ON $JB^*$-TRIPLES WHICH ARE M-IDEALS IN THEIR BIDUALS

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Abstract. The object of this paper is to investigate $JB^*$-triples which are M-ideals in their biduals.

M-ideals in their biduals is a subject of current interest (see [9], [10], [11], [12], $c_0$ and the compact operators on a Hilbert space being representative examples.

Let us recall that a Banach space $X$ is an M-ideal in its bidual (in short, M-ideal) if

$$\|\varphi\| = \|\pi(\varphi)\| + \|\varphi - \pi(\varphi)\|,$$

for all $\varphi \in X^{***}$, where $\pi$ is the canonical projection of $X$.

M-ideals can also be characterized by intersection properties of balls. For further information we refer to [2]. In particular, Á. Lima proved (see e.g. [12, p. 43] that a Banach space $X$ is M-ideal in its bidual if, and only if, $X$ has the 2-ball property in its bidual.

On the other hand, it is known (see [14, Theorem 2.6]) that Banach spaces which are M-ideals in their biduals are Asplund spaces (see [5, Chapter VII, Section 5] for a definition).

A $JB^*$-triple is a complex Banach space $J$ together with a continuous triple product $\{., ., .\} : J \times J \times J \rightarrow J$, which satisfies:

1. $\{x, y, z\}$ is bilinear and symmetric in $x$ and $z$ and conjugate-linear in $y$.
2. The Jordan identity

$$\{a, b, \{x, y, z\}\} = \{\{a, b, x\}, y, z\} - \{x, \{b, a, y\}, z\} + \{x, y, \{a, b, z\}\}.$$  

3. For each $x \in J$, the operator $x\Delta x : J \rightarrow J$ defined by $x\Delta x(\varphi) = \{x, x, \varphi\}$ is a hermitian operator with non-negative spectrum.
4. $\|x\Delta x\| = \|x\|^2$, for each $x \in J$.

As an example, any $C^*$-algebra with the triple product $\{x, y, z\} = \frac{1}{2}(xy^*z + zy^*x)$ is a $JB^*$-triple.

We recall that the bidual $J^{**}$ of a $JB^*$-triple $J$ is in a natural way a $JB^*$-triple containing $J$ as a subtriple (see [6]).

An (algebraic) ideal in a $JB^*$-triple $J$ is a complex subspace $F$ satisfying $\{x, y, F\} \subseteq F$ and $\{x, F, y\} \subseteq F$, for all $x, y \in J$. Observe that it is enough to take $x = y$ in this definition.
For the statement of our main result, let us fix some notation.

Given a closed subspace $X$ of a Banach space $Y$, and $y \in Y$, we write $P_X(y)$ for the set of best approximants of $y$ in $X$: \[ P_X(y) = \{ x \in X : \| x - y \| = \| y + X \| \}. \]

Also for $x \in X$ and $\epsilon > 0$, $B^*_x(x, \epsilon)$ will mean the open ball in $X$ with center at $x$ and radius $\epsilon$.

In what follows, we identify, if there is no ambiguity, a Banach space $Z$ with $j_Z(Z)$ and $Z^*$ with $j_Z(Z)^*$, where $j_Z$ denotes the natural injection of $Z$ into $Z^*$.

We say that the natural projection $Z^{***} \longrightarrow Z^*$ is of best approximation if, for every $\varphi \in Z^{***}$, we have that $\pi(\varphi) \in P_{Z^*}(\varphi)$.

Now, we state and prove our main result.

**Theorem 1.** Let $J$ be a $JB^*$-triple. The following assertions are equivalent:

1. $J$ is an (algebraic) ideal in $J^*$.
2. $J$ is an $M$-ideal in $J^*$.
3. There exists $t > 1$ such that \[ B^*_J(0, t\|F + J\|) \subseteq P_J(F) - P_J(F) \]
   for all $F \in J^{**}$.
4. There exists $\epsilon > 0$ such that \[ \|w\| + \epsilon\|f\| \leq \|w + f\| \]
   for all $w \in J^0$ and $f \in J^*$.
5. $J$ is an Asplund space and the natural projection $J^{***} \longrightarrow J^*$ is of best approximation.

