DIFFERENTIAL FORMS ON QUOTIENTS
BY REDUCTIVE GROUP ACTIONS

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Abstract. Let $X$ be a smooth affine algebraic variety where a reductive algebraic group $G$ acts with a smooth quotient space $Y = X//G$. We show that the algebraic differential forms on $X$ which are pull-backs of forms on $Y$ are exactly the $G$-invariant horizontal differential forms on $X$.

1. Introduction

Let $G$ be a reductive algebraic group of automorphisms of an affine algebraic variety $X$ over an algebraically closed field $k$ of characteristic zero. Denote by $k[X]$ the algebra of regular functions on $X$. The subalgebra $k[X]^G$ of $G$-invariants in $k[X]$ is finitely generated; let $Y = X//G$ be the affine algebraic variety such that $k[Y] = k[X]^G$ and let $\pi : X \to Y$ be the quotient map defined by the inclusion $k[Y] \subset k[X]$. Then we have a pull-back map $\pi^* : \Omega^*(Y) \to \Omega^*(X)$, where $\Omega^*$ denotes the algebra of Kähler differential forms, see e.g. [K].

In the present note, we describe the image of $\pi^*$ in the case where $X$ and $Y$ are smooth. In other words, we characterize the differential forms on $X$ which can be written as sums of products of invariant functions and of their differentials. Clearly, such forms are $G$-invariant and horizontal, that is, their interior product with any vector field induced by the $G$-action is zero. Let $\Omega^*_{hor}(X)^G$ be the space of $G$-invariant horizontal forms; such forms are also called basic.

Is any basic differential form in the image of $\pi^*$? This question was raised by P. Michor in the setting of a representation of a compact Lie group on a real vector space $V$, and of differential forms on $V$ with smooth coefficients (which do not coincide with Kähler differentials for the ring of smooth functions on $V$; but differential forms with polynomial coefficients coincide with Kähler differentials for the ring of polynomial functions). Michor showed that the answer is negative in general, but positive for polar representations; see [M1] and [M2].

Here we obtain the following

Theorem 1. Let $G$ be a reductive group of automorphisms of a smooth, affine variety $X$, and let $\pi : X \to Y$ be the quotient map. If $Y$ is smooth, then $\pi^* : \Omega^*(Y) \to \Omega^*_{hor}(X)^G$ is an isomorphism.

This proves a conjecture of V. L. Popov formulated in his talk at the problem session of the conference “Algebraic Groups”, Kazimierz (Poland), May 25-June 1,
1996. Some applications, and a generalization to the case of a singular quotient space, are given in §3 below. The author thanks V. L. Popov, G. Schwarz (who gave another proof of Theorem 1, in the case of a representation of a connected group) and D. Zaitsev (who gave another proof of Theorem 1, in the case where $G$ is finite) for useful discussions. Thanks are also due to the referee for suggesting simplifications of several points in the proof.

2. Proof of Theorem 1

Our approach is similar to the one in [B], §2. We begin with an observation concerning reflexive modules; this notion of commutative algebra can be defined as follows. Let $A$ be a Noetherian domain, and let $M$ be a finite $A$-module. Denote by $M' = \text{Hom}_A(M,A)$ the dual module. Then we have a natural map from $M$ to its double dual $M''$, and $M$ is called reflexive if this map is an isomorphism.

**Lemma 1.** Let $A$ be a normal Noetherian ring, let $M$, $N$ be finite $A$-modules with $M$ reflexive and $N$ torsion-free, and let $u : M \to N$ be an $A$-linear map. If $u$ is an isomorphism in codimension one, then $u$ is an isomorphism.

**Proof.** Set $Z = \text{Spec}(A)$ and let $U \subset Z$ be the open subset where $u$ is an isomorphism. Denote by $\mathcal{M}$, $\mathcal{N}$ the sheaves on $Z$ associated to $M$, $N$. Then we have a commutative square

\[
\begin{array}{ccc}
M & \to & \Gamma(U,\mathcal{M}) \\
\downarrow & & \downarrow \\
N & \to & \Gamma(U,\mathcal{N})
\end{array}
\]

By assumption, the map $\Gamma(U,\mathcal{M}) \to \Gamma(U,\mathcal{N})$ is an isomorphism. Moreover, the codimension of $Z \setminus U$ in $Z$ is at least two, and hence the map $M \to \Gamma(U,\mathcal{M})$ is an isomorphism; see [H], Proposition 1.6. Finally, the map $N \to \Gamma(U,\mathcal{N})$ is injective because $N$ is torsion-free. These three facts imply our statement.

