

ON THE GRAPH CONVERGENCE OF SUBDIFFERENTIALS OF CONVEX FUNCTIONS

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ABSTRACT. This paper provides another proof of the Attouch Theorem relating the epigraphical limit of sequences of convex functions to the set limit of the graphs of the subdifferentials.

1. INTRODUCTION

Attouch ([1] and [2]) proved that a sequence of proper lower semicontinuous convex functions on a reflexive Banach space Mosco-converges if and only if the graphs of the subdifferentials Painlevé-Kuratowski converge to the graph of the subdifferential of the Mosco-limit function and a condition that fixes the constant of integration (normalization condition) holds. Attouch and Beer generalized in [3] the result in the setting of any Banach space. They showed that in any Banach space slice convergence for proper lower semicontinuous convex functions is equivalent to Painlevé-Kuratowski convergence of the graphs of the subdifferentials plus a normalization condition. Both proofs given in Attouch [1] and Attouch and Beer [3] depend heavily on the integration theorem by Rockafellar [16] using the cyclic monotonicity of the subdifferential of convex functions. These theorems have applications in several fields like convergence problems in mechanics (see, e.g., [2]), evolution equations governed by subdifferential operators (see, e.g., [2] and [4]), numerical optimization (see, e.g., [10]), and generalized second order derivatives of convex functions (see, e.g., [11], [15] and [18]).

The aim of this paper is to provide another proof of the above theorems without using the above integration theorem by Rockafellar. This proof consists in proving in a direct way that under a normalization condition, convergence of the graphs of subdifferentials of lower semicontinuous convex functions f_n is equivalent to Mosco-convergence of these functions and their Fenchel conjugates. The key to this approach is the study of the behaviour of limits of strongly convergent sequences of elements in the graphs of subdifferentials with respect to the epigraphical limits superior and inferior of (f_n) . This has been done with the generalized Fenchel subdifferential of the epigraphical limit inferior of (f_n) since this function may be nonconvex although all the functions f_n are convex. A modification of this approach

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will be used in [9] to study the convergence of approximate subdifferentials of convex functions.

2. GRAPH CONVERGENCE OF SUBDIFFERENTIALS

We begin by recalling the graph convergence theorem of subdifferentials of convex functions proved by Attouch ([1] and [2]) for reflexive spaces.

2.1. Theorem (Attouch). *Let X be a reflexive Banach space and let $f, f_n: X \rightarrow \mathbb{R} \cup \{+\infty\}$, $n \in \mathbb{N}$, be proper lower semicontinuous convex functions. Then the following assertions are equivalent:*

- a) $f_n \xrightarrow{M} f$,
- b)

$$\left\{ \begin{array}{l} \partial f_n \xrightarrow{P.K.} \partial f \\ (N.C.): \text{ there exist } (a, a^*) \in \partial f \text{ and a sequence } (a_n, a_n^*) \in \partial f_n \text{ such that} \\ (a_n, a_n^*, f_n(a_n)) \rightarrow (a, a^*, f(a)). \end{array} \right.$$

(Here *N.C.* stands for the so-called normalization condition.)

We now make precise the notions which have been used. Let (f_n) be a sequence of functions from a topological space (X, τ) into $\mathbb{R} \cup \{-\infty, +\infty\}$. The sequential epilimit inferior of the sequence (f_n) with respect to the topology τ is the function denoted by $\tau\text{-Lief}_n$ and defined by

$$(\tau\text{-Lief}_n)(x) = \inf_{x_n \xrightarrow{\tau} x} \liminf_n f_n(x_n).$$

The sequential epilimit superior is denoted by $\tau\text{-Lsef}_n$ and defined by

$$(\tau\text{-Lsef}_n)(x) = \inf_{x_n \xrightarrow{\tau} x} \limsup_n f_n(x_n).$$

If X is a normed vector space, one says that the sequence (f_n) Mosco-converges to f ($f_n \xrightarrow{M} f$) provided

$$w\text{-Lief}_n \geq f \geq s\text{-Lsef}_n$$

where w (resp. s) denotes the weak (resp. strong) topology of X . If (g_n) and g are functions defined on the topological dual X^* of X , one says that (g_n) * -Mosco-converges to g ($g_n \xrightarrow{M^*} g$) provided

$$w^*\text{-Lief}_n \geq g \geq s\text{-Lse}g_n$$

where w^* denotes the weak star topology of X^* .