The following results are crucial for the proof of Theorem 1.

**Lemma 2** ([3, Lemma 1]). Let $X$ be a closed subspace of a Banach space $Y$ and $t \geq 0$. If \[ B^*_X(0, t\|y + X\|) \subseteq P_X(y) - P_X(y), \forall y \in Y, \]
then \[ \|w\| \leq \|h\| + (1 - t)\|h + X^0\|, \forall h \in Y^*, w \in P_{X^*}(h). \]

**Lemma 3** (see [4, Proposition 2.5 and Theorem 4.2]). Let $X$ be a Banach space and $\epsilon > 0$ such that \[ \|w\| + \epsilon\|f\| \leq \|w + f\|, \forall w \in X^0, f \in X^*. \]
Then, the following statements are true:

1. For all $\varphi \in X^{***}$, $P_{X^*}(\varphi) = \{ \pi(\varphi) \}$.
2. $X$ is an Asplund space.
3. If $Z$ is a Banach space for which the natural projection $\pi_Z$ is of best approximation and $I$ is any isometric linear mapping from $X^{**}$ onto $Z^{**}$, then $I$ is the bitranspose of an isometric linear mapping from $X$ onto $Z$.

**Proof of Lemma 3.** 1. Let $\varphi \in X^{***}$ and $x^* \in X^*$ with $\pi(\varphi) \neq x^*$. Then \[ \|\varphi - x^*\| \geq \|(\varphi - x^*) - \pi(\varphi - x^*)\| + \epsilon\|\pi(\varphi - x^*)\| > \|\varphi - \pi(\varphi)\|. \]
Therefore, \[ P_{X^*}(\varphi) = \{ \pi(\varphi) \}, \forall \varphi \in X^{***}. \]
2. If $Z$ is a separable subspace of $X$, again $Z$ satisfies (cf. [12, p.111])
\[ \|\chi\| \geq \|\chi - \pi_Z(\chi)\| + \varepsilon \|\pi_Z(\chi)\|, \forall \chi \in Z^{**}, \]
where $\pi_Z$ is the canonical projection of $Z$.

On the other hand, it is clear that, for every $\chi \in Z^{**}$,
\[ \|\chi - 2\pi_Z(\chi)\| \leq \|\chi - \pi_Z(\chi)\| + \varepsilon \|\pi_Z(\chi)\| + (1 - \varepsilon)\|\pi_Z(\chi)\| \leq (2 - \varepsilon)\|\chi\|, \]
and so, by [8, Proposition 2.8], $Z^*$ is separable, that is, $X$ is an Asplund space.

3. Let $I : X^{**} \to Z^{**}$ be an isometric isomorphism. Since $X$ and $Z$ contain no copy of $l_1$ (see [8, Proposition 2.6]), by [8, Lemma 5.6] and [7, Corollary 5.5], $I$ is $w^*-w^*$-continuous. In particular, $I^*(Z^*) = X^*$. It is clear that
\[ \|\chi + Z^*\| = \|I^*(\chi) + X^*\|, \]
for all $\chi \in Z^{***}$ (of course, $\pi_Z(\chi) \in P_{Z^*}(\chi)$), and so,
\[ I^*\pi_Z = \pi I^*. \]

Hence,
\[ I^*(Z^0) = X^0. \]

Therefore, by the Hahn-Banach theorem, $I(X) = Z$. Now, we can define $H : X \to Z$ by
\[ H(x) = j_Z^{-1}Ij_X(x), \forall x \in X. \]

The operator $H$ is continuous and $H^{**}$ coincides with $I$ on $X$. Since both operators are $w^*-w^*$-continuous, $H^{**} = I$. □

