LAPLACE TRANSFORMS AND GENERATORS
OF SEMIGROUPS OF OPERATORS

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Abstract. In this paper, a characterization for continuous functions on \((0, \infty)\) to be the Laplace transforms of \(f \in L^\infty(0, \infty)\) is obtained. It is also shown that the vector-valued version of this characterization holds if and only if the underlying Banach space has the Radon-Nikodým property. Using these characterizations, some results, different from that of the Hille-Yosida theorem, on generators of semigroups of operators are obtained.

1. Introduction

The theory of Laplace transforms plays an important role in the theory of semigroups of operators. Given a function \(F\) on \((0, \infty)\), under what conditions is \(F\) the Laplace transform of a certain function \(f\)? This problem has been investigated extensively. In [7], Widder obtained the following characterization of Laplace transforms of scalar-valued functions:

A function \(F\) on \((0, \infty)\) is the Laplace transform of \(f \in L^\infty(0, \infty)\) if and only if \(F\) is infinitely differentiable and satisfies

\[
(W_\infty) \quad \sup \left\{ \frac{1}{n!} \lambda^{n+1} F^{(n)}(\lambda) : \lambda > 0, n \in \mathbb{N} \cup \{0\} \right\} < \infty.
\]

The vector-valued version of Widder’s theorem has been investigated by Arendt among others. In [1], Arendt obtained an “integrated version of Widder’s theorem” (see [1, Theorem 1.1]), and from this generalization, the relation between the Hille-Yosida theorem and Widder’s theorem is revealed.

It is worth noting that in Widder’s characterization of Laplace transforms, condition \((W_\infty)\) involves not only the original function, but also its higher derivatives, and so in certain practical problems it may be difficult to verify condition \((W_\infty)\).

In Section 2, we give a characterization of Laplace transforms which involves only the original function but not its derivatives. Applications of this characterization can be found in [6].

In the theory of semigroups of operators, it is known that whether a linear operator \(A\) is the generator of a certain semigroup (\(C_0\)-semigroup or integrated semigroup) is related to the Laplace representation of its resolvent \(R(\lambda, A)\) (see [1], [5], [3]). In Section 3, using the results in Section 2, we obtain some characterization
results for generators of semigroups of operators. These results are different from those given by the Hille-Yosida theorem.

2. CHARACTERIZATIONS OF LAPLACE TRANSFORMS

Let \( f \in L^\infty(0, \infty) \). The Laplace transform \( F \) of \( f \) is given by

\[
F(\lambda) = \int_0^\infty e^{-\lambda t} f(t) \, dt \quad (\lambda > 0).
\]

The following result gives a characterization of those \( F \in C(0, \infty) \) that are Laplace transform of an element \( f \) in \( L^\infty(0, \infty) \). This characterization involves only the original function \( F \), not its higher derivatives.

**Theorem 2.1.** Let \( F \in C(0, \infty) \). The following assertions are equivalent.

1. \( F \) is the Laplace transform of some \( f \in L^\infty(0, \infty) \).
2. There exists a constant \( M \) such that \( |\lambda F(\lambda)| \leq M \) for a.e. \( \lambda > 0 \) and
   \[
   \left| \sum_{j=1}^\infty \frac{(-1)^{j-1}}{j-1!} e^{jn} \lambda F(j\lambda) \right| \leq M \quad \text{for a.e.} \quad \lambda > 0 \quad \text{for infinitely many} \quad n \in \mathbb{N}.
   \]
3. Same as (2), with the inequalities holding for all \( \lambda > 0 \) and all \( n \in \mathbb{N} \).

**Proof.** (1 implies 3) Put \( M = \text{ess sup}_{0 < t < \infty} |f(t)| \). It is clear that \( |\lambda F(\lambda)| \leq M \) for all \( \lambda > 0 \). Let \( \lambda > 0 \) and \( n \in \mathbb{N} \). Then

\[
\left| \sum_{j=1}^\infty \frac{(-1)^{j-1}}{j-1!} e^{jn} \lambda F(j\lambda) \right| = \left| \int_0^\infty \lambda \sum_{j=1}^\infty \frac{(-1)^{j-1}}{j-1!} e^{jn} e^{-j\lambda t} f(t) \, dt \right|
\]

\[
= \left| \int_0^\infty \lambda e^{-\lambda t} \sum_{j=1}^\infty \frac{(-1)^{j-1}}{j-1!} e^{jn} f(t) \, dt \right| 
\]

\[
\leq M.
\]

(3 implies 2) Obvious.

