A DECREASING OPERATOR FUNCTION
ASSOCIATED WITH THE FURUTA INEQUALITY

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Abstract. Let $A \geq B \geq 0$ with $A > 0$ and let $t \in [0, 1]$ and $q \geq 0$. As a generalization of a result due to Furuta, it is shown that the operator function

$$G_{p,q,t}(A, B, r, s) = A^{-r/2} \left\{ A^{r/2} B^{p} A^{-t/2} B^{s} A^{r/2} \right\}^{(q-t+r)/(p-t+s+r)} A^{-r/2}$$

is decreasing for $r \geq t$ and $s \geq 1$ if $p \geq \max\{q, t\}$. Moreover, if $1 \geq p > t$ and $q \geq t$, then $G_{p,q,t}(A, B, r, s)$ is decreasing for $r \geq 0$ and $s \geq \frac{p-t}{p-r}$. The latter result is an extension of an earlier result of Furuta.

1. Introduction

In what follows, a capital letter means a bounded linear operator on a complex Hilbert space $H$. An operator $T$ is said to be positive (in symbols: $T \geq 0$) if $(Tx, x) \geq 0$ for all $x \in H$. Also an operator $T$ is strictly positive (in symbols: $T > 0$) if $T$ is positive and invertible. As an extension of the celebrated Löwner-Heinz theorem ([13], [15]), we established the following result.

Furuta inequality ([5]). If $A \geq B \geq 0$, then for each $r \geq 0$

(i) $$(B^{r} A^{p} B^{r})^{1/q} \geq (B^{r} A^{p} B^{r})^{1/q}$$

and

(ii) $$(A^{r} A^{p} A^{r})^{1/q} \geq (A^{r} A^{p} A^{r})^{1/q}$$

hold for $p \geq 0$ and $q \geq 1$ with $(1 + 2r)q \geq p + 2r$.

Alternative proofs are given in [2], [6] and [14] and also an elementary one page proof in [7]. The domain surrounded by $p, q$ and $r$ in the Figure is the best possible one for the Furuta inequality in [16]. We remark that the Furuta inequality yields the Löwner-Heinz theorem when we put $r = 0$ in (i) or (ii) stated above: if $A \geq B \geq 0$ ensures $A^{\alpha} \geq B^{\alpha}$ for any $\alpha \in [0, 1]$. In [10, Theorem 1.1] we established the following Theorem A as extensions of the Furuta inequality.

Theorem A ([10]). If $A \geq B \geq 0$ with $A > 0$, then for each $t \in [0, 1]$ and $p \geq 1$,

$$F_{p,t}(A, B, r, s) = A^{-r/2} \left\{ A^{r/2} B^{p} A^{-t/2} B^{s} A^{r/2} \right\}^{(1-t+r)/(p-t+s+r)} A^{-r/2}$$

is decreasing for $r \geq t$ and $s \geq 1$ and $F_{p,t}(A, A, r, s) \geq F_{p,t}(A, B, r, s)$, that is, for

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each \( t \in [0, 1] \) and \( p \geq 1 \),
\[
(1.1) \quad A^{1-t+r} \geq \left\{ A^{r/2}(A^{-t/2} B^p A^{-t/2})^s A^{r/2}\right\}^{(1-t+r)/[(p-t)s+r]} A^{-r/2}
\]
holds for any \( s \geq 1 \) and \( r \) such that \( r \geq t \).

Recently a nice mean-theoretic proof of Theorem A was shown in [4]. Ando-Hiai [1] established various log majorization results to ensure excellent and useful inequalities for unitarily invariant norms and showed the following useful inequality equivalent to the main log majorization theorem: If \( A \geq B \geq 0 \) with \( A \succ 0 \), then
\[
A^r \geq \left\{ A^{r/2}(A^{-1/2} B^p A^{-1/2})^r A^{r/2}\right\}^{1/p}
\]
holds for any \( p \geq 1 \) and \( r \geq 1 \).

Theorem A interpolates this inequality by Ando-Hiai and the Furuta inequality and also extends results of [3], [8] and [9].

We write \( A \gg B \) if \( \log A \geq \log B \) for invertible positive operators \( A \) and \( B \) which is called the chaotic order [3] and related results on chaotic order are discussed in [3].

2. Statement of results

The main result in this paper is the following one which is an extension of Theorem A.