Recall also that for a sequence (C_n) of subsets of X the sequential limits inferior and superior of (C_n) with respect to the topology τ are defined by

$$\tau\text{-Li}C_n = \{x \in X: \exists x_n \in C_n, x_n \xrightarrow{\tau} x\}$$

and

$$\tau\text{-Ls}C_n = \{x \in X: \exists s(n), \exists x_{s(n)} \in C_{s(n)}, x_{s(n)} \xrightarrow{\tau} x\}.$$

The sequence (C_n) is said to be Painlevé-Kuratowski convergent to C ($C_n \xrightarrow{P.K.} C$) provided

$$\tau\text{-Ls}C_n \sqsubset C \sqsubset \tau\text{-Li}C_n.$$

If X is a normed space and $f: X \rightarrow \mathbb{R} \cup \{+\infty\}$ is convex, one denotes by ∂f the graph of the subdifferential of f :

$$\begin{aligned} \partial f &= \{(x, x^*) \in X \times X^*: x^* \in \partial f(x)\} \\ &= \{(x, x^*) \in X \times X^*: f(x) + f^*(x^*) = \langle x^*, x \rangle\} \end{aligned}$$

where f^* is the Fenchel conjugate of f , that is, $f^*(x^*) = \sup\{\langle x^*, x \rangle - f(x): x \in X\}$. We will also denote by $\partial^* f^*$ the graph of the subdifferential of f^* restricted to $X^* \times X$ (and not $X^* \times X^{**}$). So $x \in \partial^* f^*(x^*)$ is equivalent to $x^* \in \partial f(x)$.

When $\varphi: X \rightarrow \mathbb{R} \cup \{-\infty, +\infty\}$ is a nonnecessarily convex function one may extend the above definition by considering the Fenchel subdifferential notion. Recall that the Fenchel subdifferential of φ is the subset given by

$$\partial^{Fen} \varphi(x) = \{x^* \in X^*: \langle x^*, u - x \rangle + \varphi(x) \leq \varphi(u), \forall u \in X\}$$

if $x \in \text{dom } \varphi := \{x \in X: \varphi(x) < +\infty\}$, and by $\partial^{Fen} \varphi(x) = \emptyset$ if $x \notin \text{dom } \varphi$.

Recall also that φ is said to be proper provided $\text{dom } \varphi \neq \emptyset$ and $\varphi(x) > -\infty$ for every $x \in X$.

Let us now point out that the proofs of the graph convergence theorem given by Attouch ([1] and [2]) and Attouch and Beer [3] depend heavily on the following result by Rockafellar ([16] and [17]) (as well as its dual version in [3]).

2.2. Theorem (Rockafellar). *Let X be a Banach space and $f: X \rightarrow \mathbb{R} \cup \{+\infty\}$ be a proper lower semicontinuous convex function. For each $x \in X$ and $x_0 \in \text{dom } \partial f := \{u \in X: \partial f(u) \neq \emptyset\}$*

$$f(x) = \sup \left\{ f(x_0) + \sum_{i=1}^{n-1} \langle x_i^*, x_{i+1} - x_i \rangle \right\}$$

where the supremum is taken over all chains $x_1 = x_0, x_2, \dots, x_n = x$, where the intermediate points lie in $\text{dom } \partial f$, and where $x_i^* \in \partial f(x_i)$.

We are going to use another integration result and give another proof (different to those by Attouch [1] and Attouch and Beer [3]) for the theorem relating the graph convergence of subdifferentials to some specific convergence (see [2] and [5]) of the functions.

In all the sequel X will be any Banach space.

We start with the following lemma whose proof may be deduced from Rockafellar's theorem [16]. For the convenience of the reader, we will give a direct proof in the appendix by adapting Rockafellar's proof (where both functions f and g are assumed to be convex) to the case below.