(2 implies 1) Let \( f_n(t) = \sum_{j=1}^\infty \frac{(-1)^{j-1}}{j-1!} e^{jn} \frac{n}{t} F\left(\frac{jn}{t}\right) \). Then the given condition on \( F \) implies that there exist \( n_1 < n_2 < \cdots \) such that \( (f_{n_k}) \) is a bounded sequence in \( L^\infty(0, \infty) \). Since \( L^\infty(0, \infty) \) is the dual of the separable space \( L^1(0, \infty) \), \( (f_{n_k}) \) has a subsequence \( (f_{n_{k_i}}) \) which converges in the weak*-topology to \( f \in L^\infty(0, \infty) \). In particular, for every \( \lambda > 0 \),

\[
\lim_{k \to \infty} \int_0^\infty e^{-\lambda t} f_{n_{k_i}}(t) \, dt = \int_0^\infty e^{-\lambda t} f(t) \, dt.
\]

On the other hand, since

\[
\int_0^\infty \sum_{j=1}^\infty \frac{(-1)^{j-1}}{j-1!} \frac{n}{t} |F\left(\frac{jn}{t}\right)| e^{-\lambda t} \, dt < \infty
\]

and

\[
\int_0^\infty \sum_{j=1}^\infty \frac{(-1)^{j-1}}{j-1!} \frac{n}{s} |F\left(\frac{1}{s}\right)| e^{-\lambda jns} \, ds < \infty,
\]

we have

\[
\int_0^\infty f_n(t) e^{-\lambda t} \, dt = \int_0^\infty \sum_{j=1}^\infty \frac{(-1)^{j-1}}{j-1!} \frac{n}{t} F\left(\frac{jn}{t}\right) e^{-\lambda t} \, dt
\]

\[
= \sum_{j=1}^\infty \frac{(-1)^{j-1}}{j-1!} e^{jn} \int_0^\infty \frac{n}{s} F\left(\frac{1}{s}\right) e^{-\lambda jns} \, ds
\]
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\[ \int_0^\infty e^{-e^{1-x}} e^{n(1-x)} \frac{n}{s} F\left(\frac{1}{s}\right) ds \]

\[ = \int_0^\infty e^{-e^{-u}} e^{-u} \frac{n}{n+u} F\left(\frac{\lambda n}{n+u}\right) du \]

\[ = \int_0^\infty \lambda n e^{-e^{-u}} e^{-u} \frac{n}{n+u} F\left(\frac{\lambda n}{n+u}\right) du, \]

so by the dominated convergence theorem (using the condition that \(|\lambda F(\lambda)| \leq M\) a.e. \(\lambda > 0\)),

\[ \lim_{n \to \infty} \int_0^\infty f_n(t) e^{-\lambda t} dt = \int_0^\infty e^{-e^{-u}} e^{-u} F(\lambda) du = F(\lambda). \]

Hence \(F\) is the Laplace transform of \(f\).

In the proof of the above theorem, we use the following version of the dominated convergence theorem: if \(\int_X \sum_{j=1}^\infty |g_j| < \infty\), then \(\int_X \sum_{j=1}^\infty g_j = \sum_{j=1}^\infty \int_X g_j\). This kind of argument will be used in later proofs and will not be mentioned explicitly.

**Corollary 2.2.** Suppose a continuous function \(F\) on \((0, \infty)\) satisfies

\[ \sup_{\lambda > 0} |\lambda F(\lambda)| < \infty \]

and

\[ \sup_{\lambda > 0, n \in \mathbb{N}} \left| \sum_{j=1}^\infty \frac{(-1)^{j-1}}{(j-1)!} e^{\lambda j} \lambda F(j\lambda) \right| < \infty. \]

Then \(F\) is infinitely differentiable and can be extended to an analytic function on the right half-plane \(\{z \in \mathbb{C} : \text{Re} z > 0\}\).

Note that unlike Bernstein’s theorem on completely monotone functions (see [7]), the condition given in the above corollary does not involve higher derivatives of \(F\).