**Theorem 1.** Let \( A \geq B \geq 0 \) with \( A \succ 0 \). For each \( t \in [0, 1] \), \( q \geq 0 \) and \( p \geq \max\{q, t\} \),
\[
G_{p,q,t}(A, B, r, s) = A^{-r/2}\left\{ A^{r/2}(A^{-t/2} B^p A^{-t/2})^s A^{r/2}\right\}^{(q-t+r)/[(p-t)s+r]} A^{-r/2}
\]
is decreasing for \( r \geq t \) and \( s \geq 1 \).

The following result is a complementary one associated with Theorem A.

**Theorem 2.** Let \( A \geq B \geq 0 \) with \( A \succ 0 \). For each \( t \in [0, 1] \), \( 1 \geq p \geq t \) and \( q \geq t \),
\[
G_{p,q,t}(A, B, r, s) = A^{-r/2}\left\{ A^{r/2}(A^{-t/2} B^p A^{-t/2})^s A^{r/2}\right\}^{(q-t+r)/[(p-t)s+r]} A^{-r/2}
\]
is decreasing for \( r \geq 0 \) and \( s \geq \frac{q-t}{p-t} \).

The following Theorem A’ is nothing but a slight modification of Theorem A, but Theorem A’ is useful for the proof of Theorem 1.
Theorem A'. If $A \geq B \geq 0$ with $A > 0$, then for each $t \in [0,1]$, $q \in [t,1]$ and $p \geq q$,
\[ F_{p,t}(A, B, r, s) = A^{-r/2} \left\{ A^{r/2} (A^{-t/2} B_p A^{-t/2}) s A^{r/2} \right\} (q-t+r)/[(p-t)s+r] A^{-r/2} \]
is decreasing for $r \geq t$ and $s \geq 1$. Moreover for each $t \in [0,1]$, $q \in [t,1]$ and $p \geq q$, $F_{p,t}(A, A, r, s) \geq F_{p,t}(A, B, r, s)$, that is,
\[ A^{q-t+r} \geq \left\{ A^{r/2} (A^{-t/2} B_p A^{-t/2}) s A^{r/2} \right\} (q-t+r)/[(p-t)s+r] \]
holds for any $s \geq 1$ and $t \geq r$.

Corollary 3 ([3], [8], [9]). If $A \geq B \geq 0$ with $A > 0$, then for fixed $q \geq 0$,
\[ F(p, r) = A^{-r/2} (A^{-r/2} B_p A^{r/2}) (q-t+r)/[(p-t)s+r] A^{-r/2} \]
is decreasing for $r \geq 0$ and $p \geq q \geq 0$.

Corollary 4 ([11]). Let $A \geq B \geq 0$ with $A > 0$.
(i) For each $p$ and $t$ such that $1 \geq p \geq 1/2$ and $p > t \geq 0$,
\[ F_{p,t}(A, B, r, s) = A^{-r/2} \left\{ A^{r/2} (A^{-t/2} B_p A^{-t/2}) s A^{r/2} \right\} (1-t+r)/[(p-t)s+r] A^{-r/2} \]
is decreasing for $r \geq 0$ and $s \geq 1-t/p$.
(ii) For each $p$ and $t$ such that $1 \geq 1/2$ and $p \geq t \geq 0$,
\[ H_{p,t}(A, B, r, s) = A^{-r/2} \left\{ A^{r/2} (A^{-t/2} B_p A^{-t/2}) s A^{r/2} \right\} (2p-t+r)/[(p-t)s+r] A^{-r/2} \]
is decreasing for $r \geq 0$ and $s \geq 2p-t$.

3. Proofs of the results

Proof of Theorem A'. Put $A_1 = A^q$ and $B_1 = B^q$. Then $A_1 \geq B_1$ by the Löwner-Heinz theorem since $q \in [0,1]$ holds. Put $r' = r/q$, $t' = t/q$ and $p' = p/q$. Then $p' \geq 1 \geq t'$ and $r' \geq t'$ hold by the hypotheses on $p, q, t$ and $r$. Applying Theorem A on $A_1$ and $B_1$,
\[ G_{p,q,t}(A, B, r, s) = A_1^{-r/2} \left\{ A_1^{r/2} (A_1^{-t/2} B_p A_1^{-t/2}) s A_1^{r/2} \right\} (q-t+r)/[(p-t)s+r] A_1^{-r/2} \]
\[ = A_1^{-r/2} \left\{ A_1^{r/2} (A_1^{-t/2} B_1 A_1^{-t/2}) s A_1^{r/2} \right\} (1-t'+r')/[(p'-t')s+r'] A_1^{-r'/2} \]
\[ = F_{p',t'}(A_1, B_1, r', s') \]
whence $G_{p,q,t}(A, B, r, s)$ is decreasing for $r \geq t$ and $s \geq 1$ since $F_{p',t'}(A_1, B_1, r', s')$ is decreasing for $r' \geq t'$ and $s \geq 1$ by Theorem A. Moreover (2.1) is nothing but a modification of (1.1) on $A_1$ and $B_1$. Whence the proof of Theorem A' is complete.

