DIFFERENTIAL FORMS ON QUOTIENTS 
BY REDUCTIVE GROUP ACTIONS

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(Communicated by Roe Goodman)

Abstract. Let $X$ be a smooth affine algebraic variety where a reductive algebraic group $G$ acts with a smooth quotient space $Y = X//G$. We show that the algebraic differential forms on $X$ which are pull-backs of forms on $Y$ are exactly the $G$-invariant horizontal differential forms on $X$.

1. Introduction

Let $G$ be a reductive algebraic group of automorphisms of an affine algebraic variety $X$ over an algebraically closed field $k$ of characteristic zero. Denote by $k[X]$ the algebra of regular functions on $X$. The subalgebra $k[X]^G$ of $G$-invariants in $k[X]$ is finitely generated; let $Y = X//G$ be the affine algebraic variety such that $k[Y] = k[X]^G$. Then we have a pull-back map $\pi^* : \Omega^*(Y) \to \Omega^*(X)$, where $\Omega^*$ denotes the algebra of Kähler differential forms, see e.g. [K].

In the present note, we describe the image of $\pi^*$ in the case where $X$ and $Y$ are smooth. In other words, we characterize the differential forms on $X$ which can be written as sums of products of invariant functions and of their differentials.

Clearly, such forms are $G$-invariant and horizontal, that is, their interior product with any vector field induced by the $G$-action is zero. Let $\Omega^*_{\text{hor}}(X)^G$ be the space of $G$-invariant horizontal forms; such forms are also called basic.

Is any basic differential form in the image of $\pi^*$? This question was raised by P. Michor in the setting of a representation of a compact Lie group on a real vector space $V$, and of differential forms on $V$ with smooth coefficients (which do not coincide with Kähler differentials for the ring of smooth functions on $V$; but differential forms with polynomial coefficients coincide with Kähler differentials for the ring of polynomial functions). Michor showed that the answer is negative in general, but positive for polar representations; see [M1] and [M2].

Here we obtain the following

Theorem 1. Let $G$ be a reductive group of automorphisms of a smooth, affine variety $X$, and let $\pi : X \to Y$ be the quotient map. If $Y$ is smooth, then $\pi^* : \Omega^*(Y) \to \Omega^*_{\text{hor}}(X)^G$ is an isomorphism.

This proves a conjecture of V. L. Popov formulated in his talk at the problem session of the conference “Algebraic Groups”, Kazimierz (Poland), May 25-June 1,

Received by the editors October 25, 1996 and, in revised form, January 29, 1997.

1991 Mathematics Subject Classification. Primary 14L30, 22E99.
1996. Some applications, and a generalization to the case of a singular quotient space, are given in §3 below. The author thanks V. L. Popov, G. Schwarz (who gave another proof of Theorem 1, in the case of a representation of a connected group) and D. Zaitsev (who gave another proof of Theorem 1, in the case where \( G \) is finite) for useful discussions. Thanks are also due to the referee for suggesting simplifications of several points in the proof.

2. Proof of Theorem 1

Our approach is similar to the one in [B], §2. We begin with an observation concerning reflexive modules; this notion of commutative algebra can be defined as follows. Let \( A \) be a Noetherian domain, and let \( M \) be a finite \( A \)-module. Denote by \( M' = \text{Hom}_A(M,A) \) the dual module. Then we have a natural map from \( M \) to its double dual \( M'' \), and \( M \) is called reflexive if this map is an isomorphism.

**Lemma 1.** Let \( A \) be a normal Noetherian ring, let \( M, N \) be finite \( A \)-modules with \( M \) reflexive and \( N \) torsion-free, and let \( u : M \to N \) be an \( A \)-linear map. If \( u \) is an isomorphism in codimension one, then \( u \) is an isomorphism.

**Proof.** Set \( Z = \text{Spec}(A) \) and let \( U \subset Z \) be the open subset where \( u \) is an isomorphism. Denote by \( \mathcal{M}, \mathcal{N} \) the sheaves on \( Z \) associated to \( M, N \). Then we have a commutative square

\[
\begin{array}{ccc}
M & \to & \Gamma(U,\mathcal{M}) \\
\downarrow & & \downarrow \\
N & \to & \Gamma(U,\mathcal{N})
\end{array}
\]

By assumption, the map \( \Gamma(U,\mathcal{M}) \to \Gamma(U,\mathcal{N}) \) is an isomorphism. Moreover, the codimension of \( Z \setminus U \) in \( Z \) is at least two, and hence the map \( M \to \Gamma(U,\mathcal{M}) \) is an isomorphism; see [H], Proposition 1.6. Finally, the map \( N \to \Gamma(U,\mathcal{N}) \) is injective because \( N \) is torsion-free. These three facts imply our statement. 

