CAUCHY-SCHWARZ AND MEANS INEQUALITIES
FOR ELEMENTARY OPERATORS INTO NORM IDEALS

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Abstract. The Cauchy-Schwarz norm inequality for normal elementary operators
\[ \left\| \sum_{n=1}^{\infty} A_n XB_n \right\| \leq \left( \sum_{n=1}^{\infty} A_n^* A_n \right)^{1/2} \left( \sum_{n=1}^{\infty} B_n^* B_n \right)^{1/2}, \]
implies a means inequality for generalized normal derivations
\[ \left\| AX + XB \right\| \leq \left\| X \right\|^{1-\frac{1}{r}} \left\| A \right\|^{\frac{1}{p}} \left\| X \right\|^{\frac{1}{r}} \left\| B \right\|^{\frac{1}{p}}, \]
for all \( r \geq 2 \), as well as an inequality for normal contractions \( A \) and \( B \)
\[ \left\| (I - A^* A)^{\frac{1}{2}} X (I - B^* B)^{\frac{1}{2}} \right\| \leq \left\| X - AXB \right\|, \]
for all \( X \) in \( B(H) \) and for all unitarily invariant norms \( \| \cdot \| \).

1. Introduction

Let \( B(H) \) and \( C_\infty \) stand respectively for spaces of all bounded and all compact linear operators acting on a separable, infinite-dimensional, complex Hilbert space \( H \). For an \( X \in B(H) \) let \( \| X \| \) denote its norm, and for an arbitrary \( X \in C_\infty \) let \( s_1(X) \geq s_2(X) \geq \cdots \) denote the singular values of \( X \), i.e., the eigenvalues of \( |X| = (X^*X)^{1/2} \), arranged in a non-increasing order, with their multiplicities counted. Each “symmetric gauge function” \( \Phi \) on sequences gives rise to a unitarily invariant (u.i.) norm on operators defined by \( \| A \|_\Phi = \Phi(\{s_n(A)\}) \). We will denote by the symbol \( \| \cdot \| \) any such norm. Any such norm is defined on a natural subclass \( C_\| \) of \( C_\infty \) called the norm ideal associated with the norm \( \| \cdot \| \), and satisfies the invariance property \( \| UAV \| = \| A \| \) for all \( A \) in this ideal and for all unitary operators \( U, V \). Each norm ideal \( C_\| \) is closed in the topology generated by the norm \( \| \cdot \| \).

Particularly well known among unitarily invariant norms are the Schatten \( p \)-norms, defined as \( \| X \|_p = (\sum s_n^p(A))^{1/p} \) for \( 1 \leq p < \infty \), and \( \| X \|_\infty = \| X \| = s_1(X) \), which represent the norms on the Schatten \( p \)-ideals \( C_p \). The Ky Fan norms, defined as \( \| A \|_k = \Phi_k(s_1(A)) = \sum_{i=1}^{k} s_i(A) \) for \( k = 1, 2, \cdots \), represent another interesting family of unitarily invariant norms. The associated ideals \( C_\infty^{(k)} \) consist of all compact operators as every Ky Fan \( k \)-norm is equivalent to the norm in \( C_\infty \). The property saying that for all \( X \in C_\infty \) and \( Y \in C_\| \) with \( \| X \|_k \leq \| Y \|_k \) for all \( k \geq 1 \) we have...

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X ∈ C_{||-||} with ||X|| ≤ ||Y|| is known as the Ky Fan dominance property ([GK], ch. 3, §4). For a complete account of the theory of norm ideals, the reader is referred to [GK], [Sch] and [Si].

If A = (A_1, ..., A_N) and B = (B_1, ..., B_N) are N-tuples of bounded Hilbert space operators, then the elementary operator R = R_{A,B} on B(H) is defined by R(X) = \sum_{n=1}^{N} A_n X B_n. Elementary operators were introduced by Lumer and Rosenblum in [LR], who studied their spectral properties. In this setting many authors subsequently studied spectral, algebraic, metric and structural properties of elementary operators (see [F83], [F85], [F87], [FL], [McI], and the references therein).

