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DIFFERENTIAL FORMS ON QUOTIENTS BY REDUCTIVE GROUP ACTIONS

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ABSTRACT. Let X be a smooth affine algebraic variety where a reductive algebraic group G acts with a smooth quotient space $Y = X//G$. We show that the algebraic differential forms on X which are pull-backs of forms on Y are exactly the G-invariant horizontal differential forms on X.

1. INTRODUCTION

Let G be a reductive algebraic group of automorphisms of an affine algebraic variety X over an algebraically closed field k of characteristic zero. Denote by $k[X]$ the algebra of regular functions on X. The subalgebra $k[X]^G$ of G-invariants in $k[X]$ is finitely generated; let $Y = X//G$ be the affine algebraic variety such that $k[Y] = k[X]^G$ and let $\pi : X \to Y$ be the quotient map defined by the inclusion $k[Y] \subset k[X]$. Then we have a pull-back map $\pi^* : \Omega^*(Y) \to \Omega^*(X)$, where Ω^* denotes the algebra of Kähler differential forms, see e.g. [K].

In the present note, we describe the image of π^* in the case where X and Y are smooth. In other words, we characterize the differential forms on X which can be written as sums of products of invariant functions and of their differentials. Clearly, such forms are G-invariant and *horizontal*, that is, their interior product with any vector field induced by the G-action is zero. Let $\Omega^*_{\text{hor}}(X)^G$ be the space of G-invariant horizontal forms; such forms are also called *basic*.

Is any basic differential form in the image of π^*? This question was raised by P. Michor in the setting of a representation of a compact Lie group on a real vector space V, and of differential forms on V with smooth coefficients (which do not coincide with Kähler differentials for the ring of smooth functions on V; but differential forms with polynomial coefficients coincide with Kähler differentials for the ring of polynomial functions). Michor showed that the answer is negative in general, but positive for polar representations; see [M1] and [M2].

Here we obtain the following

Theorem 1. *Let G be a reductive group of automorphisms of a smooth, affine variety X, and let $\pi : X \to Y$ be the quotient map. If Y is smooth, then $\pi^* : \Omega^*(Y) \to \Omega^*_{\text{hor}}(X)^G$ is an isomorphism.*

This proves a conjecture of V. L. Popov formulated in his talk at the problem session of the conference "Algebraic Groups", Kazimierz (Poland), May 25-June 1,

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1996. Some applications, and a generalization to the case of a singular quotient space, are given in §3 below. The author thanks V. L. Popov, G. Schwarz (who gave another proof of Theorem 1, in the case of a representation of a connected group) and D. Zaitsev (who gave another proof of Theorem 1, in the case where G is finite) for useful discussions. Thanks are also due to the referee for suggesting simplifications of several points in the proof.

2. Proof of Theorem 1

Our approach is similar to the one in [B], §2. We begin with an observation concerning reflexive modules; this notion of commutative algebra can be defined as follows. Let A be a Noetherian domain, and let M be a finite A-module. Denote by $M^\vee = \mathrm{Hom}_A(M, A)$ the dual module. Then we have a natural map from M to its double dual $M^{\vee\vee}$, and M is called reflexive if this map is an isomorphism.

Lemma 1. *Let A be a normal Noetherian ring, let M, N be finite A-modules with M reflexive and N torsion-free, and let $u : M \to N$ be an A-linear map. If u is an isomorphism in codimension one, then u is an isomorphism.*

Proof. Set $Z = \mathrm{Spec}(A)$ and let $U \subset Z$ be the open subset where u is an isomorphism. Denote by $\mathcal{M}$, $\mathcal{N}$ the sheaves on Z associated to M, N. Then we have a commutative square

$$\begin{array}{ccc} M & \to & \Gamma(U,\mathcal{M}) \\ \downarrow & & \downarrow \\ N & \to & \Gamma(U,\mathcal{N}) \end{array}$$

By assumption, the map $\Gamma(U,\mathcal{M}) \to \Gamma(U,\mathcal{N})$ is an isomorphism. Moreover, the codimension of $Z \setminus U$ in Z is at least two, and hence the map $M \to \Gamma(U,\mathcal{M})$ is an isomorphism; see [H], Proposition 1.6. Finally, the map $N \to \Gamma(U,\mathcal{N})$ is injective because N is torsion-free. These three facts imply our statement. □