2.3. Lemma. *Let $g: X \rightarrow \mathbb{R} \cup \{+\infty\}$ be a lower semicontinuous function and $f: X \rightarrow \mathbb{R} \cup \{+\infty\}$ be a proper lower semicontinuous convex function. If $\partial f \sqsubset \partial^{Fen} g$, then $f = g + \text{Const}$ (where Const denotes a constant real number).*

Before giving our proof of the graph convergence theorem of subdifferentials we need to consider another lemma.

2.4. Lemma. *Let $f_n: X \rightarrow \mathbb{R} \cup \{+\infty\}$, $n \in \mathbb{N}$, be proper lower semicontinuous convex functions satisfying the hypothesis*

(H): *$s\text{-Li}_e f_n > -\infty$ and there exists a strongly convergent sequence (a_n) for which*

$$\limsup_n f_n(a_n) < +\infty.$$

Then for any sequence (x_n, x_n^*) with $(x_n, x_n^*) \in \partial f_n$ and $(x_n, x_n^*) \xrightarrow{\|\cdot\| \times \|\cdot\|^*} (x, x^*)$ one has

- a) $(w\text{-}Li_e f_n)(x) = \liminf_n f_n(x_n)$ and $(s\text{-}Ls_e f_n)(x) = \limsup_n f_n(x_n)$;
 b) $(x, x^*) \in \partial^{Fen}(w\text{-}Li_e f_n) \cap \partial(s\text{-}Ls_e f_n)$.

So under hypothesis (H) we have

$$Li \partial f_n \subset \partial^{Fen}(w\text{-}Li_e f_n) \cap \partial(s\text{-}Ls_e f_n).$$

Moreover, the same results hold if the functions f_n are defined on X^* and $w\text{-}Li_e f_n$ is replaced by $w^*\text{-}Li_e f_n$.

Remark. We consider the Fenchel subdifferential since the function $w\text{-}Li_e f_n$ need not be convex although the functions f_n are convex.

Proof of Lemma 2.4. Fix a sequence $(x_n, x_n^*) \in \partial f_n$ strongly converging to (x, x^*) in $X \times X^*$. For $y \in X$ and $y_n \xrightarrow{w} y$ one has

$$\langle x_n^*, y_n - x_n \rangle + f_n(x_n) \leq f_n(y_n)$$

and hence

$$(1) \quad \langle x^*, y - x \rangle + \liminf_n f_n(x_n) \leq \liminf_n f_n(y_n).$$

If we take $y = a$ and $y_n = a_n$ in (1) we get (using (H)) $\liminf_n f_n(x_n) < +\infty$ and hence by the first part of (H) we have that $\liminf_n f_n(x_n)$ is finite. Taking in (1) the infimum over all sequences $y_n \xrightarrow{w} y$ gives

$$(2) \quad \langle x^*, y - x \rangle + \liminf_n f_n(x_n) \leq (w\text{-}Li_e f_n)(y)$$

and setting $y = x$ in (2) we obtain

$$\liminf_n f_n(x_n) \leq (w\text{-}Li_e f_n)(x)$$

which ensures (by definition)

$$(3) \quad (w\text{-}Li_e f_n)(x) = \liminf_n f_n(x_n).$$

So it follows from (2) and (3) that $(x, x^*) \in \partial^{Fen}(w\text{-}Li_e f_n)$ and this proves the first inclusion in b). The proof of the second one is similar and the same arguments still hold for functions f_n defined on X^* . \square

Following Beer [5] we denote $\Delta\varphi := \{(x, \varphi(x), x^*) \in X \times \mathbb{R} \times X^* : x^* \in \partial\varphi(x)\}$. (Here φ is assumed to be a convex function from X into $\mathbb{R} \cup \{+\infty\}$.)

We can now establish our proof of the theorem.

2.5. Theorem. *Let $f, f_n: X \rightarrow \mathbb{R} \cup \{+\infty\}$, $n \in \mathbb{N}$, be proper lower semicontinuous convex functions. Then the following assertions are equivalent:*

- a) $f_n \xrightarrow{M} f$ and $f_n^* \xrightarrow{M^*} f^*$,
 b) $\Delta f_n \xrightarrow{P.K.} \Delta f$,
 c) $\partial f_n \xrightarrow{P.K.} \partial f$, and
 (N.C.): there exist $(a, a^*) \in \partial f$ and a sequence $(a_n, a_n^*) \in \partial f_n$ such that $(a_n, a_n^*, f_n(a_n)) \rightarrow (a, a^*, f(a))$.