Next we want to consider Laplace transforms of vector-valued functions. Given \(f \in L^\infty((0, \infty), E)\), where \(E\) is a Banach space, using the same argument as in the proof of Theorem 2.1, we see that the Laplace transform \(F\) of \(f\) satisfies

\[ (P_\infty) \sup_{\lambda > 0} \|\lambda F(\lambda)\| < \infty \quad \text{and} \quad \sup_{\lambda > 0, n \in \mathbb{N}} \left\| \sum_{j=1}^\infty \frac{(-1)^{j-1}}{(j-1)!} e^{\lambda j} \lambda F(j\lambda) \right\| < \infty. \]

We will show that the converse holds if \(E\) has the Radon-Nikodým property. In fact, this gives a characterization for Banach spaces with the Radon-Nikodým property. The idea is to show that condition \((P_\infty)\) is equivalent to Widder’s condition.

**Theorem 2.3.** Let \(E\) be a Banach space and let \(F \in C((0, \infty), E)\). The following assertions are equivalent.

1. There exists a Lipschitz continuous function \(\alpha : [0, \infty) \to E\) with \(\alpha(0) = 0\) such that

\[ F(\lambda) = \int_0^\infty \lambda e^{-\lambda t} \alpha(t) dt \quad \forall \lambda > 0. \]

2. \(F\) satisfies condition \((P_\infty)\).

3. \(F\) is infinitely differentiable and \(\sup_{\lambda > 0, n \in \mathbb{N}} \left\{ \frac{1}{n!} \lambda^{n+1} F^{(n)}(\lambda) \right\} < \infty\).
we see that

Using the proof of Theorem 2.1 together with the uniform boundedness principle, Theorem 2.1, there exists \( \tilde{f} \in L^\infty(0, \infty) \) such that

\[
\langle F(\lambda), x^* \rangle = \int_0^\infty \left( e^{-\lambda t} \int_0^t f(s) \, ds \right) \, dt = \int_0^\infty e^{-\lambda t} f(t) \, dt.
\]

Using the proof of Theorem 2.1 together with the uniform boundedness principle, we see that \( F \) satisfies condition \( (P_\infty) \).

(2 implies 1) For every \( x^* \in E^* \), we consider the function \( \lambda \mapsto \langle F(\lambda), x^* \rangle \). By Theorem 2.1, there exists \( \tilde{f} \in L^\infty(0, \infty) \) such that

\[
\langle F(\lambda), x^* \rangle = \int_0^\infty e^{-\lambda t} \tilde{f}(t) \, dt \quad \forall \lambda > 0.
\]

It follows from the proof of [1, Theorem 1.1] that there exists a function \( \alpha \) which satisfies the requirements.

The equivalence of 1 and 3 is just [1, Theorem 1.1]. \( \square \)

**Theorem 2.4.** A Banach space \( E \) has the Radon-Nikodým property if and only if every \( F \in C((0, \infty), E) \) satisfying condition \( (P_\infty) \) is the Laplace transform of some \( f \in L^\infty((0, \infty), E) \).

**Proof.** This is an immediate consequence of Theorem 2.3 and [1, Theorem 1.4]. \( \square \)

**Remark 2.1.** If \( E \) is a dual space and has the Radon-Nikodým property, then \( L^\infty((0, \infty), E) \) is a dual space [see [4]]. So given \( F \in C((0, \infty), E) \) satisfying condition \( (P_\infty) \), the bounded sequence \((f_n)\) constructed in the proof of Theorem 2.1 has a weak* limit \( f \) which is the inverse Laplace transform of \( F \).

For continuous \( f \in L^\infty((0, \infty), E) \), where \( E \) is a Banach space not necessarily possessing the Radon-Nikodým property, we have the following inversion formula.