If both \{A_n\}_{n=1}^{N} and \{B_n\}_{n=1}^{N} are families of commuting normal operators, one can easily show that the associated elementary operator R_{A,B} is normal when restricted to the Hilbert space C_2. So, in the sequel, by a normal elementary operator we will always mean such an operator, including those cases of N = ∞ with the convergence guaranteed. As was shown by G. Weiss in his paper [W83], a famous Fuglede-Putnam type theorem extends to such operators. Very important examples of normal elementary operators are so-called “pinching” operators

\[ R_T(X) = \sum_{n=1}^{∞} P_n X P_n, \]

generated by a family of mutually orthogonal self-adjoint projections \{P_n\}_{n=1}^{∞}.

2. Main results

We start with the basic Cauchy-Schwarz norm inequality for normal elementary operators. The following theorem extends “pinching” theorems 2.5.1 of [GK] and 1.19 of [Si].

**Theorem 2.1.** If \[ \sum_{n=1}^{∞} C_n^* C_n \leq 1, \sum_{n=1}^{∞} C_n C_n^* \leq 1, \sum_{n=1}^{∞} D_n^* D_n \leq 1 \text{ and } \sum_{n=1}^{∞} D_n^* C_n \leq 1 \] for some operator families \{C_n\}_{n=1}^{∞} and \{D_n\}_{n=1}^{∞}, then also \[ \sum_{n=1}^{∞} C_n Y D_n \in C_{||-||} \text{ whenever } Y \in C_{||-||} \text{ for some unitarily invariant norm } ||-||, \]

and moreover

\[ \left\| \sum_{n=1}^{∞} C_n Y D_n \right\| \leq ||Y||. \]

**Proof.** For arbitrary \( f \) and \( g \) in \( H \) a straightforward calculation gives

\[
\left| \left( \sum_{n=1}^{∞} C_n Y D_n f, g \right) \right| \leq \sum_{n=1}^{∞} ||Y|| \left\| D_n f \right\| \left\| C_n^* g \right\|
\]

\[
\leq ||Y|| \left( \sum_{n=1}^{∞} \left\| D_n f \right\|^2 \right)^{1/2} \left( \sum_{n=1}^{∞} \left\| C_n^* g \right\|^2 \right)^{1/2}
\]

\[
= ||Y|| \left( \sum_{n=1}^{∞} D_n^* D_n f, f \right)^{1/2} \left( \sum_{n=1}^{∞} C_n C_n^* g, g \right)^{1/2}
\]

\[
= ||Y|| \left( \sum_{n=1}^{∞} C_n C_n^* \right)^{1/2} \left( \sum_{n=1}^{∞} D_n^* D_n \right)^{1/2} \leq ||Y|| \left\| f \right\| \left\| g \right\|
\]

from which we conclude that

\[ \left\| \sum_{n=1}^{∞} C_n Y D_n \right\| \leq ||Y||. \]
Therefore, for all $N = 1, 2, \cdots$, for $Y \in \mathcal{C}_1$ and for all $W \in B(H)$ we have

$$|\text{tr}(\sum_{n=1}^{N} C_n Y D_n W^*)| = |\text{tr}(\sum_{n=1}^{N} C_n^* W D_n^*)|$$

$$\leq \|Y\|_1 \|\sum_{n=1}^{N} C_n^* W D_n^*\| \leq \|Y\|_1 \|W\|,$$

according to (2.2), from which we deduce that

$$\sum_{n=1}^{N} C_n Y D_n \|_1 \leq \|Y\|_1.$$  \hspace{1cm} (2.3)

