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A DECREASING OPERATOR FUNCTION ASSOCIATED WITH THE FURUTA INEQUALITY

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ABSTRACT. Let $A \geq B \geq 0$ with $A > 0$ and let $t \in [0,1]$ and $q \geq 0$. As a generalization of a result due to Furuta, it is shown that the operator function $G_{p,q,t}(A,B,r,s) = A^{-r/2}\{A^{r/2}(A^{-t/2}B^pA^{-t/2})^sA^{r/2}\}^{(q-t+r)/[(p-t)s+r]}A^{-r/2}$ is decreasing for $r \geq t$ and $s \geq 1$ if $p \geq \max\{q,t\}$. Moreover, if $1 \geq p > t$ and $q \geq t$, then $G_{p,q,t}(A,B,r,s)$ is decreasing for $r \geq 0$ and $s \geq \frac{q-t}{p-t}$. The latter result is an extension of an earlier result of Furuta.

1. INTRODUCTION

In what follows, a capital letter means a bounded linear operator on a complex Hilbert space H. An operator T is said to be positive (in symbols: $T \geq 0$) if $(Tx,x) \geq 0$ for all $x \in H$. Also an operator T is strictly positive (in symbols: $T > 0$) if T is positive and invertible. As an extension of the celebrated Löwner-Heinz theorem ([13], [15]), we established the following result.

Furuta inequality ([5]). *If $A \geq B \geq 0$, then for each $r \geq 0$*

$$(B^rA^pB^r)^{1/q} \geq (B^rB^pB^r)^{1/q} \tag{i}$$

and

$$(A^rA^pA^r)^{1/q} \geq (A^rB^pA^r)^{1/q} \tag{ii}$$

hold for $p \geq 0$ and $q \geq 1$ with $(1+2r)q \geq p+2r$.

Alternative proofs are given in [2], [6] and [14] and also an elementary one page proof in [7]. The domain surrounded by p, q and r in the Figure is the best possible one for the Furuta inequality in [16]. We remark that the Furuta inequality yields the Löwner-Heinz theorem when we put $r = 0$ in (i) or (ii) stated above: if $A \geq B \geq 0$ ensures $A^\alpha \geq B^\alpha$ for any $\alpha \in [0,1]$. In [10, Theorem 1.1] we established the following Theorem A as extensions of the Furuta inequality.

Theorem A ([10]). *If $A \geq B \geq 0$ with $A > 0$, then for each $t \in [0,1]$ and $p \geq 1$,*

$$F_{p,t}(A,B,r,s) = A^{-r/2}\{A^{r/2}(A^{-t/2}B^pA^{-t/2})^sA^{r/2}\}^{(1-t+r)/[(p-t)s+r]}A^{-r/2}$$

is decreasing for $r \geq t$ and $s \geq 1$ and $F_{p,t}(A,A,r,s) \geq F_{p,t}(A,B,r,s)$, that is, for

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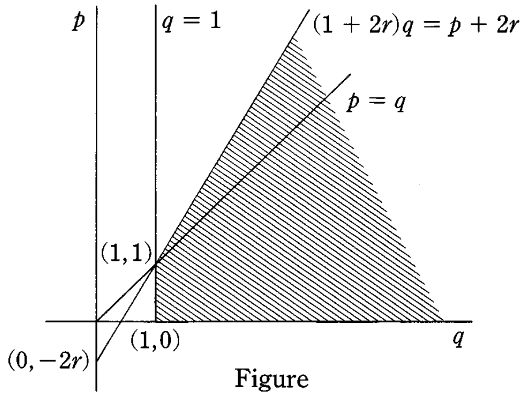

Figure

each $t \in [0,1]$ *and* $p \geq 1$,

$$A^{1-t+r} \geq \{A^{r/2}(A^{-t/2}B^pA^{-t/2})^sA^{r/2}\}^{(1-t+r)/[(p-t)s+r]} \tag{1.1}$$

holds for any $s \geq 1$ *and* r *such that* $r \geq t$.

Recently a nice mean-theoretic proof of Theorem A was shown in [4]. Ando-Hiai [1] established various log majorization results to ensure excellent and useful inequalities for unitarily invariant norms and showed the following useful inequality equivalent to the main log majorization theorem: If $A \geq B \geq 0$ with $A > 0$, then

$$A^r \geq \{A^{r/2}(A^{-1/2}B^pA^{-1/2})^rA^{r/2}\}^{1/p}$$

holds for any $p \geq 1$ and $r \geq 1$.

Theorem A interpolates this inequality by Ando-Hiai and the Furuta inequality and also extends results of [3], [8] and [9].

We write $A \gg B$ if $\log A \geq \log B$ for invertible positive operators A and B which is called the chaotic order [3] and related results on chaotic order are discussed in [3].

