( X = J \) and \( Y = J^{**} \), the implication 3) \( \Rightarrow \) 4) follows.

The implication 4) \( \Rightarrow \) 5) follows from the assertions 1 and 2 of Lemma 3.

Finally, assume that condition 5) holds for \( J \). Since \( J^* \) is the predual of the \( JBW^* \)-triple \( J^{**} \) and \( J^* \) has the RNP, it follows from [13, Theorem 11] that \( J^* \) is isometric to the dual space of a \( JB^* \)-triple \( X \) which is an ideal (so an M-ideal) in its bidual. Now applying to \( X \) the proved implication 2) \( \Rightarrow \) 4) (with \( Z = J \) and the assertion 3 of Lemma 3, \( X \) is isometric to \( J \), so \( J \) is an M-ideal in \( J^{**} \), and so 2) holds.

Remark 4. 1. Theorem 1 is true if we change the \( JB^* \)-triple \( J \) by a noncommutative \( JB^* \)-algebra \( A \) and 1) by “\( A \) is a two-sided ideal in its bidual”.

Let us recall that an n. c. (not necessarily commutative) \( JB^* \)-algebra is a complex Banach space \( A \) which is also a complex n. c. Jordan algebra (i.e. \((ab)a =
suitable triple product and the given norm. Since, for n. c. \( JB \)-algebras, M-ideals and closed two-sided ideals are the same [15, Theorem 4] and the bidual of any \( JB \)-algebra is a \( JB \)-algebra containing the given one as a subalgebra [15, Theorem 1.7], the above announced result follows directly from our theorem.

2. Assertions 2), 3) are not equivalent in an arbitrary Banach space, in fact we have the following

**Proposition 5.** For every \( r \in [0, 2[ \), there is a Banach space \( X \) failing to be M-ideal and satisfying \[
B^*_X(0, r\|x^{**} + X\|) \subseteq P_X(x^{**}) - P_X(x^{**}), \forall x^{**} \in X^{**}.
\]

**Proof.** In the first place, we recall (see [14, Theorem 1.2 and Proposition 1.5]) that a Banach space \( X \) is an M-ideal if, and only if,

\[
B^*_X(0, 2\|x^{**} + X\|) \subseteq P_X(x^{**}) - P_X(x^{**}), \forall x^{**} \in X^{**},
\]

in particular (cf. [17, Theorem 4] and [16, Proposition 3]), \( X \) is a proximinal subspace in \( X^{**} \), and, for every \( x^{**} \in X^{**} \) and \( x \in X \),

\[
\|x^{**} - x\| = \|x^{**} + X\| + d(x, P_X(x^{**}))
\]

holds.

Let us consider \( l_\infty \) with the usual norm \( \| \cdot \| \) and denote, for every \( F \in l_\infty \),

\[
| F | = \| F + c_0 \| \quad \text{and} \quad P(F) = P_{c_0}(F).
\]

Let \( 0 \leq t \leq 1 \) and consider in \( c_0 \times c_0 \) the following norm:

\[
\|(x, y)\|_t = \max\{\|x\|, \|y\| + t|\|x\||, (1 + t)|\|y\||\}, \forall x, y \in c_0.
\]

We will denote \( X_t = (c_0 \times c_0, \| \cdot \|_t) \).

It is easy to show the following assertions:

1. \( l_\infty \times l_\infty \) with the norm

\[
\|(F, G)\|_t = \max\{| |F| , |G| + t| |F| | , (1 + t)| |G| |\}, \forall F, G \in l_\infty
\]

is the bidual of \( X_t \),

2. For every \( (F, G) \in X_t^{**} \), we have that

\[
\|(F, G) + X_t\|_t = \max\{| |F| + |G| + t | |F| | , (1 + t)| |G| |\}.
\]

3. \( P(F) \times P(G) \subseteq P_t(F, G), \forall F, G \in l_\infty \), where \( P_t(F, G) = P_{c_t}(F, G) \).

We will need the following technical lemmas.

**Lemma 6.** \( X_t \) is a proximinal subspace in its bidual and satisfies the following property:

\[
B^*_X(0, 2(1 - t))\|(F, G) + X_t\|_t) \subseteq P_t(F, G) - P_t(F, G), \forall F, G \in l_\infty.
\]

**Proof.** The case \( t = 0 \) is trivial. Let \( t > 0 \) and \( x, y \in c_0 \) satisfy

\[
\|(x, y)\|_t < 2(1 - t)\|(F, G) + X_t\|_t.
\]
It is clear that one of the following assertions holds:
1. $|F| \leq |G|$.
2. $|G| + t |F| \geq |F| > |G|$.
3. $|F| > |G| + t |F|$.