If $Y \in \mathcal{C}_\infty$, let $Y = \sum_{n=1}^{\infty} s_n(Y) \langle \cdot, e_n \rangle f_n$ be a singular value decomposition for some orthonormal systems $\{e_n\}$ and $\{f_n\}$. For all $k \geq 2$ we introduce operators $Z = \sum_{n=1}^{k-1} (s_n(Y) - s_{n+1}(Y)) \sum_{j=1}^{n} \langle \cdot, e_j \rangle f_j$ and $V = s_k(Y) \sum_{n=1}^{k} \langle \cdot, e_n \rangle f_n$. We see that

$$Z = \sum_{n=1}^{k-1} \sum_{j=1}^{n} (s_n(Y) - s_{n+1}(Y)) \langle \cdot, e_j \rangle f_j$$

$$= \sum_{j=1}^{k} (s_j(Y) - s_k(Y)) \langle \cdot, e_j \rangle f_j$$

$$= \sum_{n=1}^{k} s_n(Y) \langle \cdot, e_n \rangle f_n + s_k(Y) \sum_{n=1}^{k} \langle \cdot, e_n \rangle f_n = Y - V.$$

We can also note that $s_1(Y) = \cdots = s_k(Y) = s_k(Y)$, due to orthogonality of the systems $\{e_n\}$ and $\{f_n\}$. That will allows us to conclude that for all Ky Fan $k$-norms we have

$$\left\| \sum_{n=1}^{N} C_n Y D_n \right\|_k \leq \left\| \sum_{n=1}^{N} C_n Z D_n \right\|_k + \left\| \sum_{n=1}^{N} C_n V D_n \right\|_k$$

$$\leq \|Z\|_1 + k \left\| \sum_{n=1}^{N} C_n Z D_n \right\|_\infty$$

$$\leq \sum_{n=1}^{k-1} (s_n(Y) - s_{n+1}(Y)) \sum_{j=1}^{n} \| \langle \cdot, e_j \rangle f_j \|_\infty + k \|V\|_\infty$$

$$\leq \sum_{n=1}^{k-1} n(s_n(Y) - s_{n+1}(Y)) + ks_k(Y) = \sum_{n=1}^{k-1} s_n(Y) = \|Y\|_k,$$

with (2.4) following from (2.3) and (2.5) from (2.2).

Moreover, if $Y$ is in $\mathcal{C}_\infty$ then also $\sum_{n=1}^{\infty} C_n Y D_n \in \mathcal{C}_\infty$. Indeed, elementary operators $R_N(Y) = \sum_{n=1}^{N} C_n Y D_n$ acting on $\mathcal{C}_\infty^{(k)}$ represent a bounded family, because $\|R_N(Y)\|_k \leq \|Y\|_k$ for all $Y \in \mathcal{C}_\infty$ by (2.6). Also, for one dimensional operators
f \otimes g and M > N we have

\[\|R_M(f \otimes g) - R_N(f \otimes g)\|_k \leq \|\sum_{n=N+1}^{M} D_n^* f \otimes C_n g\|_1\]

\[\leq \sum_{n=N+1}^{M} \|D_n^* f\| \|C_n g\| \leq \left(\sum_{n=N+1}^{M} C_n C_n^*\right)^{1/2} g\| \left(\sum_{n=N+1}^{M} D_n^* D_n\right)^{1/2} f\|,
\]

which \(\to 0\) as \(M, N \to \infty\). Therefore \(R_N(Y)\) converge in \(C^k\) for all finite dimensional \(Y\) to a compact operator. By the uniform boundedness principle the same is true for all \(Y \in C^k\), due to its separability. So (2.1) holds for all Ky Fan \(k\)-norms, and we therefore invoke the Ky Fan dominance property to conclude that (2.1) holds for all unitarily invariant norms, as required.

In the sequel we will refer to a family \(\{A_n\}_{n=1}^{\infty}\) in \(B(H)\) as square summable if \(\sum_{n=1}^{\infty} \|A_n f\|^2 < \infty\) for all \(f \in H\). Though this means just the weak convergence of \(\sum_{n=1}^{\infty} A_n^* A_n\), an appeal to the resonance principle shows that \(\sum_{n=1}^{\infty} A_n^* A_n\) actually defines a bounded Hilbert space operator, and due to the monotonicity of its partial sums, the convergence is moreover strong. For such families the following Cauchy-Schwarz inequality holds:

**Theorem 2.2.** For a square summable families \(\{A_n\}_{n=1}^{\infty}\) and \(\{B_n\}_{n=1}^{\infty}\) of commuting normal operators

\[(2.7) \quad \left\| \sum_{n=1}^{\infty} A_n X B_n \right\| \leq \left\| \sum_{n=1}^{\infty} A_n^* A_n^{1/2} X \sum_{n=1}^{\infty} B_n^* B_n^{1/2} \right\|,
\]

for all \(X \in B(H)\) and for all u.i. norms \(\|.\|\). If \(C_1\|\cdot\|\) is separable and \(X \in C_1\), then the left-hand side sum converges in the norm of this ideal.