We will check that Lemma 1 applies to the ring $k[Y]$ and to the map $\pi^* : \Omega^*(Y) \to \Omega^*_{\text{hor}}(X)^G$. Observe that both $\Omega^*(Y)$ and $\Omega^*_{\text{hor}}(X)^G$ are $k[Y]$-modules, and that $\pi^*$ is $k[Y]$-linear. Because $Y$ is smooth, $\Omega^*(Y)$ is locally free over $k[Y]$, and hence reflexive. Moreover, the $k[X]$-module $\Omega^*(X)$ is finitely generated and torsion-free. Therefore, the same holds for the $k[Y]$-module $\Omega^*(X)^G$ (see [P-V], Theorem 3.24) and for its submodule $\Omega^*_{\text{hor}}(X)^G$.

So it suffices to check that $\pi^*$ is an isomorphism in codimension one. For this, we will use Luna’s slice theorem, see [L1]. First recall a

**Definition.** An equivariant morphism $f : X' \to X$ of affine $G$-varieties is excellent if

(i) $f$ is étale,
(ii) the induced morphism $f//G : X'//G := Y' \to Y$ is étale, and
(iii) the morphism $(f,\pi) : X' \to X \times_Y Y'$ is an isomorphism.

Let $y \in Y$. The fiber $\pi^{-1}(y)$ contains a unique closed orbit, say $G \cdot x$, and the isotropy group $G_x := H$ is reductive. By Luna’s slice theorem, we have a locally closed smooth, affine, $H$-stable subvariety $S \subset X$ which contains $x$, with the following properties:

(1) $N := T_xS$ is an $H$-complement to $T_x(G \cdot x)$ in $T_xX$.
(2) The canonical map $G \times^H S \to X$ is excellent.
(3) There is an excellent $H$-morphism $S \to N$ sending $x$ to the origin in $N$. (Hence the induced map $G \times H \to G \times H \to N$ is excellent.)

We will also need the following

**Lemma 2.** Let $f : X' \to X$ be excellent, and set $Y' := X'/\!/G$. Then both maps $k[Y'] \otimes_{k[Y]} \Omega^*_\text{hor}(X)^G \to \Omega^*_\text{hor}(X')^G$ and $k[Y'] \otimes_{k[Y]} \Omega^*(Y) \to \Omega^*(Y')$ are isomorphisms of $k[Y']$-modules.

**Proof.** Because $f/\!/G$ is etale, the natural map $u : k[Y'] \otimes_{k[Y]} \Omega^1(Y) \to \Omega^1(Y')$ is surjective (indeed, the cokernel of $u$ is the space of relative differentials $\Omega^1(Y'/Y)$ which vanishes because $f/\!/G$ is unramified). Moreover, because $Y$ is smooth and $f/\!/G$ is etale, $Y'$ is smooth, too. Therefore, the $k[Y']$-modules $k[Y'] \otimes_{k[Y]} \Omega^1(Y)$ and $\Omega^1(Y')$ are locally free of rank $\dim(Y') = \dim(Y)$. It follows that $u$ is an isomorphism. This implies the second statement.

Because $k[X'] = k[X] \otimes_{k[Y]} k[Y']$ and $f$ is etale, there are equivariant isomorphisms $k[Y'] \otimes_{k[Y]} \Omega^*(X) \to k[X] \otimes_{k[X]} \Omega^*(X) \to \Omega^*(X')$. This implies the first statement. 

Using the above and Lemma 2 twice, one reduces Theorem 1 to spaces of the form $G \times H \times N$. Now, by Lemma 4.1 in [M1], restriction to $N \subset G \times H \times N$ induces an isomorphism $\Omega^*_\text{hor}(G \times H \times N)^G \simeq \Omega^*_\text{hor}(N)^H$, so we can reduce to the case of representations. Moreover, writing $N = N^H \oplus V$ as an $H$-representation, we can reduce to considering the action of $H$ on $V$, since the extra trivial factor just tensors the algebras $k[V]$ and $k[V]^H$ with $k[N^H]$ and the spaces $\Omega^*_\text{hor}(V)^H$ and $\Omega^*(V/\!/H)$ with $\Omega^*(N^H)$.