We will check that Lemma 1 applies to the ring \( k[Y] \) and to the map \( \pi^* : \Omega^*(Y) \to \Omega^*_{\text{hor}}(X)^G \). Observe that both \( \Omega^*(Y) \) and \( \Omega^*_{\text{hor}}(X)^G \) are \( k[Y] \)-modules, and that \( \pi^* \) is \( k[Y] \)-linear. Because \( Y \) is smooth, \( \Omega^*(Y) \) is locally free over \( k[Y] \), and hence reflexive. Moreover, the \( k[X] \)-module \( \Omega^*(X) \) is finitely generated and torsion-free. Therefore, the same holds for the \( k[Y] \)-module \( \Omega^*(X)^G \) (see [P-V], Theorem 3.24) and for its submodule \( \Omega^*_{\text{hor}}(X)^G \).

So it suffices to check that \( \pi^* \) is an isomorphism in codimension one. For this, we will use Luna’s slice theorem, see [L1]. First recall a

**Definition.** An equivariant morphism \( f : X' \to X \) of affine \( G \)-varieties is excellent if

(i) \( f \) is étale,
(ii) the induced morphism \( f//G : X'//G := Y' \to Y \) is étale, and
(iii) the morphism \( (f,\pi) : X' \to X \times_Y Y' \) is an isomorphism.

Let \( y \in Y \). The fiber \( \pi^{-1}(y) \) contains a unique closed orbit, say \( G \cdot x \), and the isotropy group \( G_x := H \) is reductive. By Luna’s slice theorem, we have a locally closed smooth, affine, \( H \)-stable subvariety \( S \subset X \) which contains \( x \), with the following properties:

1. \( N := T_xS \) is an \( H \)-complement to \( T_x(G \cdot x) \) in \( T_xX \).
2. The canonical map \( G \times^H S \to X \) is excellent.
There is an excellent $H$-morphism $S \to N$ sending $x$ to the origin in $N$. (Hence the induced map $G \times H S \to G \times H N$ is excellent.)

We will also need the following

**Lemma 2.** Let $f : X' \to X$ be excellent, and set $Y' := X'//G$. Then both maps $k[Y'] \otimes_{k[Y]} \Omega^*_{\text{hor}}(X) \to \Omega^*_{\text{hor}}(X')^G$ and $k[Y'] \otimes_{k[Y]} \Omega^* (Y) \to \Omega^* (Y')$ are isomorphisms of $k[Y']$-modules.

**Proof.** Because $f//G$ is etale, the natural map $u : k[Y'] \otimes_{k[Y]} \Omega^1 (Y) \to \Omega^1 (Y')$ is surjective (indeed, the cokernel of $u$ is the space of relative differentials $\Omega^1 (Y'/Y)$ which vanishes because $f//G$ is unramified). Moreover, because $Y$ is smooth and $f//G$ is etale, $Y'$ is smooth, too. Therefore, the $k[Y']$-modules $k[Y'] \otimes_{k[Y]} \Omega^1 (Y)$ and $\Omega^1 (Y')$ are locally free of rank $\dim(Y') = \dim(Y)$. It follows that $u$ is an isomorphism. This implies the second statement.

Because $k[X] = k[X] \otimes_{k[Y]} k[Y']$ and $f$ is etale, there are equivariant isomorphisms $k[Y'] \otimes_{k[Y]} \Omega^* (X) \to k[X] \otimes_{k[X]} \Omega^* (X) \to \Omega^* (X')$. This implies the first statement. 

Using the above and Lemma 2 twice, one reduces Theorem 1 to spaces of the form $G \times H N$. Now, by Lemma 4.1 in [M1], restriction to $N \subset G \times H N$ induces an isomorphism $\Omega^*_{\text{hor}}(G \times H N)^G \simeq \Omega^*_{\text{hor}}(N)^H$, so we can reduce to the case of representations. Moreover, writing $N = N^H \oplus V$ as an $H$-representation, we can reduce to considering the action of $H$ on $V$, since the extra trivial factor just tensors the algebras $k[V]$ and $k[V]^H$ with $k[N^H]$ and the spaces $\Omega^*_{\text{hor}}(V)^H$ and $\Omega^* (V//H)$ with $\Omega^* (N^H)$.