Proof of Theorem 1. The equivalence 1) $\iff$ 2) follows from the well-known fact that the closed ideals of a $JB^*$-triple $J$ are precisely the M-ideals of $J$ (see [1, Theorem 3.2]). The implications 2) $\Rightarrow$ 3) $\Rightarrow$ 4) $\Rightarrow$ 5) have sense for any Banach space $J$ and actually are true in this more general context. Indeed, 2) $\Rightarrow$ 3) (with $t = 2$) have been proved in [14, Theorem 1.2]. Next, we show that 3) $\Rightarrow$ 4). Assume that $\pi$ is the natural projection of $J$. Note that
\[ \|\varphi - (\varphi - \pi(\varphi))\| = \|\pi(\varphi)\| - \|\varphi - w\|, \forall \varphi \in J^{***}, w \in J^0 \]
so,
\[ \varphi - \pi(\varphi) \in P_{J^0}(\varphi) \text{ and } \|\pi(\varphi)\| = \|\varphi + J^0\|. \]
(In particular, if $w \in J^0$, then $w \in P_{J^0}(f + w), \forall f \in J^*$.)

Therefore, by Lemma 2, taking $\varepsilon = t - 1$, $X = J$ and $Y = J^{**}$, the implication 3) $\Rightarrow$ 4) follows.

The implication 4) $\Rightarrow$ 5) follows from the assertions 1 and 2 of Lemma 3.

Finally, assume that condition 5) holds for $J$. Since $J^*$ is the predual of the $JWB^*$-triple $J^{**}$ and $J^*$ has the RNP, it follows from [13, Theorem 11] that $J^*$ is isometric to the dual space of a $JB^*$-triple $X$ which is an ideal (so an M-ideal) in its bidual. Now applying to $X$ the proved implication 2) $\Rightarrow$ 4) (with $Z = J$) and the assertion 3 of Lemma 3, $X$ is isometric to $J$, so $J$ is an M-ideal in $J^{**}$, and so 2) holds. □

Remark 4. 1. Theorem 1 is true if we change the $JB^*$-triple $J$ by a noncommutative $JB^*$-algebra $A$ and 1) by “$A$ is a two-sided ideal in its bidual”.

Let us recall that an n. c. (not necessarily commutative) $JB^*$-algebra is a complex Banach space $A$ which is also a complex n. c. Jordan algebra (i.e. $(ab)a =
Let $0 \leq t \leq 1$ and consider in $c_0 \times c_0$ the following norm:

$$
\|(x, y)\|_t = \max\{\|x\|, (1 + t)\|x\|, \|y\|, (1 + t)\|y\|\}, \forall x, y \in c_0.
$$

We will denote $X_t = (c_0 \times c_0, \| \cdot \|_t)$.

It is easy to show the following assertions:

1. $l_\infty \times l_\infty$ with the norm

$$
\|(F, G)\|_{l_\infty} = \max\{\|F\|, \|G\|, (1 + t)\|F\|, (1 + t)\|G\|\}, \forall F, G \in l_\infty
$$

is the bidual of $X_t$.

2. For every $(F, G) \in X_t^{**}$, we have that

$$
\|(F, G) + X_t\|_t = \max\{|F|, G|, (1 + t)\|F\|, (1 + t)\|G\|\}.
$$

3. $P(F) \times P(G) \subseteq P_t(F, G), \forall F, G \in l_\infty$, where $P_t(F, G) = P_{X_t}(F, G)$.

We will need the following technical lemmas.

**Lemma 6.** $X_t$ is a proximinal subspace in its bidual and satisfies the following property:

$$
B_{X_t}^{X_t}(0, 2(1 - t))|(F, G) + X_t|_t \subseteq P_t(F, G) - P_t(F, G), \forall F, G \in l_\infty.
$$

**Proof.** The case $t = 0$ is trivial. Let $t > 0$ and $x, y \in c_0$ satisfy

$$
\|(x, y)\|_t < 2(1 - t)|(F, G) + X_t|_t.
$$
It is clear that one of the following assertions holds:

1. $|F| \leq |G|$
2. $|G| + t \mid F \geq |F| \mid G|$
3. $|F| \mid G| + t \mid F|$

Case 1. $(\|F, G\| + X_i) = (1 + t) \mid G\).