Moreover, by Lemma 1, we may replace the quotient $Y$ by any open subset $U$ such that $Y \setminus U$ has codimension at least two in $Y$. Recall that $Y$ is stratified by slice types $(H, N)$; see [L1]. If $y \in Y$ is in the open stratum (resp. in a stratum of codimension one), then $V/\!/H$ is a point (resp. is one-dimensional). So we can finally reduce to the case of a reductive group $H \subset GL(V)$, where $V^H = 0$ and $\dim(V/\!/H) \leq 1$. Then the proof is concluded by the following lemmas.

**Lemma 3.** In the case where $V/\!/H$ is a point, the spaces $\Omega^p_{\text{hor}}(V)^H$ vanish for all $p \geq 1$.

**Proof.** We may assume that $H$ is connected. First consider the case where $H$ is a torus. Then, because $V/\!/H$ is a point, all weights of $H$ in $V$ lie in an open half-space. So the weights of $\wedge^p V^*$ lie in the opposite half-space. Observing that $\Omega^p(V) \simeq S^*(V^*) \otimes \wedge^p V^*$, where $S^*(V^*)$ is the symmetric algebra of $V^*$, we obtain $\Omega^p(V)^H = 0$ for $p \geq 1$, which implies our statement.

In the general case of reductive $H$, let $T \subset H$ be a maximal torus. By the Hilbert-Mumford criterion, there exists a $T$-submodule $W \subset V$ such that $V = HW$ and that all weights of $T$ in $W$ lie in an open half-space. Because the map

$$f : H \times W \to V$$

$$\begin{array}{c}
(g, w) \\
\mapsto gw
\end{array}$$

is surjective, we can find $w \in W$ such that the differential

$$df_{e,w} : \text{Lie}(H) \times W \to V$$

$$\begin{array}{c}
(\xi, z) \\
\mapsto \xi w + z
\end{array}$$

is surjective, where $\text{Lie}(H)$ denotes the Lie algebra of $H$. In other words, we have $V = \text{Lie}(H)w + W$. It follows that $\wedge^p V = \sum_{q=0}^p (\wedge^{p-q} \text{Lie}(H)w) \wedge (\wedge^q W)$. Let $\omega \in$
we have \( \Omega^p_{hor}(V)^H \). Because \( \omega \) is horizontal, it vanishes on \( \sum_{q=0}^{p-1} (\wedge^p \text{Lie}(H) w) \wedge (\wedge^q W) \).

Moreover, restriction of \( \omega \in S^*(V^*) \otimes \wedge^p V^* \) to \( W \) is a \( T \)-equivariant morphism from \( W \) to \( \wedge^p V^* \). By the first step of the proof, this morphism vanishes on the subspace \( \wedge^p W \) of \( \wedge^p V^* \). So restriction of \( \omega \) to \( W \) is trivial. Because \( V = HW \) and \( \omega \) is \( H \)-invariant, we conclude that \( \omega = 0 \).

**Lemma 4.** In the case where \( V//H \) is one-dimensional, the map \( \Omega^1(V//H) \rightarrow \Omega^1_{hor}(V)^H \) is an isomorphism, and moreover \( \Omega^p_{hor}(V)^H = 0 \) for all \( p \geq 2 \).

**Proof.** The algebra \( k[V]^H \) is generated by a non-constant homogeneous function \( f \). We have \( \Omega^1(V//H) = k[f] df \) and \( \Omega^p(V//H) = 0 \) for \( p \geq 2 \). By Lemma 3 and our reductions, the \( k[f] \)-linear map \( \Omega^*(V//H) \rightarrow \Omega^*_{hor}(V)^H \) is an isomorphism over the field \( k(f) \). Because the \( k[f] \)-module \( \Omega^*_{hor}(V)^H \) is torsion-free, it follows that \( \Omega^p_{hor}(V)^H = 0 \) for \( p \geq 2 \) and that \( \Omega^1_{hor}(V)^H = k[f] \omega \) for some homogeneous form \( \omega \). Let \( m \) be the degree of \( f \), and let \( n \) be the degree of \( \omega \). Because \( df \) is in \( k[f] \omega \), we have \( m-1 = rm + n \) for some integer \( r \geq 0 \). This forces \( r = 0 \) and then \( \omega \) is a scalar multiple of \( df \).

### 3. Some Applications and a Generalization

Here are two applications of Theorem 1 which were pointed out by V. L. Popov.