We will check that Lemma 1 applies to the ring $k[Y]$ and to the map $\pi^* : \Omega^*(Y) \to \Omega^*_{\mathrm{hor}}(X)^G$. Observe that both $\Omega^*(Y)$ and $\Omega^*_{\mathrm{hor}}(X)^G$ are $k[Y]$-modules, and that π^* is $k[Y]$-linear. Because Y is smooth, $\Omega^*(Y)$ is locally free over $k[Y]$, and hence reflexive. Moreover, the $k[X]$-module $\Omega^*(X)$ is finitely generated and torsion-free. Therefore, the same holds for the $k[Y]$-module $\Omega^*(X)^G$ (see [P-V], Theorem 3.24) and for its submodule $\Omega^*_{\mathrm{hor}}(X)^G$.

So it suffices to check that π^* is an isomorphism in codimension one. For this, we will use Luna's slice theorem, see [L1]. First recall a

Definition. An equivariant morphism $f : X' \to X$ of affine G-varieties is *excellent* if

(i) f is étale,

(ii) the induced morphism $f//G : X'//G := Y' \to Y$ is étale, and

(iii) the morphism $(f, \pi) : X' \to X \times_Y Y'$ is an isomorphism.

Let $y \in Y$. The fiber $\pi^{-1}(y)$ contains a unique closed orbit, say $G \cdot x$, and the isotropy group $G_x := H$ is reductive. By Luna's slice theorem, we have a locally closed smooth, affine, H-stable subvariety $S \subset X$ which contains x, with the following properties:

(1) $N := T_xS$ is an H-complement to $T_x(G \cdot x)$ in T_xX.

(2) The canonical map $G \times^H S \to X$ is excellent.

(3) There is an excellent H-morphism $S \to N$ sending x to the origin in N. (Hence the induced map $G \times^H S \to G \times^H N$ is excellent.)

We will also need the following

Lemma 2. *Let $f : X' \to X$ be excellent, and set $Y' := X'//G$. Then both maps $k[Y'] \otimes_{k[Y]} \Omega^*_{\mathrm{hor}}(X)^G \to \Omega^*_{\mathrm{hor}}(X')^G$ and $k[Y'] \otimes_{k[Y]} \Omega^*(Y) \to \Omega^*(Y')$ are isomorphisms of $k[Y']$-modules.*

Proof. Because $f//G$ is etale, the natural map $u : k[Y'] \otimes_{k[Y]} \Omega^1(Y) \to \Omega^1(Y')$ is surjective (indeed, the cokernel of u is the space of relative differentials $\Omega^1(Y'/Y)$ which vanishes because $f//G$ is unramified). Moreover, because Y is smooth and $f//G$ is etale, Y' is smooth, too. Therefore, the $k[Y']$-modules $k[Y'] \otimes_{k[Y]} \Omega^1(Y)$ and $\Omega^1(Y')$ are locally free of rank $\dim(Y') = \dim(Y)$. It follows that u is an isomorphism. This implies the second statement.

Because $k[X'] = k[X] \otimes_{k[Y]} k[Y']$ and f is etale, there are equivariant isomorphisms $k[Y'] \otimes_{k[Y]} \Omega^*(X) \to k[X'] \otimes_{k[X]} \Omega^*(X) \to \Omega^*(X')$. This implies the first statement. ∎

Using the above and Lemma 2 twice, one reduces Theorem 1 to spaces of the form $G \times^H N$. Now, by Lemma 4.1 in [M1], restriction to $N \subset G \times^H N$ induces an isomorphism $\Omega^*_{\mathrm{hor}}(G \times^H N)^G \simeq \Omega^*_{\mathrm{hor}}(N)^H$, so we can reduce to the case of representations. Moreover, writing $N = N^H \oplus V$ as an H-representation, we can reduce to considering the action of H on V, since the extra trivial factor just tensors the algebras $k[V]$ and $k[V]^H$ with $k[N^H]$ and the spaces $\Omega^*_{\mathrm{hor}}(V)^H$ and $\Omega^*(V//H)$ with $\Omega^*(N^H)$.