By assumption on $(x, y)$,

$$\|y\| < 2 |G|$$

(and so, by (1), there are $y_1, y_2 \in P(G)$ with $y = y_1 - y_2$, and

$$\|x\| < 2(1 - t^2) |G|.$$  

If $(1 - t^2) |G| < |F|$, then, by (1), there are $x_1, x_2 \in P(F)$ such that $x = x_1 - x_2$, and so, $(x_i, y_i) \in P(F) \times P(G) \subseteq P_i(F, G)$, for $i = 1, 2$.

In the other case, if $\alpha = \frac{|F|}{(1 - t^2)|G|}$, then, by (1), there are $z_1, z_2 \in P(F)$ such that $\alpha x = z_1 - z_2$.

Let $x_1 = z_1 + \frac{1 - \alpha}{2}x$ and $x_2 = z_2 - \frac{1 - \alpha}{2}x$. It is clear that $x = x_1 - x_2$ and

$$\|F - x_i\| \leq \|F - z_i\| + \frac{1 - \alpha}{2} \|x\|$$

$$\leq |F| + (1 - \alpha)(1 - t^2) |G| = (1 - t^2) |G|,$$ for $i = 1, 2$

and so,

$$\|(F, G) - (x_i, y_i)\|_t \leq (1 + t) \mid G\), for $i = 1, 2$;

that is, in any case, $(x_i, y_i) \in P_i(F, G)$, for $i = 1, 2$, as required.

Case 2. $(\|F, G\| + X_i) = |G| + t \mid F\|)$.

By assumption,

$$(1 + t)\|y\| < 2(1 - t)(|G| + t \mid F\|) \leq 2 |G|,$$

and

$$\|x\| < 2(1 - t)(|G| + t \mid F\|) \leq 2 |F|.$$  

Therefore, by (1), $(x, y) \in P(F) \times P(G) - P(F) \times P(G) \subseteq P_i(F, G) - P_i(F, G)$, as required.

Case 3. $(\|F, G\| + X_i) = \mid F\|)$.

Again, by assumption on $(x, y)$,

$$\|x\| < 2 |F|$$

(and so, by (1), there are $x_1, x_2 \in P(F)$ such that $x = x_1 - x_2$, and

$$\|y\| < 2(1 - t) |F|.$$  

If $\alpha = \frac{|G|}{(1 - t^2)|F|}$, then, by (1), there are $z_1, z_2 \in P(G)$ such that $\alpha y = z_1 - z_2$.

Let $y_1 = z_1 + \frac{1 - \alpha}{2}y$ and $y_2 = z_2 - \frac{1 - \alpha}{2}y$. It is clear that $y = y_1 - y_2$ and

$$\|(F, G) - (x_i, y_i)\|_t \leq |F|,$$ for $i = 1, 2$;

that is, $(x_i, y_i) \in P_i(F, G)$, for $i = 1, 2$, as required.  

\[\square\]
Lemma 7. \(X_t\) is an M-ideal if, and only if, \(t = 0\).

Proof. If \(t = 0\), then \(X_0\) is an M-ideal.

Suppose that \(t > 0\). Let \((F,G) \in X_t^{**}\) such that
\[0 \in P(F) \setminus P(G)\text{ and } |G| + t |F| \geq |F| > \|G\|.

Then,
\[\|(F,G) + X_t\|_t = |G| + t |F| \text{ and } \|(F,G)\|_t = \|G\| + t\|F\|.

In this case, it is easy to show that \(P_t(F,G) = P(F) \times P(G)\),
and
\[d_t(0, P_t(F,G)) \geq (1 + t)d(0, P(G)).\]

Therefore, by (2), we have that
\[\|(F,G) + X_t\|_t + d_t(0, P_t(F,G)) \geq |G| + t |F| + (1 + t)d(0, P(G))
= \|G\| + t |F| + td(0, P(G)) > \|(F,G)\|_t.

In particular, again by (2), \(X_t\) is not an M-ideal.

Now, to conclude the proof of Proposition 5, it is enough to take \(r = 2(1 - t)\),
with \(0 < t \leq 1\). \(\square\)

References


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