Moreover, by Lemma 1, we may replace the quotient $Y$ by any open subset $U$ such that $Y \setminus U$ has codimension at least two in $Y$. Recall that $Y$ is stratified by slice types $(H, N)$; see [L1]. If $y \in Y$ is in the open stratum (resp. in a stratum of codimension one), then $V//H$ is a point (resp. is one-dimensional). So we can finally reduce to the case of a reductive group $H \subset \text{GL}(V)$, where $V^H = 0$ and $\dim(V//H) \leq 1$. Then the proof is concluded by the following lemmas.

**Lemma 3.** In the case where $V//H$ is a point, the spaces $\Omega^p_{\text{hor}} (V)^H$ vanish for all $p \geq 1$.

**Proof.** We may assume that $H$ is connected. First consider the case where $H$ is a torus. Then, because $V//H$ is a point, all weights of $H$ in $V$ lie in an open half-space. So the weights of $\wedge^p V^*$ lie in the opposite half-space. Observing that $\Omega^p (V) \simeq S^* (V^*) \otimes \wedge^p V^*$, where $S^* (V^*)$ is the symmetric algebra of $V^*$, we obtain $\Omega^p (V)^H = 0$ for $p \geq 1$, which implies our statement.

In the general case of reductive $H$, let $T \subset H$ be a maximal torus. By the Hilbert-Mumford criterion, there exists a $T$-submodule $W \subset V$ such that $V = HW$ and that all weights of $T$ in $W$ lie in an open half-space. Because the map

$$
\begin{align*}
f : H \times W & \to V \\
(g, w) & \mapsto g w
\end{align*}
$$

is surjective, we can find $w \in W$ such that the differential

$$
\begin{align*}
df_{x, w} : \text{Lie}(H) \times W & \to V \\
(\xi, z) & \mapsto \xi w + z
\end{align*}
$$

is surjective, where $\text{Lie}(H)$ denotes the Lie algebra of $H$. In other words, we have $V = \text{Lie}(H) w + W$. It follows that $\wedge^p V = \sum_{q=0}^p (\wedge^{p-q} \text{Lie}(H) w) \wedge (\wedge^q W)$. Let $\omega \in V$
we have a commutative diagram

\[
\begin{array}{ccc}
\Omega^p_h(V)^H & \to & \Omega^p_h(V//H) \\
\downarrow & & \downarrow \\
\Omega^p_h(X)^G & \to & \Omega^p_h(X//G)
\end{array}
\]

where the top horizontal map is pull-back by the morphism \(X//G \to X//H\). But the latter is an isomorphism by the Luna-Richardson restriction theorem; see [L2], Corollaire 4, and [L-R]. So, by Theorem 1, both vertical arrows are isomorphisms.

Finally, Theorem 1 can be extended to the case where the quotient space \(Y\) may be singular. Then we have to replace the \(k[Y]\)-module of Kähler differentials \(\Omega^*(Y)\) by its double dual \(\Omega^*(Y)^{\vee\vee}\), the module of Zariski-Lipman differential forms. By adapting the previous argument, we obtain the following

\textbf{Lemma 4.} In the case where \(V//H\) is one-dimensional, the map \(\Omega^1(V//H) \to \Omega^1_{hor}(V)^H\) is an isomorphism, and moreover \(\Omega^p_{hor}(V)^H = 0\) for all \(p \geq 2\).

\textit{Proof.} The algebra \(k[V]^H\) is generated by a non-constant homogeneous function \(f\). We have \(\Omega^1(V//H) = k[f] df\) and \(\Omega^p(V//H) = 0\) for \(p \geq 2\). By Lemma 3 and our reductions, the \(k[f]\)-linear map \(\Omega^*(V//H) \to \Omega^*_{hor}(V)^H\) is an isomorphism over the field \(k(f)\). Because the \(k[f]\)-module \(\Omega^*_{hor}(V)^H\) is torsion-free, it follows that \(\Omega^p_{hor}(V)^H = 0\) for \(p \geq 2\) and that \(\Omega^1_{hor}(V)^H = k[f] \omega\) for some homogeneous form \(\omega\). Let \(m\) be the degree of \(f\), and let \(n\) be the degree of \(\omega\). Because \(df\) is in \(k[f] \omega\), we have \(m - 1 = rm + n\) for some integer \(r \geq 0\). This forces \(r = 0\) and then \(\omega\) is a scalar multiple of \(df\).