Proof. a) \Rightarrow b). For the inclusion $\Delta f \sqsupset Li\Delta f_n$ we follow the proof by Attouch [1], [2]. Fix $(x, f(x), x^*) \in \Delta f$. There exist (by properties of Mosco-convergence, see [2] and [6] for example) $u_n \xrightarrow{\|\cdot\|} x$ and $u_n^* \xrightarrow{\|\cdot\|^*} x^*$ such that $f_n(u_n) \rightarrow f(x)$ and $f_n^*(u_n^*) \rightarrow f^*(x^*)$.

Put $\varepsilon_n := f_n^*(u_n^*) + f_n(u_n) - \langle u_n^*, u_n \rangle \geq 0$. Then (ε_n) converges to $f^*(x^*) + f(x) - \langle x^*, x \rangle = 0$ and $u_n^* \in \partial \varepsilon_n f_n(u_n)$ (the ε_n -approximate subdifferential). By the Brondsted-Rockafellar Lemma [8] there exists $(x_n, x_n^*) \in \partial f_n$ such that $\|x_n - u_n\| \leq \sqrt{\varepsilon_n}$ and $\|x_n^* - u_n^*\| \leq \sqrt{\varepsilon_n}$. This ensures $(x_n, x_n^*) \xrightarrow{\|\cdot\|} (x, x^*)$.

Noting now that a) ensures that (H) holds, we obtain by Lemma 2.4

$$\limsup_n f_n(x_n) = f(x) = \liminf_n f_n(x_n)$$

and hence $(x, f(x), x^*) \in Li\Delta f_n$. So the inclusion is proved.

It remains to show that $Ls\Delta f_n \sqsubset \Delta f$. Fix now $(x, r, x^*) \in Ls\Delta f_n$ and consider a subsequence $(x_k, f_k(x_k), x_k^*) \rightarrow (x, r, x^*)$ with $x_k^* \in \partial f_k(x_k)$ (here $k \in K \subset \mathbb{N}$). By Lemma 2.4 one has $(x, x^*) \in \partial f$ and $f_k(x_k) \rightarrow f(x)$ and hence $r = f(x)$ and $(x, r, x^*) \in \Delta f$. This completes the proof of a) \Rightarrow b).

b) \Rightarrow c). It is obvious that b) ensures that $\partial f \sqsubset Li\partial f_n$. Fix now $(x, x^*) \in Ls\partial f_n$, and choose a subsequence $(x_k, x_k^*) \in \partial f_k$ for $k \in K \subset \mathbb{N}$ with $(x_k, x_k^*) \rightarrow (x, x^*)$. Taking another subsequence if necessary we may suppose by Lemma 2.4 $f_k(x_k) \rightarrow r \in \mathbb{R}$. So by the assumption we get $(x, r, x^*) \in \Delta f$ and hence $(x, x^*) \in \partial f$. Then we conclude $Ls\partial f_n \sqsubset \partial f$.

c) \Rightarrow a). It is not difficult to see that the normalization condition (N.C.) ensures the hypothesis (H) in Lemma 2.4. So we have by this lemma

$$\partial f \sqsubset Li\partial f_n \sqsubset \partial^{Fen}(w-Li_e f_n) \cap \partial(s-Ls_e f_n)$$

and by Lemma 2.3 we may deduce

$$(1) \quad f = w-Li_e f_n + C_1 = s-Ls_e f_n + C_2$$

where C_1 and C_2 are two constant real numbers. Moreover, Lemma 2.4 once again and condition (N.C.) ensure the following equality in \mathbb{R} :

$$(2) \quad (w-Li_e f_n)(a) = f(a) = (s-Ls_e f_n)(a).$$

It follows, from (1) and (2), that $C_1 = C_2 = 0$ and hence (1) becomes $f_n \xrightarrow{M} f$. To show $f_n^* \xrightarrow{M^*} f^*$ it is enough to note that

$$\partial f_n \xrightarrow{P.K.} \partial f \Leftrightarrow \partial^* f_n^* \xrightarrow{P.K.} \partial^* f^*$$

and that (N.C.) ensures the condition (N.C*.):

there exist $(a^*, a) \in \partial^* f^*$ and a sequence $(a_n^*, a_n) \in \partial^* f_n^*$ such that $(a_n^*, a_n, f_n^*(a_n^*)) \rightarrow (a^*, a, f^*(a^*))$, and to use similar arguments since Lemma 2.3 still holds for functions defined on X^* (see the appendix). □