2. Statement of results

The main result in this paper is the following one which is an extension of Theorem A.

Theorem 1. *Let* $A \geq B \geq 0$ *with* $A > 0$. *For each* $t \in [0,1]$, $q \geq 0$ *and* $p \geq \max\{q,t\}$,

$$G_{p,q,t}(A,B,r,s) = A^{-r/2}\{A^{r/2}(A^{-t/2}B^pA^{-t/2})^sA^{r/2}\}^{(q-t+r)/[(p-t)s+r]}A^{-r/2}$$

is decreasing for $r \geq t$ *and* $s \geq 1$.

The following result is a complementary one associated with Theorem A.

Theorem 2. *Let* $A \geq B \geq 0$ *with* $A > 0$. *For each* $t \in [0,1]$, $1 \geq p \geq t$ *and* $q \geq t$,

$$G_{p,q,t}(A,B,r,s) = A^{-r/2}\{A^{r/2}(A^{-t/2}B^pA^{-t/2})^sA^{r/2}\}^{(q-t+r)/[(p-t)s+r]}A^{-r/2}$$

is decreasing for $r \geq 0$ *and* $s \geq \frac{q-t}{p-t}$.

The following Theorem A′ is nothing but a slight modification of Theorem A, but Theorem A′ is useful for the proof of Theorem 1.

Theorem A′. *If $A \geq B \geq 0$ with $A > 0$, then for each $t \in [0,1]$, $q \in [t,1]$ and $p \geq q$,*

$$F_{p,t}(A,B,r,s) = A^{-r/2}\{A^{r/2}(A^{-t/2}B^pA^{-t/2})^sA^{r/2}\}^{(q-t+r)/[(p-t)s+r]}A^{-r/2}$$

is decreasing for $r \geq t$ and $s \geq 1$. Moreover for each $t \in [0,1]$, $q \in [t,1]$ and $p \geq q$, $F_{p,t}(A,A,r,s) \geq F_{p,t}(A,B,r,s)$, that is,

$$A^{q-t+r} \geq \{A^{r/2}(A^{-t/2}B^pA^{-t/2})^sA^{r/2}\}^{(q-t+r)/[(p-t)s+r]} \tag{2.1}$$

holds for any $s \geq 1$ and $r \geq t$.

Corollary 3 ([3], [8], [9])**.** *If $A \geq B \geq 0$ with $A > 0$, then for fixed $q \geq 0$,*

$$F(p,r) = A^{-r/2}(A^{-r/2}B^pA^{r/2})^{(q+r)/(p+r)}A^{-r/2}$$

is decreasing for $r \geq 0$ and $p \geq q \geq 0$.

Corollary 4 ([11])**.** *Let $A \geq B \geq 0$ with $A > 0$.*

(i) *For each p and t such that $1 \geq p \geq \frac{1}{2}$ and $p > t \geq 0$,*

$$F_{p,t}(A,B,r,s) = A^{-r/2}\{A^{r/2}(A^{-t/2}B^pA^{-t/2})^sA^{r/2}\}^{(1-t+r)/[(p-t)s+r]}A^{-r/2}$$

is decreasing for $r \geq 0$ and $s \geq \frac{1-t}{p-t}$.

(ii) *For each p and t such that $\frac{1}{2} \geq p > t \geq 0$,*

$$H_{p,t}(A,B,r,s) = A^{-r/2}\{A^{r/2}(A^{-t/2}B^pA^{-t/2})^sA^{r/2}\}^{(2p-t+r)/[(p-t)s+r]}A^{-r/2}$$

is decreasing for $r \geq 0$ and $s \geq \frac{2p-t}{p-t}$.

3. Proofs of the results

Proof of Theorem A′. Put $A_1 = A^q$ and $B_1 = B^q$. Then $A_1 \geq B_1$ by the Löwner-Heinz theorem since $q \in [0,1]$ holds. Put $r' = \frac{r}{q}$, $t' = \frac{t}{q}$ and $p' = \frac{p}{q}$. Then $p' \geq 1 \geq t'$ and $r' \geq t'$ hold by the hypotheses on p, q, t and r. Applying Theorem A on A_1 and B_1,

$$\begin{aligned} G_{p,q,t}(A,B,r,s) &= A^{-r/2}\{A^{r/2}(A^{-t/2}B^pA^{-t/2})^sA^{r/2}\}^{(q-t+r)/[(p-t)s+r]}A^{-r/2} \\ &= A_1^{-r'/2}\{A_1^{r'/2}(A_1^{-t'/2}B_1^{p'}A_1^{-t'/2})^sA_1^{r'/2}\}^{(1-t'+r')/[(p'-t')s+r']}A_1^{-r'/2} \\ &= F_{p',t'}(A_1,B_1,r',s') \end{aligned}$$

whence $G_{p,q,t}(A,B,r,s)$ is decreasing for $r \geq t$ and $s \geq 1$ since $F_{p',t'}(A_1,B_1,r',s')$ is decreasing for $r' \geq t'$ and $s \geq 1$ by Theorem A. Moreover (2.1) is nothing but a modification of (1.1) on A_1 and B_1. Whence the proof of Theorem A′ is complete.