Moreover, by Lemma 1, we may replace the quotient Y by any open subset U such that $Y \setminus U$ has codimension at least two in Y. Recall that Y is stratified by slice types (H, N); see [L1]. If $y \in Y$ is in the open stratum (resp. in a stratum of codimension one), then $V//H$ is a point (resp. is one-dimensional). So we can finally reduce to the case of a reductive group $H \subset \mathrm{GL}(V)$, where $V^H = 0$ and $\dim(V//H) \leq 1$. Then the proof is concluded by the following lemmas.

Lemma 3. *In the case where $V//H$ is a point, the spaces $\Omega^p_{\mathrm{hor}}(V)^H$ vanish for all $p \geq 1$.*

Proof. We may assume that H is connected. First consider the case where H is a torus. Then, because $V//H$ is a point, all weights of H in V lie in an open half-space. So the weights of $\wedge^p V^*$ lie in the opposite half-space. Observing that $\Omega^p(V) \simeq S^\bullet(V^*) \otimes \wedge^p V^*$, where $S^\bullet(V^*)$ is the symmetric algebra of V^*, we obtain $\Omega^p(V)^H = 0$ for $p \geq 1$, which implies our statement.

In the general case of reductive H, let $T \subset H$ be a maximal torus. By the Hilbert-Mumford criterion, there exists a T-submodule $W \subset V$ such that $V = HW$ and that all weights of T in W lie in an open half-space. Because the map

$$\begin{array}{cccc} f: & H \times W & \to & V \\ & (g, w) & \mapsto & gw \end{array}$$

is surjective, we can find $w \in W$ such that the differential

$$\begin{array}{cccc} df_{e,w}: & Lie(H) \times W & \to & V \\ & (\xi, z) & \to & \xi w + z \end{array}$$

is surjective, where $Lie(H)$ denotes the Lie algebra of H. In other words, we have $V = Lie(H)w + W$. It follows that $\wedge^p V = \sum_{q=0}^p (\wedge^{p-q} Lie(H)w) \wedge (\wedge^q W)$. Let $\omega \in$

$\Omega^p_{\rm hor}(V)^H$. Because ω is horizontal, it vanishes on $\sum_{q=0}^{p-1}(\wedge^{p-q}Lie(H)w)\wedge(\wedge^q W)$. Moreover, restriction of $\omega\in S^\bullet(V^*)\otimes\wedge^p V^*$ to W is a T-equivariant morphism from W to $\wedge^p V^*$. By the first step of the proof, this morphism vanishes on the subspace $\wedge^p W$ of $\wedge^p V$. So restriction of ω to W is trivial. Because $V=HW$ and ω is H-invariant, we conclude that $\omega=0$. □

Lemma 4. *In the case where $V//H$ is one-dimensional, the map $\Omega^1(V//H)\to\Omega^1_{\rm hor}(V)^H$ is an isomorphism, and moreover $\Omega^p_{\rm hor}(V)^H=0$ for all $p\geq 2$.*

Proof. The algebra $k[V]^H$ is generated by a non-constant homogeneous function f. We have $\Omega^1(V//H)=k[f]\,df$ and $\Omega^p(V//H)=0$ for $p\geq 2$. By Lemma 3 and our reductions, the $k[f]$-linear map $\Omega^*(V//H)\to\Omega^*_{\rm hor}(V)^H$ is an isomorphism over the field $k(f)$. Because the $k[f]$-module $\Omega^*_{\rm hor}(V)^H$ is torsion-free, it follows that $\Omega^p_{\rm hor}(V)^H=0$ for $p\geq 2$ and that $\Omega^1_{\rm hor}(V)^H=k[f]\omega$ for some homogeneous form ω. Let m be the degree of f, and let n be the degree of ω. Because df is in $k[f]\omega$, we have $m-1=rm+n$ for some integer $r\geq 0$. This forces $r=0$ and then ω is a scalar multiple of df. □

3. Some applications and a generalization

Here are two applications of Theorem 1 which were pointed out by V. L. Popov. First, we recover the following classical result of L. Solomon [So].