**Theorem 2.5.** Let \( E \) be a Banach space. Let \( f : (0, \infty) \to E \) be a bounded continuous function and \( F \) its Laplace transform. Then

\[
f(t) = \lim_{n \to \infty} \sum_{j=1}^{\infty} \frac{(-1)^{j-1}}{(j-1)!} e^{jnt} n F(jn) \quad \forall t > 0,
\]

the convergence is uniform on compact subsets of \((0, \infty)\), and uniform on bounded subsets of \((0, \infty)\) if \( f(0+) \) exists, and in this case,

\[
f(0+) = (1 - e^{-1})^{-1} \lim_{n \to \infty} \sum_{j=1}^{\infty} \frac{(-1)^{j-1}}{(j-1)!} F(jn).
\]

**Proof.** Let \( t \geq 0 \) and \( n \in \mathbb{N} \). Then

\[
\lim_{n \to \infty} \sum_{j=1}^{\infty} \frac{(-1)^{j-1}}{(j-1)!} e^{jnt} n F(jn) = \lim_{n \to \infty} \int_0^\infty n \sum_{j=1}^{\infty} \frac{(-1)^{j-1}}{(j-1)!} e^{jnt} e^{-jnr} f(r) \, dr
\]

\[
= \lim_{n \to \infty} \int_0^\infty n e^{-nt} e^u f(u) \, du
\]

\[
= \lim_{n \to \infty} \int_{-nt}^\infty e^{-u} e^{-u} f(\frac{nt+u}{n}) \, du
\]

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where the last equality follows from the dominated convergence theorem and the condition that $f$ is continuous. Since $f$ is uniformly continuous on $[a, b]$ for $0 < a < b < \infty$ (on $(0, b]$ if $f(0+)$ exists), the convergence given in the last equality is uniform on $[a, b]$ (on $(0, b]$ if $f(0+)$ exists).

\[\]

Remark 2.2. Using the same idea as in the above proof, we see that the sequence $(f_n)$ constructed in the proof of Theorem 2.1 converges to $f$ for all $t > 0$ if $f$ is continuous. However, we cannot consider the convergence at $t = 0$ for this sequence.

3. SEMIGROUPS OF OPERATORS

Let $E$ be a Banach space. The space of all bounded linear operators from $E$ into itself is denoted by $\mathcal{B}(E)$. A family $(S(t))_{t > 0} \subset \mathcal{B}(E)$ is said to be a semigroup if $S(s + t) = S(s)S(t)$ for all $s, t > 0$. If $(S(t))_{t > 0}$ is a strongly continuous semigroup and $\text{SOT-lim}_{t \to 0^+} S(t) = I := S(0)$, $(S(t))_{t \geq 0}$ is called a $C_0$-semigroup.

Proposition 3.1. Let $E$ be a Banach space. Let $A : \mathcal{D}(A) \subset E \longrightarrow E$ be a closed linear operator and let $w \in \mathbb{R}$. If there exists a strongly continuous semigroup $(S(t))_{t > 0} \subset \mathcal{B}(E)$ satisfying $\|S(t)\| \leq Me^{wt}$ for all $t > 0$, where $M$ is a constant, such that for all $x \in E$,

$$R(\lambda, A)x = \int_0^\infty e^{-\lambda t}S(t)x \, dt \quad \forall \lambda > w,$$

then $(w, \infty) \subset \rho(A)$ and the function $F : (0, \infty) \longrightarrow \mathcal{B}(E)$ defined by

$$F(\lambda) = R(w + \lambda, A)$$

satisfies condition $(P_\infty)$. The converse is true if $E$ has the Radon-Nikodým property.

Proof. The condition on $(S(t))_{t > 0}$ implies that $F$ is the Laplace transform (in the strong operator topology) of the bounded function $t \mapsto e^{-wt}S(t)$. Hence $F$ satisfies condition $(P_\infty)$.

Conversely, if $F$ satisfies condition $(P_\infty)$, by Theorem 2.3, it satisfies the Hille-Yosida condition, namely,

$$\sup_{\lambda > 0, m \in \mathbb{N}} \|(\lambda R(\lambda, A - w))^m\| < \infty.$$

Hence by [1, Theorem 6.2], there exists a strongly continuous semigroup $(T(t))_{t > 0}$ satisfying $\sup_{t > 0} \|T(t)\| < \infty$ such that $R(\lambda, A - w)x = \int_0^\infty e^{-\lambda t}T(t)x \, dt$ for all $\lambda > 0$, $x \in E$. Hence $(S(t) = e^{wt}T(t))_{t > 0}$ is the required semigroup.

