PROCEEDINGS OF THE
AMERICAN MATHEMATICAL SOCIETY
Volume 126, Number 9, September 1998, Pages 2569–2577
S 0002-9939(98)04348-2

CENTRAL EXTENSIONS OF SOME LIE ALGEBRAS

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(Communicated by Roe Goodman)

Abstract. We consider three Lie algebras: *Der* $\mathbb{C}((t))$, the Lie algebra of all derivations on the algebra $\mathbb{C}((t))$ of formal Laurent series; the Lie algebra of all differential operators on $\mathbb{C}((t))$; and the Lie algebra of all differential operators on $\mathbb{C}((t)) \otimes \mathbb{C}^n$. We prove that each of these Lie algebras has an essentially unique nontrivial central extension.

The Lie algebra of all derivations on the Laurent polynomial algebra $\mathbb{C}[t, t^{-1}]$ can also be characterized as the Lie algebra of vector fields on the circle. The analogous object over a field F of characteristic $p > 0$, *Der* $F[t]/(t^p)$, is called the Witt algebra [C], and this name is sometimes applied to *Der* $\mathbb{C}[t, t^{-1}]$ as well. It is known [Bl] that *Der* $F[t]/(t^p)$ has an essentially unique nontrivial one-dimensional central extension, and also [GF] that *Der* $\mathbb{C}[t, t^{-1}]$ has an essentially unique nontrivial one-dimensional central extension. The proofs of these facts are similar. The nontrivial one-dimensional central extension of *Der* $\mathbb{C}[t, t^{-1}]$ is called the Virasoro algebra. It is one of the fundamental objects in representation theory as well as in theoretical physics.

For a positive integer n, the Lie algebra of all differential operators on $\mathbb{C}[t, t^{-1}] \otimes \mathbb{C}^n$ has a nontrivial one-dimensional central extension, and the extended Lie algebra is related to the representation theory of affine Lie algebras [KP]. It is proved in [L] that this extension is essentially unique (also see [F]). When $n = 1$, the Lie algebra of all differential operators on the Laurent polynomial ring $\mathbb{C}[t, t^{-1}]$ can also be characterized as the Lie algebra of differential operators on the circle; the corresponding extension is referred to, particularly in the physics literature, as $\mathcal{W}_{1+\infty}$. Some representations of $\mathcal{W}_{1+\infty}$ have been studied recently (see, e.g., [KR], [FKRW]). In [FKRW], it is shown that some representations of $\mathcal{W}_{1+\infty}$ have natural structures of vertex operator algebras (see, e.g., [Bo] and [FLM] for definitions).

Each of these constructions involves the Laurent polynomial algebra $\mathbb{C}[t, t^{-1}]$. This algebra is, of course, contained in $\mathbb{C}((t))$, the algebra of formal Laurent series. In this paper, we consider the Lie algebras obtained by replacing $\mathbb{C}[t, t^{-1}]$ by $\mathbb{C}((t))$ in each of these constructions. We show that each of the resulting Lie algebras has an essentially unique nontrivial one-dimensional central extension.

We work over the field of complex numbers, though all results hold over any field of characteristic zero.

Received by the editors September 13, 1996 and, in revised form, February 4, 1997.
1991 *Mathematics Subject Classification.* Primary 17B65, 17B56; Secondary 17B66.
Key words and phrases. Lie algebra, central extension, 2-cocycle.
The second author was supported in part by NSF Grant DMS-9401851.

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1. Some basic definitions

Let L and $\hat{L}$ be two Lie algebras over $\mathbb{C}$. The Lie algebra $\hat{L}$ is said to be a one-dimensional central extension of L if there is a Lie algebra exact sequence $0 \to \mathbb{C}c \to \hat{L} \to L \to 0$, where $\mathbb{C}c$ is the one-dimensional trivial Lie algebra and the image of $\mathbb{C}c$ is contained in the center of $\hat{L}$. It is well-known that $\hat{L}$ is a one-dimensional central extension of L if and only if $\hat{L}$ is the direct sum of L and $\mathbb{C}c$ as vector spaces and the Lie bracket $[\cdot,\cdot]_1$ in $\hat{L}$ is given by

$$\begin{aligned}&[x,y]_1 = [x,y] + \varphi(x,y)c,\\ &[x,c]_1 = 0\end{aligned}$$

for all $x, y \in L$, where $[\cdot,\cdot]$ is the Lie bracket in L and $\varphi : L \times L \to \mathbb{C}$ is a bilinear form on L satisfying the following conditions:

$$\text{(i)}\quad \varphi(x,y) = -\varphi(y,x), \tag{1}$$

$$\text{(ii)}\quad \varphi([x,y],z) + \varphi([y,z],x) + \varphi([z,x],y) = 0 \tag{2}$$

for all $x, y, z \in L$. The bilinear form φ is called a 2-cocycle on L. A central extension is trivial if $\hat{L}$ is the direct sum of a subalgebra M and $\mathbb{C}c$ as Lie algebras, where the subalgebra M is isomorphic to L. A 2-cocycle φ corresponding to a trivial central extension is called a 2-coboundary, or a trivial 2-cocycle, and is given by a linear function f from L to $\mathbb{C}$:

$$\varphi(x,y) = f([x,y])$$

for all $x, y \in L$. The 2-coboundary defined by f is denoted by α_f. The set of all 2-cocycles on L is a vector space, denoted by $Z^2(L,\mathbb{C})$. The set of all 2-coboundaries is a subspace of $Z^2(L,\mathbb{C})$, denoted by $B^2(L,\mathbb{C})$. The quotient space $Z^2(L,\mathbb{C})/B^2(L,\mathbb{C})$ is called the 2nd cohomology group of L with coefficients in $\mathbb{C}$, and denoted by $H^2(L,\mathbb{C})$. If $\dim H^2(L,\mathbb{C})=1$, we say that L has an essentially unique nontrivial one-dimensional central extension. We say that 2-cocycles φ, ψ are equivalent if $\varphi - \psi$ is a 2-coboundary.