Lemma 1. (i) *Let $A > 0$ and B be an invertible operator. For any real number λ*

$$(BAB^*)^\lambda = BA^{1/2}(A^{1/2}B^*BA^{1/2})^{\lambda-1}A^{1/2}B^*.$$

(ii) *If $A \geq 0$ and $\|Y\| \leq 1$, then $(Y^*AY)^\alpha \geq Y^*A^\alpha Y$ holds for any $\alpha \in [0,1]$.*

(iii) *If $A^s \geq B^s$ for $A, B > 0$ and some $s > 0$, then $A \gg B$.*

(i) of Lemma 1 is shown in [10], (ii) is a simple corollary of [12] and (iii) is easily obtained since $\log t$ is an operator monotone function.

Lemma 2 ([3], [9])**.** *$A \gg B$ holds if and only if for any fixed $q \geq 0$,*
$$H_q(A,B,\beta,\alpha) = A^{-\beta/2}(A^{\beta/2}B^{\alpha}A^{\beta/2})^{(q+\beta)/(\alpha+\beta)}A^{-\beta/2}$$
is decreasing for $\beta \geq 0$ and $\alpha \geq q$.

Proposition 1. *Let $A \geq B \geq 0$ with $A > 0$. For each $t \in [0,1]$, $q \geq 0$, $p \geq \max\{q,t\}$ and $s \geq 1$,*
$$G_{p,q,t}(A,B,r,s) = A^{-r/2}\{A^{r/2}(A^{-t/2}B^pA^{-t/2})^sA^{r/2}\}^{(q-t+r)/[(p-t)s+r]}A^{-r/2}$$
is decreasing for $r \geq t$.

Proof. Putting $q = t = r$ in (2.1) of Theorem A′, we have

$$A^t \geq \{A^{t/2}(A^{-t/2}B^pA^{-t/2})^sA^{t/2}\}^{t/[(p-t)s+t]} \quad \text{for any } p \geq t \text{ and } s \geq 1. \tag{3.1}$$

Giving p, t and s satisfying the hypotheses in Proposition 1, put $\alpha_0 = (p-t)s+t$ and $M = \{A^{t/2}(A^{-t/2}B^pA^{-t/2})^sA^{t/2}\}^{1/\alpha_0}$. Therefore (3.1) and (iii) in Lemma 1 ensure $A \gg M$ and the hypotheses on p, q, t and s imply
$$\alpha_0 - q = (p-t)s - (q-t) \geq (p-t)(s-1) \geq 0,$$
so that $\alpha_0 \geq q$ holds. Put $\beta = r - t \geq 0$. Then applying Lemma 2,
$$\begin{aligned}
G_{p,q,t}(A,B,r,s) &= A^{-r/2}\{A^{r/2}(A^{-t/2}B^pA^{-t/2})^sA^{r/2}\}^{(q-t+r)/[(p-t)s+r]}A^{-r/2}\\
&= A^{-r/2}\{A^{(r-t)/2}A^{t/2}(A^{-t/2}B^pA^{-t/2})^sA^{t/2}A^{(r-t)/2}\}^{(q-t+r)/[(p-t)s+r]}A^{-r/2}\\
&= A^{-t/2}A^{-\beta/2}\{A^{\beta/2}M^{\alpha_0}A^{\beta/2}\}^{(q+\beta)/(\alpha_0+\beta)}A^{-\beta/2}A^{-t/2}\\
&= A^{-t/2}H_q(A,M,\beta,\alpha_0)A^{-t/2},
\end{aligned}$$
so that $G_{p,q,t}(A,B,r,s)$ is decreasing for $r \geq t$ since $H_q(A,M,\beta,\alpha_0)$ is decreasing for $\beta \geq 0$ by Lemma 2 and $\beta = r - t$. Whence the proof is complete.

Proposition 2. *Let $A \geq B \geq 0$ with $A > 0$. For each $t \in [0,1]$ and $p, r \geq t$,*
$$A^{r/2}(A^{-t/2}B^pA^{-t/2})^{s_1}A^{r/2} \geq \{A^{r/2}(A^{-t/2}B^pA^{-t/2})^{s_2}A^{r/2}\}^{[(p-t)s_1+r]/[(p-t)s_2+r]}$$
holds for $s_2 \geq s_1 \geq 1$.