Remark 3.1. The converse is also true if $A$ is densely defined. In this case, the strongly continuous semigroup $(S(t))_{t > 0}$ can be extended to a $C_0$ semigroup $(S(t))_{t \geq 0}$ (see Corollary 3.7).
Proposition 3.2. Let $E$ be a Banach space. Let $w \in \mathbb{R}$. Suppose $A : \mathcal{D}(A) \subset E \rightarrow E$ is the generator of a $C_0$-semigroup $(S(t))_{t \geq 0}$ with $\|S(t)\| \leq Me^{wt}$ for all $t \geq 0$, where $M$ is a constant. Then for every $x \in E$, we have

\[ S(t)x = e^{wt} \lim_{n \to \infty} n \sum_{j=1}^{\infty} \frac{(-1)^{j-1}}{(j-1)!} e^{jnt} R(jn + w, A)x \quad \text{for } t > 0, \]

\[ (1 - e^{-1})x = \lim_{n \to \infty} n \sum_{j=1}^{\infty} \frac{(-1)^{j-1}}{(j-1)!} R(jn + w, A)x, \]

and the convergence is uniform on $(0, b]$ for $0 < b < \infty$.

Proof. This is an immediate consequence of Theorem 2.5.

Let $n \in \mathbb{N}$. A strongly continuous family $(S(t))_{t \geq 0} \subset \mathcal{B}(E)$ is called an $n$-times integrated semigroup if $S(0) = 0$ and, for all $x \in E$,

\[ S(t)S(s)x = \frac{1}{(n-1)!} \left[ \int_t^{s+t} (s+t-r)^{n-1} S(r)x \, dr - \int_0^s (s+t-r)^{n-1} S(r)x \, dr \right] \quad \forall s, t \geq 0. \]

For convenience, a $C_0$-semigroup is also called a 0-times integrated semigroup.

An $n$-times integrated semigroup $(S(t))_{t \geq 0}$ (where $n \in \mathbb{N}$) is said to be

1. exponentially bounded if there exist constants $M, w$ such that $\|S(t)\| \leq Me^{wt}$ for all $t \geq 0$;
2. non-degenerate if $S(t)x = 0$ for all $t \geq 0$ implies $x = 0$;
3. locally Lipschitz if there exist constants $M, w$ such that $\|S(t+h) - S(t)\| \leq Me^{(t+h)w}h$ for all $t, h \geq 0$.

Given a non-degenerate, exponentially bounded $n$-times integrated semigroup $(S(t))_{t \geq 0}$ (where $n \in \mathbb{N}$), there exists a unique operator $A$ and there exists $a \in \mathbb{R}$ with $(a, \infty) \subset \rho(A)$ such that $R(\lambda, A)x = \int_0^\infty \lambda^n e^{-\lambda t} S(t)x \, dt$ for all $\lambda > a, x \in E$. This unique operator is called the generator of $(S(t))_{t \geq 0}$. Since we are mainly interested in generators, for $n \in \mathbb{N}$, a non-degenerate, exponentially bounded $n$-times integrated semigroup will be called an $n$-times integrated semigroup for simplicity.

It should be pointed out that for an $n$-times integrated semigroup $(S(t))_{t \geq 0}$ (n $\in \mathbb{N}$) with $\|S(t)\| \leq Me^{wt}$ for all $t \geq 0$, the constant $w$ must be non-negative. This follows from the equality

\[ S(t)x = \frac{t^n}{n!}x + \int_0^t S(s)Ax \, ds, \]

which holds for all $x \in \mathcal{D}(A)$ and $t \geq 0$. Similarly, if $(S(t))_{t \geq 0}$ is locally Lipschitz with $\|S(t+h) - S(t)\| \leq Me^{w(t+h)}h$ for all $t, h \geq 0$, the constant $w$ must be non-negative.

If $A$ generates an $n$-times integrated semigroup $(S(t))_{t \geq 0}$, then for every $\lambda \in \mathbb{C}$, $A - \lambda$ generates an $n$-times integrated semigroup $(\tilde{S}(t))_{t \geq 0}$, where

\[ \tilde{S}(t)x = e^{-\lambda t} S(t)x + \sum_{k=1}^n \lambda^k \binom{n}{k} \int_0^t \cdots \int_0^t e^{-\lambda u_1} S(u_1) x \, du_1 \cdots du_k \quad \forall x \in E. \]

(To see this, it suffices to check that $\int_0^\infty e^{-u} \tilde{S}(t)x \, dt = \frac{1}{u^n} R(u, A - \lambda)x$.) The following two lemmas give the relation between the locally Lipschitz constants of $(S(t))_{t \geq 0}$ and $(\tilde{S}(t))_{t \geq 0}$. 