Case 1. $\|\left((F,G) + X_i\right)\|_t = (1 + t) |G|$.

By assumption on $(x, y)$,

$$\|y\| < 2 |G|$$

(and so, by (1), there are $y_1, y_2 \in P(G)$ with $y = y_1 - y_2$), and

$$\|x\| < 2(1 - t^2) |G|.$$ 

If $(1 - t^2) |G| < |F|$, then, by (1), there are $x_1, x_2 \in P(F)$ such that $x = x_1 - x_2$, and so, $(x_1, y_i) \in P(F \times P(G) \subseteq P_t(F, G)$, for $i = 1, 2$.

In the other case, if $\alpha = \frac{|F|}{(1 - t^2) |G|}$, then, by (1), there are $z_1, z_2 \in P(F)$ such that $\alpha x = z_1 - z_2$.

Let $x_1 = z_1 + \frac{1-\alpha}{2} x$ and $x_2 = z_2 - \frac{1-\alpha}{2} x$. It is clear that $x = x_1 - x_2$ and

$$\|F - x_i\| \leq \|F - z_i\| + \frac{1-\alpha}{2} \|x\|$$

$$\leq |F| + (1 - \alpha)(1 - t^2) |G| = (1 - t^2) |G|,$$ for $i = 1, 2$

and so,

$$\|\left((F,G) - (x_i, y_i)\right)\|_t \leq (1 + t) |G|,$$ for $i = 1, 2$;

that is, in any case, $(x_i, y_i) \in P_t(F, G)$, for $i = 1, 2$, as required.

Case 2. $\|\left((F,G) + X_i\right)\|_t = |G| + t |F|)$.

By assumption,

$$(1 + t)\|y\| < 2(1 - t)(|G| + t |F|)) \leq 2 |G|,$$

and

$$\|x\| < 2(1 - t)(|G| + t |F|) \leq 2 |F|.$$ 

Therefore, by (1), $(x, y) \in P(F \times P(G) - P(F \times P(G) \subseteq P_t(F, G) - P_t(F, G)$, as required.

Case 3. $\|\left((F,G) + X_i\right)\|_t = |F|)$.

Again, by assumption on $(x, y)$,

$$\|x\| < 2 |F|$$

(and so, by (1), there are $x_1, x_2 \in P(F)$ such that $x = x_1 - x_2$), and

$$\|y\| < 2(1 - t) |F|.$$ 

If $\alpha = \frac{|G|}{(1 - t^2) |F|}$, then, by (1), there are $z_1, z_2 \in P(G)$ such that $\alpha y = z_1 - z_2$.

Let $y_1 = z_1 + \frac{1-\alpha}{2} y$ and $y_2 = z_2 - \frac{1-\alpha}{2} y$. It is clear that $y = y_1 - y_2$ and

$$\|\left((F,G) - (x_i, y_i)\right)\|_t \leq |F|,$$ for $i = 1, 2$;

that is, $(x_i, y_i) \in P_t(F, G)$, for $i = 1, 2$, as required. 

\[ \square \]
Lemma 7. \( X_t \) is an M-ideal if, and only if, \( t = 0 \).

Proof. If \( t = 0 \), then \( X_0 \) is an M-ideal.

Suppose that \( t > 0 \). Let \((F,G) \in X_t^{**}\) such that
\[
0 \in P(F) \setminus P(G) \quad \text{and} \quad |G| + t|F| \geq |F| > \|G\|.
\]
Then,
\[
\| (F,G) + X_t \|_t = |G| + t|F| \quad \text{and} \quad \| (F,G) \|_t = \|G\| + t\|F\|.
\]
In this case, it is easy to show that
\[
P_t(F,G) = P(F) \times P(G),
\]
and
\[
d_t(0, P_t(F,G)) \geq (1 + t)d(0, P(G)).
\]
Therefore, by (2), we have that
\[
\| (F,G) + X_t \|_t + d_t(0, P_t(F,G))
\]
\[
\geq |G| + t|F| + (1 + t)d(0, P(G))
\]
\[
= \|G\| + t|F| + td(0, P(G)) > \|(F,G)\|_t.
\]
In particular, again by (2), \( X_t \) is not an M-ideal.

Now, to conclude the proof of Proposition 5, it is enough to take \( r = 2(1 - t) \), with \( 0 < t \leq 1 \).

References


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