Proof. There is no loss of generality in assuming that $B > 0$. Put $q = t$ in (2.1) of Theorem A′. Then we have for $t \in [0,1]$

$$A^r \geq \{A^{r/2}(A^{-t/2}B^pA^{-t/2})^{s_2}A^{r/2}\}^{r/[(p-t)s_2+r]} \quad \text{for } s_2 \geq 1,\ r \geq t \quad \text{and} \quad p \geq t. \tag{3.2}$$

Put $Z^2 = \{A^{r/2}(A^{-t/2}B^pA^{-t/2})^{s_2}A^{r/2}\}^{r/[(p-t)s_2+r]}$ and also put $Y = ZA^{-r/2}$. Then $A^{r/2}Z^{-1}Y = I$ and $\|Y\| \leq 1$ since $A^r \geq Z^2$ holds by (3.2). Therefore we obtain

$$\begin{aligned}
&A^{r/2}(A^{-t/2}B^pA^{-t/2})^{s_1}A^{r/2}\\
&= A^{r/2}\{Y^*Z^{-1}A^{r/2}(A^{-t/2}B^pA^{-t/2})^{s_2}A^{r/2}Z^{-1}Y\}^{s_1/s_2}A^{r/2}\\
&\geq A^{r/2}Y^*\{Z^{-1}A^{r/2}(A^{-t/2}B^pA^{-t/2})^{s_2}A^{r/2}Z^{-1}\}^{s_1/s_2}YA^{r/2} \quad \text{by (ii) of Lemma 1}\\
&= Z\{A^{r/2}(A^{-t/2}B^pA^{-t/2})^{s_2}A^{r/2}\}^{(p-t)s_1/[(p-t)s_2+r]}Z\\
&= \{A^{r/2}(A^{-t/2}B^pA^{-t/2})^{s_2}A^{r/2}\}^{[(p-t)s_1+r]/[(p-t)s_2+r]},
\end{aligned} \tag{3.3}$$
so the proof is complete.

Proposition 3. *Let $A \geq B \geq 0$ with $A > 0$. For each $t \in [0,1,]$, $q \geq 0$, $p \geq \max\{q,t\}$ and $r \geq t$,*

$$G_{p,q,t}(A,B,r,s) = A^{-r/2}\{A^{r/2}(A^{-t/2}B^pA^{-t/2})^sA^{r/2}\}^{(q-t+r)/[(p-t)s+r]}A^{-r/2}$$

is decreasing for $s \geq 1$.

Proof. Raise each side of (3.3) to the power $\frac{q-t+r}{(p-t)s_1+r} \in [0,1]$. By the Löwner-Heinz theorem this yields

$$\begin{aligned}\{A^{r/2}&(A^{-t/2}B^pA^{-t/2})^{s_1}A^{r/2}\}^{(q-t+r)/[(p-t)s_1+r]}\\ &\geq \{A^{r/2}(A^{-t/2}B^pA^{-t/2})^{s_2}A^{r/2}\}^{(q-t+r)/[(p-t)s_2+r]}\end{aligned}$$

for $s_2 \geq s_1 \geq 1$, so the proof is complete.

Proof of Theorem 1. The proof follows from Proposition 1 and Proposition 3.

Proof of Theorem 2. Put $M = A^{-t/2}B^pA^{-t/2}$, $N = A^{p-t}$, $\alpha = \frac{q-t}{p-t}$ and $\beta = \frac{r}{p-t}$. Then

$$M = A^{-t/2}B^pA^{-t/2} \leq A^{-t/2}A^pA^{-t/2} = A^{p-t} = N$$

holds by the Löwner-Heinz theorem. Therefore applying Lemma 2,

$$\begin{aligned}G_{p,q,t}(A,B,r,s) &= A^{-r/2}\{A^{r/2}(A^{-t/2}B^pA^{-t/2})^sA^{r/2}\}^{(q-t+r)/[(p-t)s+r]}A^{-r/2}\\ &= N^{-\beta/2}\{N^{\beta/2}M^sN^{\beta/2}\}^{(\alpha+\beta)/(s+\beta)}N^{-\beta/2}\\ &= H_\alpha(N,M,\beta,s).\end{aligned}$$

$G_{p,q,t}(A,B,r,s)$ is decreasing for $r \geq 0$ and $s \geq \frac{q-t}{p-t}$ since $H_\alpha(N,M,\beta,s)$ is decreasing for $\beta \geq 0$ and $s \geq \alpha$ by Lemma 2.

Proof of Corollary 3. We have only to put $t = 0$ and replace ps by p in Theorem 1.

Proof of Corollary 4. Theorem 2 easily implies (i) and (ii).

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