Corollary 1. *Let $G\subset\mathrm{GL}(V)$ be a finite group generated by pseudo-reflections. Then any G-invariant differential form on V is the pull-back of a differential form on V/G.*

Indeed, the quotient space V/G is smooth, and Theorem 1 applies.

Another application is an algebraic version of a recent result of P. Michor: let G be a Lie group of isometries of a smooth Riemannian manifold M. Assume that the G-action on M admits a section Σ, and denote by $W(\Sigma)=N_G(\Sigma)/Z_G(\Sigma)$ the corresponding "generalized Weyl group". Then restriction of differential forms with smooth coefficients induces an isomorphism $\Omega^*_{\rm hor}(M)^G\to\Omega^*(\Sigma)^{W(\Sigma)}$; see [M1], Theorem 3.7, and [M2], Theorem 2.

Corollary 2. *Let X be a smooth irreducible affine variety with an action of a reductive group G. Denote by $H\subset G$ a principal isotropy group of X, by W the quotient group $N_G(H)/H$, and by $\Sigma\subset X^H$ the closure of the subset of principal points. If the quotient $X//G$ is smooth, then restriction to Σ induces an isomorphism $\Omega^*_{\rm hor}(X)^G\simeq\Omega^*_{\rm hor}(\Sigma)^W$.*

Indeed, we have a commutative diagram

$$\begin{array}{ccc}\Omega^*(X//G) & \to & \Omega^*(\Sigma//W)\\ \downarrow & & \downarrow\\ \Omega^*_{\rm hor}(X)^G & \to & \Omega^*_{\rm hor}(\Sigma)^W\end{array}$$

where the top horizontal map is pull-back by the morphism $\Sigma//W\to X//G$. But the latter is an isomorphism by the Luna-Richardson restriction theorem; see [L2], Corollaire 4, and [L-R]. So, by Theorem 1, both vertical arrows are isomorphisms.

Finally, Theorem 1 can be extended to the case where the quotient space Y may be singular. Then we have to replace the $k[Y]$-module of Kähler differentials $\Omega^*(Y)$ by its double dual $\Omega^*(Y)^{\vee\vee}$, the module of *Zariski-Lipman differential forms*. By adapting the previous argument, we obtain the following

Theorem 2. *Let G be a reductive group acting on a smooth affine variety X, and let $\pi : X \to Y$ be the quotient map. If no divisor in X is mapped by π to a subvariety of codimension at least two in Y, then $\pi^* : \Omega^*(Y) \to \Omega^*_{\mathrm{hor}}(X)^G$ factors through an isomorphism $\Omega^*(Y)^{\vee\vee} \to \Omega^*_{\mathrm{hor}}(X)^G$.*

In particular, the Zariski-Lipman differential forms on a quotient of a smooth variety X by a finite group are exactly the invariant differential forms on X.

Proof. The assumption that no divisor in X is contracted by π implies that the $k[Y]$-module $\Omega^*_{\mathrm{hor}}(X)^G$ is reflexive, using [H] Proposition 1.6. Therefore, π^* factors through a map $\Omega^*(Y)^{\vee\vee} \to \Omega^*_{\mathrm{hor}}(X)^G$. Furthermore, this map is an isomorphism, by Lemma 1, smoothness in codimension one of Y, and Theorem 1. ☐

We ignore whether the assumption that π contracts no divisor is necessary for Theorem 2 to hold. Observe that the Zariski-Lipman differentials cannot be replaced by Kähler differentials in this statement, as shown by the following example.

Let G be a cyclic group of order n acting on $X = k^2$ by scalar multiplication by n-th roots of unity. Then the algebra $k[X]^G$ is generated by all monomials of total degree n in the coordinates x_1, x_2. Therefore, the differentials of these monomials are the minimal system of homogeneous generators of the $k[Y]$-module $\Omega^1(Y)$; this system consists of $n+1$ elements. But the $k[Y]$-module $\Omega^1_{\mathrm{hor}}(X)^G = \Omega^1(X)^G$ is minimally generated by $2n$ homogeneous elements, products of monomials of degree $n-1$ by dx_1, dx_2. So the quotient $\Omega^1(X)^G/\pi^*\Omega^1(Y)$ has dimension $n-1$.

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