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Lemma 3.3. Let $n \in \mathbb{N}$. Suppose $A$ generates an $n$-times integrated semigroup $(S(t))_{t \geq 0}$ satisfying
\[ \|S(t+h) - S(t)\| \leq Mh \quad \forall t, h \geq 0, \]
where $M$ is a constant. Then for every $\lambda > 0$, $A + \lambda$ generates an $n$-times integrated semigroup $(\tilde{S}(t))_{t \geq 0}$ with the property that given any $\epsilon > 0$, there exists a constant $\tilde{M}$ such that
\[ \|\tilde{S}(t+h) - \tilde{S}(t)\| \leq \tilde{M} e^{(\lambda+\epsilon)(t+h)}h \quad \forall t, h \geq 0. \]

Proof. Let $\lambda, \epsilon > 0$. Take $M_1 > 0$ such that $\|S(t)\| \leq M_1 e^{\lambda t}$ for all $t \geq 0$. Then for every $t, h \geq 0$, we have
\[
\|\tilde{S}(t+h) - \tilde{S}(t)\| \leq \|e^{\lambda(t+h)}S(t+h) - e^{\lambda t}S(t)\|
+ \sum_{k=1}^{n} \lambda^k \left( \frac{n}{k} \right) \int_t^{t+h} \int_0^{u_k} \cdots \int_0^{u_2} e^{\lambda u_1} \|S(u_1)\| \, du_1 \cdots \, du_k
\leq e^{\lambda(t+h)}\|S(t+h) - S(t)\| + (e^{\lambda(t+h)} - e^{\lambda t})\|S(t)\|
+ \sum_{k=1}^{n} \lambda^k \left( \frac{n}{k} \right) \int_t^{t+h} \int_0^{u_k} \cdots \int_0^{u_2} M_1 e^{(\lambda+\epsilon)u_1} \, du_1 \cdots \, du_k
\leq Me^{\lambda(t+h)}h + e^{\lambda(t+h)}\lambda h M_1 e^{\epsilon t} + \sum_{k=1}^{n} \lambda^k \left( \frac{n}{k} \right) (\lambda + \epsilon)^{1-k} e^{(\lambda+\epsilon)(t+h)}h
\leq \left[ M + \lambda M_1 + M_1 \sum_{k=1}^{n} \lambda^k \left( \frac{n}{k} \right) (\lambda + \epsilon)^{1-k} \right] e^{(\lambda+\epsilon)(t+h)}h. \]

Lemma 3.4. Let $n = 1$ or 2. Suppose $A$ generates an $n$-times integrated semigroup $(S(t))_{t \geq 0}$ satisfying
\[ \|S(t+h) - S(t)\| \leq Me^{w(t+h)}h \quad \forall t, h \geq 0, \]
where $M, w$ are constants. Then for every $\lambda > w$, $A - \lambda$ generates an $n$-times integrated semigroup $(\tilde{S}(t))_{t \geq 0}$ satisfying
\[ \|\tilde{S}(t+h) - \tilde{S}(t)\| \leq \tilde{M} h \quad \forall t, h \geq 0, \]
where $\tilde{M}$ is a constant.

Proof. Let $\lambda > w$. Take $\epsilon > 0$ such that $\lambda > w + \epsilon$. It follows from the condition on $(S(t))_{t \geq 0}$ that there exists $M_1 > 0$ such that $\|S(t)\| \leq M_1 e^{(w+\epsilon)t}$ for all $t \geq 0$. So for every $t, h \geq 0$, we have
\[
\|\tilde{S}(t+h) - \tilde{S}(t)\| \leq \|e^{-\lambda(t+h)}S(t+h) - e^{-\lambda t}S(t)\| + 2\lambda \int_t^{t+h} e^{-\lambda r} \|S(r)\| \, dr
+ \lambda^2 \int_t^{t+h} \int_0^{s} e^{-\lambda r} \|S(r)\| \, dr \, ds
\leq e^{-\lambda(t+h)}\|S(t+h) - S(t)\| + |e^{-\lambda(t+h)} - e^{-\lambda t}|M_1 e^{(w+\epsilon)t}
+ 2\lambda \int_t^{t+h} M_1 e^{(w+\epsilon-\lambda)r} \, dr + \lambda^2 \int_t^{t+h} \int_0^{s} M_1 e^{(w+\epsilon-\lambda)r} \, dr \, ds
\leq (e^{(w-\lambda)(t+h)}Mh + \lambda e^{-\lambda h}M_1 e^{(w+\epsilon)t}) + 2\lambda M_1 h + \lambda^2 M_1 (\lambda - w - \epsilon)^{-1}h
\leq [M + 3\lambda M_1 + \lambda^2 M_1 (\lambda - w - \epsilon)^{-1}]h. \]