This theorem also shows that the Mosco-convergence of (f_n) to f in a nonreflexive Banach space does not ensure the graph convergence of the subdifferential. Indeed in any such space there exist sequences (f_n) that M -converge and such that the sequences of Fenchel conjugates (f_n^*) do not M^* -converge (see [7]). However in reflexive spaces, a theorem of Mosco [14] says that the M -convergence of proper lower semicontinuous convex functions and that their conjugates are equivalent. So the Attouch theorem (see [1] and [2]) is a direct consequence of Theorem 2.5.

Before relating Theorem 2.5 to the Attouch-Beer Theorem (see [3]), let us establish the following lemma which also has its own interest. We first need some facts about the notion of slice convergence.

By Theorem 8.2.2 in Beer [6], in the class of proper lower semicontinuous convex functions a sequence $(f_n)_n$ slice-converges to f iff for any nonempty open subsets U and V in X and X^* respectively and any real number $a > 0$ there exists N such that

$$\begin{aligned} \text{Epi } f \cap (U \times] - \infty, \alpha[) \neq \emptyset &\Rightarrow \text{Epi } f_n \cap (U \times] - \infty, \alpha[) \neq \emptyset \quad \forall n \geq N, \\ \text{Epi } f^* \cap (V \times] - \infty, \alpha[) \neq \emptyset &\Rightarrow \text{Epi } f_n^* \cap (V \times] - \infty, \alpha[) \neq \emptyset \quad \forall n \geq N. \end{aligned}$$

2.6. Lemma. *Let $f, f_n: X \rightarrow \mathbb{R} \cup \{+\infty\}$, $n \in \mathbb{N}$, be proper lower semicontinuous convex functions. Then the following assertions are equivalent:*

- a) $f_n \xrightarrow{M} f$ and $f_n^* \xrightarrow{M^*} f^*$,
- b) $(f_n)_n$ slice converges to f ,
- c) $f \geq s\text{-}Ls_e f_n$ and $f^* \geq s\text{-}Ls_e f_n^*$,
- d) for each $x \in X$ and each $x^* \in X^*$ there exist $x_n \xrightarrow{\|\cdot\|} x$ and $x_n^* \xrightarrow{\|\cdot\|} x^*$ such that $f(x) \geq \lim_n f_n(x_n)$ and $f^*(x^*) \geq \lim_n f_n^*(x_n^*)$,
- e) for each $x \in \text{dom } \partial f$ and each $x^* \in \text{dom } \partial^* f^*$ there exist $x_n \xrightarrow{\|\cdot\|} x$ and $x_n^* \xrightarrow{\|\cdot\|} x^*$ such that $f(x) \geq \lim_n f_n(x_n)$ and $f^*(x^*) \geq \lim_n f_n^*(x_n^*)$.

Proof. The implications c) \Rightarrow d) and d) \Rightarrow e) are obvious and a) \Rightarrow b) follows easily from the characterization above of slice convergence. So it is enough to prove the implications e) \Rightarrow c), b) \Rightarrow c) and c) \Rightarrow a).

e) \Rightarrow c). Assertion e) obviously ensures $f(x) \geq (s\text{-}Ls_e f_n)(x)$ and $f^*(x^*) \geq (s\text{-}Ls_e f_n^*)(x^*)$ for any $x \in \text{dom } \partial f$ and any $x^* \in \text{dom } \partial^* f^*$. Hence, c) follows from Lemma 3.1 in the appendix.

