A SHORT PROOF OF A CHARACTERIZATION OF REFLEXIVITY OF JAMES

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(Communicated by Dale Alspach)

Abstract. A short direct proof is given to a well-known intrinsic characterization of reflexivity due to R. C. James.

The following famous intrinsic geometric characterization of reflexivity is due to R. C. James [4] (cf. also e.g. [1, p. 51], [2, p. 58] or [3]).

Theorem. A Banach space $X$ is reflexive if and only if there is a $\theta \in (0, 1)$ such that if $(x_n)_{n=1}^{\infty}$ is a sequence of elements of the unit sphere of $X$, $S_X$, with $\|u\| > \theta$ for all $u \in \text{conv}\{x_1, x_2, \ldots\}$, then there are $n_0 \in \mathbb{N}$, $u \in \text{conv}\{x_1, x_2, \ldots, x_{n_0}\}$ and $v \in \text{conv}\{x_{n_0+1}, x_{n_0+2}, \ldots\}$ such that $\|u - v\| \leq \theta$.

This note provides a short and easy direct proof of the James theorem. It does not rely on Helly’s theorem (like the proof in [4]) nor the Šmulian-Eberlein theorem (like the proof in [5, pp. 95–99]).

Proof of the Theorem. Necessity. The following proof is traditional. We present it here for the sake of completeness. Let $X$ be reflexive. Fix any $\theta > 0$, consider any $(x_n)_{n=1}^{\infty} \subset S_X$, and denote $K_n = \text{conv}\{x_{n+1}, x_{n+2}, \ldots\}$. Since $(K_n)_{n=0}^{\infty}$ is a nested sequence of weakly compact sets, there is $x \in \bigcap_{n=0}^{\infty} K_n$. Since $x \in K_0$, there are $n_0 \in \mathbb{N}$ and $u \in \text{conv}\{x_1, \ldots, x_{n_0}\}$ such that $\|x - u\| < \theta/2$. Since $x \in K_{n_0}$, there is $v \in \text{conv}\{x_{n_0+1}, x_{n_0+2}, \ldots\}$ such that $\|x - v\| < \theta/2$. Hence $\|v - u\| < \theta$.

 Sufficiency. Denote $B_\theta = \{F \in X^{**} : \|F\| \leq \theta\}$. This is a weak* closed set. If $X \neq X^{**}$, then by Riesz’ lemma there is $F_\theta \in S_{X^{**}} \backslash (B_\theta + \{x\})$ for all $x \in X$. Note that $F_\theta$ is in the weak* closure of $S_X$ (by Goldstine’s theorem and weak* lower-semicontinuity of the norm). Pick any $x_0 \in S_X$. Since the weak* open set $X^{**} \backslash (B_\theta + \{x_0\})$ contains $F_\theta$, it also contains a convex weak* neighbourhood $V_1$ of $F_\theta$, which means

$$\|v - x_0\| > \theta \quad \forall v \in V_1.$$ 

Since $F_\theta \in X^{**} \backslash B_\theta$, we can assume that $V_1 \subset X^{**} \backslash B_\theta$. Pick any $x_1 \in V_1 \cap S_X$. Since $F_\theta \in X^{**} \backslash (B_\theta + \text{conv}\{x_0, x_1\})$, there is a convex weak* neighbourhood $V_2 \subset V_1$ of $F_\theta$ such that

$$\|v - u\| > \theta \quad \forall u \in \text{conv}\{x_0, x_1\}, \quad \forall v \in V_2.$$
Pick any \( x_2 \in V_2 \cap S_X \) and continue as above. The sequences of convex sets \( V_1 \supset V_2 \supset \cdots \) and elements \((x_n)_{n=1}^{\infty} \subset S_X\) satisfy \( x_n \in V_n \) and
\[
\|v-u\| > \theta \quad \forall u \in \text{conv}\{x_1, \ldots, x_n\}, \quad \forall v \in V_{n+1}.
\]
Since \( \text{conv}\{x_n, x_{n+1}, \ldots\} \subset V_n \subset X^{**} \setminus B_\theta \), this contradicts the assumption. \( \square \)

**REFERENCES**


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