Lemma 3.5. Let $E$ be a Banach space and let $w, M \geq 0$. Suppose $F : [0, \infty) \to E$ satisfies $\limsup_{t \to 0^+} h^{-1} \|F(t + h) - F(t)\| \leq Me^{wh}$ for all $t \geq 0$. Then

\[ \|F(t + h) - F(t)\| \leq Me^{w(t+h)}h \quad \text{for all } t, h \geq 0. \]

Proof. It suffices to prove the result for the case where $E = \mathbb{R}$. First, we note that $F$ is Lipschitz continuous on every bounded interval in $[0, \infty)$. Indeed, for every $\eta > 0$, take $M_1 > Me^{\eta}$; then we have $\limsup_{t \to 0^+} h^{-1} \|F(t + h) - F(t)\| < M_1$ for all $t \in [0, \eta)$. From this it follows that $\|F(t + h) - F(t)\| \leq M_1 h$ whenever $0 \leq t < t + h \leq \eta$.

Next, since $F$ is absolutely continuous on bounded intervals in $[0, \infty)$,

\[ \int_0^t F'(s) \, ds = F(t) - F(0) \quad \text{for all } t \geq 0. \]

Hence for $t, h \geq 0$,

\[ |F(t + h) - F(t)| = | \int_t^{t+h} F'(s) \, ds | \leq \int_t^{t+h} M e^{ws} \, ds \leq M e^{w(t+h)}h. \]

\[ \square \]

Theorem 3.6. Let $E$ be a Banach space and let $A : \mathcal{D}(A) \subseteq E \to E$ be a linear operator.

1. Let $n \in \mathbb{N} \cup \{0\}$. Suppose there exists $w \geq 0$ such that $(w, \infty) \subset \rho(A)$ and the function $F : (0, \infty) \to \mathcal{B}(E)$ defined by

\[ F(\lambda) = \frac{1}{\lambda^n} R(w + \lambda, A) \]

satisfies condition $(P_\infty)$. Then $A$ generates an $(n+1)$-times integrated semigroup $(S(t))_{t \geq 0}$ with the property that, given any $w_1 > w$, there exists $M_1 > 0$ such that

\[ \limsup_{h \to 0^+} h^{-1} \|S(t + h) - S(t)\| \leq M_1 e^{w_1 t} \quad \forall t \geq 0. \]

2. Let $n = 0$ or $1$. Suppose $A$ generates an $(n+1)$-times integrated semigroup $(S(t))_{t \geq 0}$ satisfying

\[ \limsup_{h \to 0^+} h^{-1} \|S(t + h) - S(t)\| \leq M_1 e^{w_1 t} \quad \forall t \geq 0, \]

where $M_1, w_1$ are constants. Then $(w_1, \infty) \subset \rho(A)$ and for every $w > w_1$, the function $F_w : (0, \infty) \to \mathcal{B}(E)$ defined by

\[ F_w(\lambda) = \frac{1}{\lambda^n} R(w + \lambda, A) \]

satisfies condition $(P_\infty)$.