b) \Rightarrow c). We fix $x \in X$ and we are going to prove $f(x) \geq (s\text{-}Ls_e f_n)(x)$. Obviously we may suppose $r := f(x) \in \mathbb{R}$. Then for each integer $k \geq 1$ one has $\text{Epi } f \cap \mathbb{B}(x, \frac{1}{k}) \times] - \infty, r + \frac{1}{k}[\neq \emptyset$ and hence by the characterization above of slice convergence, one may find an increasing sequence of integers (p_k) such that $p_k \geq k$ and elements $(u_{n,k}, r_{n,k}) \in \text{Epi } f_n \cap \mathbb{B}(x, \frac{1}{k}) \times] - \infty, r + \frac{1}{k}[$ for all $n \geq p_k$. If for each $k \geq 1$ and each integer $n \in [p_k, p_{k+1}[$ one sets $x_n = u_{n,k}$, then one has

$$f_n(x_n) \leq r_{n,k} \leq r + \frac{1}{k} = f(x) + \frac{1}{k} \quad \text{and} \quad x_n \in \mathbb{B}\left(x, \frac{1}{k}\right).$$

Therefore $x_n \xrightarrow{\|\cdot\|} x$ and $f(x) \geq \limsup_n f_n(x_n) \geq (s\text{-}Ls_e f_n)(x)$.

Similarly we complete the proof to obtain assertion c).

c) \Rightarrow a). We start with the proof of the inequality

$$(1) \quad w^*\text{-}Lief_n^* \geq (s\text{-}Ls_e f_n)^*.$$

Fix any $x \in X$ and choose $x_n \xrightarrow{\|\cdot\|} x$ such that $(s\text{-}Ls_e f_n)(x) = \lim f_n(x_n)$. Consider any $x^* \in X^*$ and $x_n^* \xrightarrow{w^*} x^*$. By the Fenchel inequality we have

$$f_n^*(x_n^*) \geq \langle x_n^*, x_n \rangle - f_n(x_n)$$

and hence

$$\liminf_n f_n^*(x_n^*) \geq \liminf_n (\langle x_n^*, x_n \rangle - f_n(x_n)) = \langle x^*, x \rangle - (s\text{-}Ls_e f_n)(x).$$

Therefore, we have

$$\begin{aligned} (w^* \text{-} Li_e f_n^*)(x^*) &= \inf_{\substack{w^* \\ x_n^* \rightarrow x^*}} \liminf_n f_n^*(x_n^*) \\ &\geq \sup_{x \in X} (\langle x^*, x \rangle - (s\text{-}Ls_e f_n)(x)) \\ &= (s\text{-}Ls_e f_n)^*(x^*), \end{aligned}$$

which is the desired inequality. Similarly, we have

$$(2) \quad w\text{-}Li_e f_n \geq (s\text{-}Ls_e f_n^*)^*.$$

Now assertion c) and the inequalities (1) and (2) imply

$$s\text{-}Ls_e f_n^* \leq f^* \leq (s\text{-}Ls_e f_n)^* \leq w^*\text{-}Li_e f_n^*$$

and

$$s\text{-}Ls_e f_n \leq f \leq (s\text{-}Ls_e f_n^*)^* \leq w\text{-}Li_e f_n$$

and hence $f_n^* \xrightarrow{M^*} f^*$ and $f_n \xrightarrow{M} f$. So the proof is complete. \square

Remark. The proof of a) \Rightarrow b), b) \Rightarrow c) and c) \Rightarrow a) is valid in any normed linear space.

The equivalence b) \Leftrightarrow e) (from which c) and d) follow easily) was already proved by a different method in Attouch-Beer [3].

Now Lemma 2.6 allows us to reformulate Theorem 2.5 with the statement of Attouch-Beer's theorem [3] and to also get a theorem by Beer [5]. The equivalence a) \Leftrightarrow c) is proved by Attouch-Beer [3] and a) \Leftrightarrow b) is similar to Theorem 4.7 in Beer [5].

2.7. Corollary (see [3] and [5]). *Let $f, f_n: X \rightarrow \mathbb{R} \cup \{+\infty\}$, $n \in \mathbb{N}$, be proper lower semicontinuous convex functions. Then the following assertions are equivalent:*

- a) $(f_n)_n$ slice converges to f ,
- b) $\Delta f_n \xrightarrow{P.K.} \Delta f$,
- c) $\partial f_n \xrightarrow{P.K.} \partial f$, and

(N.C.): *there exist $(a, a^*) \in \partial f$ and a sequence $(a_n, a_n^*) \in \partial f_n$ such that $(a_n, a_n^*, f_n(a_n)) \rightarrow (a, a^*, f(a))$.*

3. APPENDIX

In this section we are going to give a direct proof of Lemma 2.3.