**Corollary 1.** Let \( G \subset \text{GL}(V) \) be a finite group generated by pseudo-reflections. Then any \( G \)-invariant differential form on \( V \) is the pull-back of a differential form on \( V/G \).

Indeed, the quotient space \( V/G \) is smooth, and Theorem 1 applies.

Another application is an algebraic version of a recent result of P. Michor: let \( G \) be a Lie group of isometries of a smooth Riemannian manifold \( M \). Assume that the \( G \)-action on \( M \) admits a section \( \Sigma \), and denote by \( W(\Sigma) = N_G(\Sigma)/Z_G(\Sigma) \) the corresponding “generalized Weyl group”. Then restriction of differential forms with smooth coefficients induces an isomorphism \( \Omega^*_\text{hor}(M)^G \rightarrow \Omega^*_\text{hor}(\Sigma)^{W(\Sigma)} \); see [M1], Theorem 3.7, and [M2], Theorem 2.

**Corollary 2.** Let \( X \) be a smooth irreducible affine variety with an action of a reductive group \( G \). Denote by \( H \subset G \) a principal isotropy group of \( X \), by \( W \) the quotient group \( N_G(H)/H \), and by \( \Sigma \subset X^H \) the closure of the subset of principal points. If the quotient \( X//G \) is smooth, then restriction to \( \Sigma \) induces an isomorphism \( \Omega^*_\text{hor}(X)^G \simeq \Omega^*_\text{hor}(\Sigma)^W \).

Indeed, we have a commutative diagram

\[
\begin{array}{ccc}
\Omega^*(X//G) & \rightarrow & \Omega^*(\Sigma//W) \\
\downarrow & & \downarrow \\
\Omega^*_\text{hor}(X)^G & \rightarrow & \Omega^*_\text{hor}(\Sigma)^W
\end{array}
\]

where the top horizontal map is pull-back by the morphism \( \Sigma//W \rightarrow X//G \). But the latter is an isomorphism by the Luna-Richardson restriction theorem; see [L2], Corollaire 4, and [L-R]. So, by Theorem 1, both vertical arrows are isomorphisms.

Finally, Theorem 1 can be extended to the case where the quotient space \( Y \) may be singular. Then we have to replace the \( k[Y] \)-module of Kähler differentials \( \Omega^*(Y) \) by its double dual \( \Omega^*(Y)^{vv} \), the module of Zariski-Lipman differential forms. By adapting the previous argument, we obtain the following
Theorem 2. Let $G$ be a reductive group acting on a smooth affine variety $X$, and let $\pi : X \to Y$ be the quotient map. If no divisor in $X$ is mapped by $\pi$ to a subvariety of codimension at least two in $Y$, then $\pi^* : \Omega^*(Y) \to \Omega^*_\text{hor}(X)^G$ factors through an isomorphism $\Omega^*(Y)^\vee \to \Omega^*_\text{hor}(X)^G$.

In particular, the Zariski-Lipman differential forms on a quotient of a smooth variety $X$ by a finite group are exactly the invariant differential forms on $X$.

Proof. The assumption that no divisor in $X$ is contracted by $\pi$ implies that the $k[Y]$-module $\Omega^*_\text{hor}(X)^G$ is reflexive, using [H] Proposition 1.6. Therefore, $\pi^*$ factors through a map $\Omega^*(Y)^\vee \to \Omega^*_\text{hor}(X)^G$. Furthermore, this map is an isomorphism, by Lemma 1, smoothness in codimension one of $Y$, and Theorem 1. $\square$

We ignore whether the assumption that $\pi$ contracts no divisor is necessary for Theorem 2 to hold. Observe that the Zariski-Lipman differentials cannot be replaced by Kähler differentials in this statement, as shown by the following example.

Let $G$ be a cyclic group of order $n$ acting on $X = k^2$ by scalar multiplication by $n$-th roots of unity. Then the algebra $k[X]^G$ is generated by all monomials of total degree $n$ in the coordinates $x_1, x_2$. Therefore, the differentials of these monomials are the minimal system of homogeneous generators of the $k[Y]$-module $\Omega^1(Y)$; this system consists of $n+1$ elements. But the $k[Y]$-module $\Omega^1_{\text{hor}}(X)^G = \Omega^1(X)^G$ is minimally generated by $2n$ homogeneous elements, products of monomials of degree $n-1$ by $dx_1, dx_2$. So the quotient $\Omega^1(X)^G/\pi^*\Omega^1(Y)$ has dimension $n-1$.

References


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