**Proof.** First, we need a suitable factorization for Hilbert space operators \(A_n\) and \(B_n\). Let \(A = (\sum_{n=1}^{\infty} A_n^* A_n)^{1/2}\) and \(B = (\sum_{n=1}^{\infty} B_n^* B_n)^{1/2}\), and let \(P\) and \(Q\) denote respectively the orthogonal projections on \(R(A^2)\) and \(R(B^2)\). If for a given \(f \in H\) we have that \(P f = \lim_k \to \infty A g_k\) for some sequence \(\{g_k\}\) in \(H\), then \(\lim_k \to \infty A_n g_k\) exists for all \(n \geq 1\) and does not depend on the chosen sequence. Indeed,

\[\|A_n g_k - A_n g_l\| \leq \|A(g_k - g_l)\| \to 0\] as \(k, l \to \infty\), and also \(\|A_n g_k - A_n h_k\| \leq \|A(g_k - h_k)\| \to 0\) as \(k \to \infty\) whenever \(\lim_k \to \infty A h_k = Pf\) for some other sequence \(\{h_k\}\). Thus we can correctly introduce operators \(C_n, n = 1, 2, \ldots\), by \(C_n f = \lim_k \to \infty A_n g_k\), where \(\{g_k\}\) is any sequence in \(H\) such that \(\lim_k \to \infty A g_k = Pf\). Let us note that due to our definition every \(C_n\) vanishes on \(N(A)\), i.e., \(C_n = C_n P\), and also \(C_n A = AC_n = A_n\). Moreover, \(\sum_{n=1}^{\infty} C_n C_n^* = P\). Indeed, \(\sum_{n=1}^{\infty} C_n^* C_n A^2 = \sum_{n=1}^{\infty} A_n^* A_n = A^2\) implies \(\sum_{n=1}^{\infty} C_n^* C_n P = P\), which together with the fact that \(C_n (I - P) = 0\) gives the desired conclusion. For all \(m, n = 1, 2, \ldots\), \(C_m\) and \(C_n\) commute on \(R(A^2)\) and \(N(A^2)\), and so also on all of \(H\). Thus \(\{C_n\}_{n=1}^{\infty}\) is a commuting family of normal contractions which realize the factorizations \(C_n A = AC_n = A_n\), with \(\sum_{n=1}^{\infty} C_n^* C_n = P\), and which commute with the family \(\{A_n\}_{n=1}^{\infty}\). Similarly we get a commuting family \(\{D_n\}_{n=1}^{\infty}\) of normal contractions which also commute with \(\{B_n\}_{n=1}^{\infty}\) and satisfy \(D_n B = BD_n = B_n\) and \(\sum_{n=1}^{\infty} D_n^* D_n = Q\). One could easily derive the next explicit
\( C_n = A_n A^\dagger = A^\dagger A_n \), where \( A^\dagger \) denotes a (densely defined) Moore-Penrose (generalized) inverse for \( A \).

For \( Y = AXB \in C_{1,1} \) (there is nothing to prove in the opposite case), an application of Theorem 2.1 gives

\[
\left\| \sum_{n=1}^\infty A_n X B_n \right\| = \left\| \sum_{n=1}^\infty C_n Y D_n \right\|
\]

(2.8)

\[
\leq \| Y \| = \left\| \left( \sum_{n=1}^\infty A_n^* A_n \right)^{1/2} X \left( \sum_{n=1}^\infty B_n^* B_n \right)^{1/2} \right\|
\]

which proves the first part of theorem.