3.1. Lemma. *Let $g: X \rightarrow \mathbb{R} \cup \{+\infty\}$ and $k: X^* \rightarrow \mathbb{R} \cup \{+\infty\}$ be proper lower semicontinuous functions and $f: X \rightarrow \mathbb{R} \cup \{+\infty\}$ be a proper lower semicontinuous convex function. Then*

- a) $f \geq g$ whenever $f(x) \geq g(x)$ for every $x \in \text{dom } \partial f$;
- b) $f^* \geq k$ whenever $f^*(x^*) \geq k(x^*)$ for every $x^* \in \text{dom } \partial^* f^*$.

Proof. The lemma is a direct consequence of the lower semicontinuity of g and k and of the Brøndsted-Rockafellar theorem [8] ensuring for any $x \in \text{dom } f$ and any $x^* \in \text{dom } f^*$

$$f(x) = \liminf_{\substack{u \rightarrow x \\ u \in \text{dom } \partial f}} f(u) \quad \text{and} \quad f^*(x^*) = \liminf_{\substack{u^* \rightarrow x^* \\ u^* \in \text{dom } \partial^* f^*}} f^*(u^*). \quad \square$$

In the proof of Lemma 3.3 (used in Section 2) we will employ the following result where for any function $\varphi: X^{**} \rightarrow \mathbb{R} \cup \{+\infty\}$ we denote by φ_X the restriction of φ to X .

3.2. Lemma. *Let $h: X \rightarrow \mathbb{R} \cup \{+\infty\}$ and $k: X^* \rightarrow \mathbb{R} \cup \{+\infty\}$ be proper. Then*

- a) $x^* \in \partial^{Fen} h(x) \Rightarrow h(x) = (h_X^{**})(x)$ and $x^* \in \partial(h_X^{**})(x)$;
- b) $x \in \partial^{Fen} k(x^*) \Rightarrow k(x^*) = (k_X^*)^*(x^*)$ and $x \in \partial^*(k_X^*)^*(x^*)$.

Moreover, if h and k are lower semicontinuous with respect to the norm and the weak star topology respectively, then

- c) h is convex if and only if $\partial(h_X^{**}) \neq \emptyset$ and $\partial(h_X^{**}) \subset \partial^{Fen} h$;
- d) k is convex if and only if $\partial(k_X^*)^* \neq \emptyset$ and $\partial^*(k_X^*)^* \subset \partial^{Fen} k$.

Proof. Assertions a) and b) are not difficult to prove and the implications “ \Rightarrow ” in c) and d) are direct consequences of the Brondsted-Rockafellar theorem [8]. So let us prove the converse ones. Fix any $x \in \text{dom } \partial(h_X^{**})$ and any $x^* \in \text{dom } \partial(k_X^*)^*$ (since these sets are nonempty by hypothesis). Then, by assumption,

$$\partial^{Fen} h(x) \neq \emptyset \quad \text{and} \quad \partial^{Fen} k(x^*) \neq \emptyset$$

and hence $h(x) = h_X^{**}(x)$ and $k(x^*) = (k_X^*)^*(x^*)$ by assertions a) and b). Therefore, it follows, from Lemma 3.1, that $h_X^{**} \geq h$ and $(k_X^*)^* \geq k$. By the Fenchel inequality, we get $h = h_X^{**}$ and $k = (k_X^*)^*$ and hence h and k are convex. \square

3.3. Lemma. *Let $g: X \rightarrow \mathbb{R} \cup \{+\infty\}$ and $h: X^* \rightarrow \mathbb{R} \cup \{+\infty\}$ be proper lower semicontinuous functions and $f: X \rightarrow \mathbb{R} \cup \{+\infty\}$ be a proper lower semicontinuous convex function. Then*

- a) $f = g + \text{Const} \Leftrightarrow \partial f \subset \partial^{Fen} g$;
- b) $f^* = h + \text{Const} \Leftrightarrow \partial^* f^* \subset \partial^{Fen} h$.