3. Some applications and a generalization

Here are two applications of Theorem 1 which were pointed out by V. L. Popov.

First, we recover the following classical result of L. Solomon [So].

\textbf{Corollary 1.} Let \(G \subset \text{GL}(V)\) be a finite group generated by pseudo-reflections. Then any \(G\)-invariant differential form on \(V\) is the pull-back of a differential form on \(V//G\).

Indeed, the quotient space \(V//G\) is smooth, and Theorem 1 applies.

Another application is an algebraic version of a recent result of P. Michor: let \(G\) be a Lie group of isometries of a smooth Riemannian manifold \(M\). Assume that the \(G\)-action on \(M\) admits a section \(\Sigma\), and denote by \(W(\Sigma) = N_G(\Sigma)/Z_G(\Sigma)\) the corresponding “generalized Weyl group”. Then restriction of differential forms with smooth coefficients induces an isomorphism \(\Omega^*_{hor}(M)^G \to \Omega^*(\Sigma)^W(\Sigma)\); see [M1], Theorem 3.7, and [M2], Theorem 2.

\textbf{Corollary 2.} Let \(X\) be a smooth irreducible affine variety with an action of a reductive group \(G\). Denote by \(H \subset G\) a principal isotropy group of \(X\), by \(W\) the quotient group \(N_G(H)/H\), and by \(\Sigma \subset X^H\) the closure of the subset of principal points. If the quotient \(X//G\) is smooth, then restriction to \(\Sigma\) induces an isomorphism \(\Omega^*_{hor}(X)^G \simeq \Omega^*_{hor}(\Sigma)^W\).

Indeed, we have a commutative diagram

\[
\begin{array}{ccc}
\Omega^*(X//G) & \to & \Omega^*(\Sigma//W) \\
\downarrow & & \downarrow \\
\Omega^*_{hor}(X)^G & \to & \Omega^*_{hor}(\Sigma)^W
\end{array}
\]
Theorem 2. Let $G$ be a reductive group acting on a smooth affine variety $X$, and let $\pi : X \to Y$ be the quotient map. If no divisor in $X$ is mapped by $\pi$ to a subvariety of codimension at least two in $Y$, then $\pi^* : \Omega^*(Y) \to \Omega^*_{\text{hor}}(X)^G$ factors through an isomorphism $\Omega^*(Y)^{\vee \vee} \to \Omega^*_{\text{hor}}(X)^G$.

In particular, the Zariski-Lipman differential forms on a quotient of a smooth variety $X$ by a finite group are exactly the invariant differential forms on $X$.

Proof. The assumption that no divisor in $X$ is contracted by $\pi$ implies that the $k[Y]$-module $\Omega^*_{\text{hor}}(X)^G$ is reflexive, using [H] Proposition 1.6. Therefore, $\pi^*$ factors through a map $\Omega^*(Y)^{\vee \vee} \to \Omega^*_{\text{hor}}(X)^G$. Furthermore, this map is an isomorphism, by Lemma 1, smoothness in codimension one of $Y$, and Theorem 1.

We ignore whether the assumption that $\pi$ contracts no divisor is necessary for Theorem 2 to hold. Observe that the Zariski-Lipman differentials cannot be replaced by Kähler differentials in this statement, as shown by the following example.

Let $G$ be a cyclic group of order $n$ acting on $X = k^2$ by scalar multiplication by $n$-th roots of unity. Then the algebra $k[X]^G$ is generated by all monomials of total degree $n$ in the coordinates $x_1, x_2$. Therefore, the differentials of these monomials are the minimal system of homogeneous generators of the $k[Y]$-module $\Omega^1(Y)$; this system consists of $n + 1$ elements. But the $k[Y]$-module $\Omega^1_{\text{hor}}(X)^G = \Omega^1(X)^G$ is minimally generated by $2n$ homogeneous elements, products of monomials of degree $n - 1$ by $dx_1, dx_2$. So the quotient $\Omega^1(X)^G / \pi^* \Omega^1(Y)$ has dimension $n - 1$.

References


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