Finally, if \( C_{1,1} \) is separable, then for all \( N = 1, 2, \ldots \), an application of the just proven part of theorem combined with the arithmetic-geometric means inequality in [BhD] gives

\[
\left\| \sum_{n=N}^\infty A_n X B_n \right\| \leq \left( \sum_{n=N}^\infty A_n^* A_n \right)^{1/2} X \left( \sum_{n=N}^\infty B_n^* B_n \right)^{1/2}
\]

(2.9)

\[
\leq \frac{1}{2} \left( \sum_{n=N}^\infty C_n^* C_n \right) A X B + A X B \left( \sum_{n=N}^\infty D_n^* D_n \right)
\]

We see by (2.8) that \( \{\sum_{n=N}^\infty C_n^* C_n\}_{N=1}^\infty \) and \( \{\sum_{n=N}^\infty D_n^* D_n\}_{N=1}^\infty \) represent bounded sequences of selfadjoint operators which strongly converge to 0 as \( N \rightarrow \infty \). As \( AXB \in C_{1,1} \), which is separable, then the right-hand side of (2.9) tends to 0 as \( N \rightarrow \infty \) by Theorem 3.6.3. of [GK]. The conclusion follows.

**Corollary 2.1.** For normal \( A \) and \( B \) in \( B(H) \) and for all real \( r \geq 2 \),

\[
\left\| AX + XB \right\| \leq \left( \frac{1 + |A|^r}{2} \right)^{1/2} X \left( \frac{1 + |B|^r}{2} \right)^{1/2}
\]

as well as

\[
\left\| X + AXB \right\| \leq \left( \frac{1 + |A|^r}{2} \right)^{1/2} X \left( \frac{1 + |B|^r}{2} \right)^{1/2}
\]

for all \( X \in B(H) \) and for all u.i. norms \( \| \cdot \| \).

**Proof.** \( \{A, I\} \) and \( \{I, B\} \) are families of normal commuting operators, and so for \( r = 2 \) the desired conclusion follows by Theorem 2.2. For \( r > 2 \) the mapping \( t \rightarrow t^{1/2} \) is operator monotone by a well known Heinz therem, and therefore this is an operator concave mapping [see [BSh]]. Specifically, \( \frac{1 + |A|^2}{2} \leq \left( \frac{1 + |A|^r}{2} \right)^{1/2} \), from which we obtain

\[
\left\| \left( \frac{1 + |A|^2}{2} \right)^{1/2} \right\| \left( \frac{1 + |A|^r}{2} \right)^{1/2} \leq 1 \text{ and similarly } \left\| \left( \frac{1 + |B|^2}{2} \right)^{1/2} \right\| \left( \frac{1 + |B|^r}{2} \right)^{1/2} \leq 1.
\]

Therefore

\[
\left\| \left( \frac{1 + |A|^2}{2} \right)^{1/2} X \left( \frac{1 + |B|^2}{2} \right)^{1/2} \right\| \leq \left\| \left( \frac{1 + |A|^r}{2} \right)^{1/2} X \left( \frac{1 + |B|^r}{2} \right)^{1/2} \right\|
\]

which completes the proof.
Corollary 2.2. For normal $A$ and $B$ in $B(H)$ the inequality
\begin{equation}
\label{eq:2.12}
\left\| \frac{AX + XB}{2} \right\| \leq \|X\|^{1 - \frac{1}{2}} \left\| \frac{|A^r X + X |B^r |}{2} \right\|^{\frac{1}{2}}
\end{equation}
holds for all real $r \geq 2$, for all u.i. norms $\| \cdot \|$ and for all $X \in C_1$.\hfill