Lemma 1. (i) Let $A > 0$ and $B$ be an invertible operator. For any real number $\lambda$
\[ (BAB^*)^\lambda = BA^{1/2} (A^{1/2} B B A^{1/2})^\lambda \]
\[ = BA^{1/2} (A^{1/2} B B A^{1/2})^\lambda \]
is increasing. (ii) If $A \geq 0$ and $\|Y\| \leq 1$, then $(Y^* AY)^\alpha \geq Y^* A^\alpha Y$ holds for any $\alpha \in [0,1]$.
(iii) If $A^* \geq B^*$ for $A, B > 0$ and some $s > 0$, then $A \gg B$.

(i) of Lemma 1 is shown in [10], (ii) is a simple corollary of [12] and (iii) is easily obtained since $\log t$ is an operator monotone function.
Lemma 2 ([3], [9]). $A \gg B$ holds if and only if for any fixed $q \geq 0,$
\[ H_q(A, B, \beta, \alpha) = A^{-\beta/2}(A^{\beta/2}B^\alpha A^{\beta/2})^{(q+\beta)/(\alpha+\beta)} A^{-\beta/2} \]
is decreasing for $\beta \geq 0$ and $\alpha \geq q.$

Proposition 1. Let $A \geq B \geq 0$ with $A > 0.$ For each $t \in [0, 1],$ $q \geq 0,$ $p \geq \max\{q, t\}$ and $s \geq 1,$
\[ G_{p,q,t}(A, B, r, s) = A^{-r/2}\{A^{r/2}(A^{-t/2}B^p A^{-t/2})^s A^{r/2}\}^{(q-t+r)/[(p-t)s+r]} A^{-r/2} \]
is decreasing for $r \geq t.$

Proof. Putting $q = t = r$ in (2.1) of Theorem A', we have
\[ (3.1) \quad A^t \geq \{A^{t/2}(A^{-t/2}B^p A^{-t/2})^s A^{r/2}\}^{(q-t+r)/[(p-t)s+r]} A^{-r/2} \]
for any $p \geq t$ and $s \geq 1.$

Giving $p, t$ and $s$ satisfying the hypotheses in Proposition 1, put $\alpha_0 = (p-t)s+t$ and $M = \{A^{t/2}(A^{-t/2}B^p A^{-t/2})^s A^{r/2}\}^{1/\alpha_0}.$ Therefore (3.1) and (iii) in Lemma 1 ensure $A \gg M$ and the hypotheses on $p, q, t$ and $s$ imply
\[ (\alpha_0 - q = (p-t)s - (q-t) \geq (p-t)(s-1) \geq 0, \]
so that $\alpha_0 \geq q$ holds. Put $\beta = r - t \geq 0.$ Then applying Lemma 2,
\[ G_{p,q,t}(A, B, r, s) = A^{-r/2}\{A^{r/2}(A^{-t/2}B^p A^{-t/2})^s A^{r/2}\}^{(q-t+r)/[(p-t)s+r]} A^{-r/2} \]
\[ = A^{-r/2}\{A^{r/2}(A^{-t/2}B^p A^{-t/2})^s A^{r/2}\}^{(q-t+r)/[(p-t)s+r]} A^{-r/2} \]
\[ = A^{-t/2}H_q(A, M, \beta, \alpha_0)A^{-t/2}, \]
so that $G_{p,q,t}(A, B, r, s)$ is decreasing for $r \geq t$ since $H_q(A, M, \beta, \alpha_0)$ is decreasing for $\beta \geq 0$ by Lemma 2 and $\beta = r - t.$ Whence the proof is complete.