Proof. a) The implication “ \Rightarrow ” being obvious, let us prove the reverse one. The proof is heavily inspired by techniques in Rockafellar [16] where both functions are assumed to be convex. Set $j = 1/2\|\cdot\|^2$. As j is continuous and as $\partial f \subset \partial^{Fen} g$ (by assumption), we have

$$\partial(f + j) = \partial f + \partial j \subset \partial^{Fen} g + \partial j \subset \partial^{Fen}(g + j)$$

and hence

$$(1) \quad \partial^*(f + j)^* \subset \partial^*(g + j)^*.$$

As $(f + j)^* = f^* \nabla j^*$ is finite and continuous on X^* (see Moreau [13]), it follows from (1) that the lower semicontinuous convex function $(g + j)^*$ is also finite on all X^* , hence (see [13]) continuous on X^* . We then claim

$$(2) \quad \partial(f + j)^* \subset \partial(g + j)^*.$$

Indeed let $x^{**} \in \partial(f + j)^*(x)$, i.e.

$$(3) \quad (f + j)^{**}(x^{**}) + (f + j)^*(x^*) = \langle x^{**}, x^* \rangle.$$

One knows (by Moreau [13]) that there exists a generalized sequence (x_i) in X weakly- w^{**} converging to x^{**} for which $(f + j)^{**}(x^{**}) = \lim_i (f + j)(x_i)$. Put $\varepsilon_i := (f + j)^*(x^*) + (f + j)(x_i) - \langle x^*, x_i \rangle \geq 0$.

Then, $x^* \in \partial_{\varepsilon_i}(f + j)(x_i)$ and $\varepsilon_i \rightarrow 0$ by (3). By the Brondsted-Rockafellar theorem, there exists $(z_i, z_i^*) \in \partial(f + j)$ satisfying $\|z_i - x_i\| \leq \sqrt{\varepsilon_i}$, $\|z_i^* - x^*\| \leq \sqrt{\varepsilon_i}$.

Then $z_i \xrightarrow{w^{**}} x^{**}$, $z_i^* \xrightarrow{\|\cdot\|} x^*$ and, for $i \geq i_0$, $z_i \in \partial(f + j)^*(z_i^*) \subset \gamma \mathbb{B}_X$, where \mathbb{B}_X denotes the closed unit ball of X (since $(f + j)^*$ is locally Lipschitz). Then the

subnet $(z_i)_{i \geq i_0}$ is bounded in norm and it is not difficult to see that this boundedness and relation (1) ensure $x^{**} \in \partial(g+j)^*(x^*)$. So the claim is proved. Now, it is easy to see that (2) and the locally Lipschitzian behavior of $(f+j)^*$ and $(g+j)^*$ ensure

$$(f+j)^* = (g+j)^* + \text{Const.}$$

So

$$(4) \quad f+j = (f+j)_X^{**} = (g+j)_X^{**} + \text{Const}$$

and hence (taking the assumption into account)

$$\partial(g+j)_X^{**} = \partial(f+j) \sqsubset \partial^{Fen}(g+j)$$

which ensures by Lemma 3.2 and (4)

$$g+j = (g+j)_X^{**} = f+j + \text{Const},$$

that is, $f = g + \text{Const}$ and assertion a) is proved.

b) Let us prove the implication “ \Leftarrow ” of assertion b). It is easily seen that the assumption $\partial^* f^* \sqsubset \partial^{Fen} h$ ensures $\partial f \sqsubset \partial(h_X^*)$ (since one can verify that $x \in \partial^{Fen} h(x^*) \Rightarrow x^* \in \partial(h_X^*)(x)$). Then by assertion a), one has $f = h_X^* + \text{Const}$ and hence $f^* = (h_X^*) + \text{Const}$. Therefore $f^* = h + \text{Const}$ since $(h_X^*)^* = h$ by the above inclusion and Lemma 3.2. So the proof of the lemma is complete. \square

Remark. We refer the reader to [19] where the general perturbation inclusion

$$\partial f(x) \sqsubset \partial g(x) + \varepsilon \mathbb{B}_{X^*}$$

is considered.

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