Proof. By Corollary 2.1, for all $t > 0$,
\begin{align*}
\left\| \frac{AX + XB}{2} \right\| &= t^{-1} \left\| \frac{tAX + X tB}{2} \right\| \\
&\leq t^{-1} \left\| \left(1 + \frac{|tA|^r}{2} \right) X \left(1 + \frac{|tB|^r}{2} \right) \right\|,
\end{align*}
and therefore
\begin{equation}
\left\| \frac{AX + XB}{2} \right\| \leq t^{-1} \|X\|^{1 - \frac{2}{t}} \left\| \left(1 + \frac{|tA|^r}{2} \right) X \left(1 + \frac{|tB|^r}{2} \right) \right\|^{\frac{2}{t}},
\end{equation}
by [Ki], because $\frac{2}{t} < 1$. Therefore, the arithmetic-geometric mean inequality implies
\begin{align*}
\left\| \frac{AX + XB}{2} \right\| &\leq \frac{1}{2t} \|X\|^{1 - \frac{2}{t}} \left\| \frac{1 + |tA|^r}{2} X + X \frac{1 + |tB|^r}{2} \right\|^{\frac{2}{t}} \\
&\leq \frac{1}{2} \|X\|^{1 - \frac{2}{t}} \left( t^{- \frac{2}{t}} \|X\| + t^{\frac{2}{t}} \left\| \frac{|A^r X + X |B^r |}{2} \right\| \right)^{\frac{2}{t}}.
\end{align*}
As the right-hand side equals $\|X\|^{1 - \frac{2}{t}} \left\| \frac{|A^r X + X |B^r |}{2} \right\|^{\frac{1}{2}}$, which attains its minimum
for $t = \|X\|^{\frac{1}{2}} \left\| \frac{|A^r X + X |B^r |}{2} \right\|^{- \frac{1}{2}}$, the conclusion follows. \hfill

Theorem 2.3. For normal contractions $A$ and $B$ the inequality
\begin{equation}
\label{eq:2.14}
\left\| \left( I - A^* A \right)^{\frac{1}{2}} X (I - B^* B)^{\frac{1}{2}} \right\| \leq \|X - AXB\|,
\end{equation}
holds for all $X \in B(H)$ and for all unitarily invariant norms $\| \cdot \|$.\hfill

Proof. First, we note that $\text{s-lim}_{n \to \infty} A^n (I - A^* A)^{\frac{1}{2}} = 0$. Indeed, by a spectral theorem, for every $f \in H$ there is a positive, finite Borel measure $\mu$ concentrated on $D = \{ z \in C : |z| \leq 1 \}$ such that $\|A^n (I - A^* A)^{\frac{1}{2}} f\|^2 = \int_D |z|^2 |1 - |z|^2| d\mu_f(z)$, whence the desired conclusion follows by Lebesgue’s dominating convergence theorem. Therefore
\begin{equation*}
\text{w-} \lim_{n \to \infty} (I - A^* A)^{\frac{1}{2}} (X - A^n X B^n) (I - B^* B)^{\frac{1}{2}} = (I - A^* A)^{\frac{1}{2}} X (I - B^* B)^{\frac{1}{2}}.
\end{equation*}
So by Theorem 2.2 we get
\[
\left\| (I - A^*A)^{\frac{1}{2}}X(I - B^*B)^{\frac{1}{2}} \right\|
\]
\[
= \left\| \lim_{n \to \infty} (I - A^*A)^{\frac{1}{2}}(X - A^nXB^n)(I - B^*B)^{\frac{1}{2}} \right\|
\]
\[
= \left\| \sum_{k=0}^{\infty} (I - A^*A)^{\frac{1}{2}}A^k(X - AXB)B^k(I - B^*B)^{\frac{1}{2}} \right\|
\]
\[
\leq \left\| \left( \sum_{k=0}^{\infty} (I - |A|^2)|A|^{2k} \right)^{\frac{1}{2}}(X - AXB) \left( \sum_{k=0}^{\infty} |B|^{2k}(I - |B|^2) \right)^{\frac{1}{2}} \right\|
\]
\[
= \| (I - P)(X - AXB)(I - Q) \| \leq \| X - AXB \|
\]
(2.15)
where \( P \) and \( Q \) are the orthogonal projections on \( \text{Ker}(I - A^*A) \) and \( \text{Ker}(I - B^*B) \) respectively. This concludes the proof. \( \square \)

REFERENCES