Proof. (1) By Theorem 2.3, there exists a constant $M > 0$ and a function $T : [0, \infty) \to \mathcal{B}(E)$ satisfying $T(0) = 0$ and $\|T(t + h) - T(t)\| \leq Mh$ for all $t, h \geq 0$ such that for all $x \in E$,

\[ R(\lambda, A - w)x = \int_0^\infty \lambda^{n+1} e^{-\lambda t} T(t)x \, dt \quad \forall \lambda > 0. \]

By [1, Theorem 3.1], $(T(t))_{t \geq 0}$ is an $(n+1)$-times integrated semigroup with generator $A - w$. Hence by Lemma 3.3, $A$ generates an $(n+1)$-times integrated semigroup $(S(t))_{t \geq 0}$ with the required property.
Remark 3.2. The second assertion in the above theorem does not hold if \( n \geq 2 \). For example, in \( \mathbb{R} \) or \( \mathbb{C} \), \( A = -1 \) generates a 3-times integrated semigroup \( (S(t))_{t \geq 0} \) satisfying \( \limsup_{t \to 0} h^{-1} \|S(t+h) - S(t)\| \leq 2e^t \) for all \( t \geq 0 \). However, \( F_w(\lambda) = \frac{1}{\lambda^w} \) does not satisfy condition \( (P_\infty) \) for any \( w \).

Corollary 3.7. Let \( A : D(A) \subset E \longrightarrow E \) be closed and densely defined and let \( n \in \mathbb{N} \cup \{0\} \). If there exists \( w > 0 \) such that \( (w, \infty) \subset \rho(A) \) and the function \( F : (0, \infty) \longrightarrow \mathbb{B}(E) \) defined by

\[
F(\lambda) = \frac{1}{\lambda^n} R(w + \lambda, A)
\]

satisfies condition \( (P_\infty) \), then \( A \) generates an \( n \)-times integrated semigroup. The converse is true for \( n = 0, 1 \).

Proof. If \( A \) satisfies the given condition, then by Theorem 3.6, \( A \) generates a locally Lipschitz \((n+1)\)-times integrated semigroup. Hence by [1, Corollary 4.2], \( A \) generates an \( n \)-times integrated semigroup.

Conversely, for \( n = 0 \) or \( 1 \), if \( A \) generates an \( n \)-times integrated semigroup \((S(t))_{t \geq 0}\) with \( \|S(t)\| \leq Me^{w_1 t} \) for all \( t \geq 0 \), where \( M, w_1 \) are constants, then \( A \) generates an \((n+1)\)-times integrated semigroup \((S(t) = \int_0^t S(r) dr)_{t \geq 0}\) (in the strong operator topology) satisfying \( \|S(t+h) - S(t)\| \leq Me^{w_1 (t+h)} h \) for all \( t, h \geq 0 \). Hence the required result follows from Theorem 3.6.

To close our discussion, we give the following example studied in [6].

Example 3.1. Let \( E = L^1(0, R) \times L^1(0, R) \), where \( R \) is a positive constant (larger than the life span of human beings). Let \( A : D(A) \subset E \longrightarrow E \) be given by

\[
A \varphi = (-\varphi_1 - (\mu + \delta) \varphi_1 + \sigma \varphi_2, -\varphi_2 - (\bar{\mu} + \sigma) \varphi_2 + \sigma \varphi_1),
\]

where \( D(A) \) consists of all \( \varphi = (\varphi_1, \varphi_2) \in E \) with \( \varphi_1, \varphi_2 \) absolutely continuous and satisfying

\[
\varphi_1(0) = \beta \int_0^R h(r)k(r) \varphi_1(r) dr + \tilde{\beta} \int_0^R h(r) \tilde{k}(r) \varphi_2(r) dr,
\]

\[
\varphi_2(0) = \alpha \int_0^R h(r)k(r) \varphi_1(r) dr + \tilde{\alpha} \int_0^R h(r) \tilde{k}(r) \varphi_2(r) dr,
\]

and \( \mu, \bar{\mu}, \sigma, \delta, k, \tilde{k}, h, \tilde{h} \) are nonnegative measurable functions on \([0, R]\) \((\mu, \bar{\mu} \) are the age specific mortality moduli of normal and disabled people; \( 0 \leq \sigma(r), \delta(r) \leq 1 \) represent the recovery rate and disabled rate at age \( r \); \( 0 < k(r), \tilde{k}(r) < 1 \) represent the proportion of the female population and that of the female disabled population of age \( r \); \( h, \tilde{h} \) with \( L^1 \)-norm equal to 1 are the birth modes of females and disabled females respectively) and \( \alpha, \tilde{\alpha}, \beta, \bar{\beta} \) are constants (which, in fact, depend on government population policy). Then \( A \) satisfies the conditions given in Corollary 3.7 for \( n = 0 \) (for details, see [6]) and thus generates a \( C_0 \)-semigroup.
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