Proposition 2. Let $A \geq B \geq 0$ with $B > 0.$ For each $t \in [0, 1]$ and $p, r \geq t,$
\[ A^{r/2}(A^{-t/2}B^p A^{-t/2})^{s_1} A^{r/2} \geq \{A^{r/2}(A^{-t/2}B^p A^{-t/2})^{s_2} A^{r/2}\}^{[(p-t)s_1+r]/[(p-t)s_2+r]} \]
holds for $s_2 \geq s_1 \geq 1.$

Proof. There is no loss of generality in assuming that $B > 0.$ Put $q = t$ in (2.1) of Theorem A'. Then we have for $t \in [0, 1]$
\[ (3.2) \quad A^t \geq \{A^{t/2}(A^{-t/2}B^p A^{-t/2})^{s_2} A^{r/2}\}^{r/[(p-t)s_2+r]} \quad \text{for} \quad s_2 \geq 1, \quad r \geq t \quad \text{and} \quad p \geq t. \]

Put $Z^2 = \{A^{r/2}(A^{-t/2}B^p A^{-t/2})^{s_2} A^{r/2}\}^{r/[(p-t)s_2+r]}$ and also put $Y = ZA^{-r/2}.$ Then $A^{r/2}Z^{-1}Y = I$ and $\|Y\| \leq 1$ since $A^t \geq Z^2$ holds by (3.2). Therefore we obtain
\[ (3.3) \quad A^{r/2}(A^{-t/2}B^p A^{-t/2})^{s_1} A^{r/2} \]
\[ = A^{r/2}\{Y^* Z^{-1}A^{r/2}(A^{-t/2}B^p A^{-t/2})^{s_2} A^{r/2} Z^{-1}Y\}^{s_1/s_2} A^{r/2} \]
\[ \geq A^{r/2}Y^* \{Z^{-1}A^{r/2}(A^{-t/2}B^p A^{-t/2})^{s_2} A^{r/2} Z^{-1}\}^{s_1/s_2} YA^{r/2} \]
by (ii) of Lemma 1
\[ = Z\{A^{r/2}(A^{-t/2}B^p A^{-t/2})^{s_2} A^{r/2}\}^{(p-t)s_1/[(p-t)s_2+r]} Z \]
\[ = \{A^{r/2}(A^{-t/2}B^p A^{-t/2})^{s_2} A^{r/2}\}^{[(p-t)s_1+r]/[(p-t)s_2+r]}, \]
so the proof is complete.
1. We have only to put $\beta$ for $r \geq 0$. For each $t \in [0,1]$, $q \geq 0$, $p \geq \max\{q,t\}$ and $r \geq t$,

$$G_{p,q,t}(A, B, r, s) = A^{-r/2} \{ A^{r/2}(A^{-t/2}B^pA^{-t/2})^s A^{r/2}\}^{(q-t+r)/[(p-t)+r]}A^{-r/2}$$

is decreasing for $s \geq 1$.

**Proof.** Raise each side of (3.3) to the power $\frac{q-t+r}{(p-t)+r} \in [0,1]$. By the Löwner-Heinz theorem this yields

$$\{ A^{r/2}(A^{-t/2}B^pA^{-t/2})^s A^{r/2}\}^{(q-t+r)/[(p-t)+r]}$$

for $s_2 \geq s_1 \geq 1$, so the proof is complete.

**Proof of Theorem 1.** The proof follows from Proposition 1 and Proposition 3.

**Proof of Theorem 2.** Put $M = A^{-t/2}B^pA^{-t/2}$, $N = A^{p-t}$, $\alpha = \frac{q-t}{p-t}$ and $\beta = \frac{r}{p-t}$. Then

$$M = A^{-t/2}B^pA^{-t/2} \leq A^{-t/2}A^pA^{-t/2} = A^{p-t} = N$$

holds by the Löwner-Heinz theorem. Therefore applying Lemma 2,

$$G_{p,q,t}(A, B, r, s) = A^{-r/2} \{ A^{r/2}(A^{-t/2}B^pA^{-t/2})^s A^{r/2}\}^{(q-t+r)/[(p-t)+r]}A^{-r/2}$$

$$= N^{-\beta/2}(N^{\beta/2}M^{\beta/2})^{(\alpha+\beta)/(s+\beta)}N^{-\beta/2}$$

$$= H_{a}(N, M, \beta, s).$$

$G_{p,q,t}(A, B, r, s)$ is decreasing for $r \geq 0$ and $s \geq \frac{q-t}{p-t}$ since $H_{a}(N, M, \beta, s)$ is decreasing for $\beta \geq 0$ and $s \geq \alpha$ by Lemma 2.

**Proof of Corollary 3.** We have only to put $t = 0$ and replace $ps$ by $p$ in Theorem 1.

**Proof of Corollary 4.** Theorem 2 easily implies (i) and (ii).

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