The following two lemmas will be used in the proofs of our main results.

Lemma 1. *Let L be a Lie algebra and S a subset of L such that S spans L and for each $x \in S$, $x = [y_x, z_x]$ for some $y_x, z_x \in L$. If a 2-cocycle φ satisfies $\varphi(y_x, z_x) = 0$ for all $x \in S$, then either $\varphi = 0$ or φ is nontrivial.*

Proof. Suppose that φ is trivial, so that $\varphi = \alpha_f$ for some linear function f. Then for each $x \in S$,

$$f(x) = f([y_x, z_x]) = \varphi(y_x, z_x) = 0.$$

Thus $f = 0$ since S spans L. This implies that $\varphi = \alpha_f = 0$. □

Lemma 2. *Let L be a Lie algebra and φ a 2-cocycle on L. Suppose there are linear endomorphisms E and F of L such that*

$$\varphi(Ex, y) = \varphi(x, Fy)$$

for all $x, y \in L$, E is surjective and F is locally nilpotent (i.e., for any $y \in L$, there is a positive integer n such that $F^n y = 0$). Then the 2-cocycle φ is 0.

Proof. For $x, y \in L$, let n be a positive integer such that $F^n y = 0$. Since E is surjective, we have $x' \in L$ such that $x = E^n x'$. Thus,

$$\varphi(x,y) = \varphi(E^n x', y) = \varphi(x', F^n y) = 0.$$

□

Let A be a (not necessarily associative) algebra. A linear map $\delta\colon A \to A$ is called a derivation, if $\delta(ab) = \delta(a)b + a\delta(b)$ for all $a, b \in A$.

Now consider the algebra of all formal Laurent series

$$\mathbb{C}((t)) = \left\{ \sum_{i\in\mathbb{Z}, i\geq n} a_i t^i \middle| a_i \in \mathbb{C}, n \in \mathbb{Z} \right\}.$$

It is known that the set of all derivations on $\mathbb{C}((t))$ is

$$\mathcal{A} = \left\{ f(t)\frac{d}{dt} \middle| f(t) \in \mathbb{C}((t)) \right\},$$

where $\frac{d}{dt}$ is the formal derivation defined by $\frac{d}{dt}\colon\ \mathbb{C}((t)) \to \mathbb{C}((t)),\ \sum a_i t^i \longmapsto \sum i a_i t^{i-1}$. For convenience, we denote $\frac{d}{dt}$ by D and denote $\frac{d}{dt}f(t)$ by $f'(t)$. For $f(t) = \sum a_i t^i \in \mathbb{C}((t))$, define $Res\ f(t) = a_{-1}$. If $Res\ f(t) = 0$, we can define the formal integral of $f(t)$ as

$$\sum_{i\neq -1} \frac{a_i}{i+1} t^{i+1},$$

denoted by $\int f(t)$. Then $\mathcal{A}$ is a Lie algebra under the bracket operation:

$$\begin{aligned} &[f(t)D, g(t)D] \\ &= (f(t)D) \circ (g(t)D) - (g(t)D) \circ (f(t)D) \\ &= f(t)g'(t)D - g(t)f'(t)D \end{aligned}$$

for $f(t), g(t) \in \mathbb{C}((t))$, where the $\circ$ is the composition of operators.

More generally, consider the space of all differential operators on the algebra of formal Laurent series $\mathbb{C}((t))$:

$$\mathcal{B} = \operatorname{span} \left\{ f(t)D^l \middle| l \in \mathbb{N}, f(t) \in \mathbb{C}((t)) \right\}.$$

$\mathcal{B}$ is a Lie algebra under the bracket operation:

$$\begin{aligned} [f(t)D^l, g(t)D^k] &= (f(t)D^l) \circ (g(t)D^k) - (g(t)D^k) \circ (f(t)D^l) \\ = \sum_{i=0}^{l} \binom{l}{i} f(t) \left(D^{l-i} g(t)\right) D^{k+i} &- \sum_{j=0}^{k} \binom{k}{j} g(t) \left(D^{k-j} f(t)\right) D^{l+j}. \end{aligned}$$

Furthermore, we may consider $\mathcal{C} = \mathcal{B} \otimes gl_n(\mathbb{C})$, the space of differential operators on $\mathbb{C}((t)) \otimes \mathbb{C}^n$. Note that $\mathcal{C} \subset End\ (\mathbb{C}((t)) \otimes \mathbb{C}^n)$. Define the bracket operation on $\mathcal{C}$ by linearity and the commutator

$$\begin{aligned} &[f(t)D^l \otimes A, g(t)D^k \otimes B] \\ &= (f(t)D^l \otimes A) \circ (g(t)D^k \otimes B) - (g(t)D^k \otimes B) \circ (f(t)D^l \otimes A) \\ &= \sum_{i=0}^{l} \binom{l}{i} f(t) \left(D^{l-i} g(t)\right) D^{k+i} \otimes AB - \sum_{j=0}^{k} \binom{k}{j} g(t) \left(D^{k-j} f(t)\right) D^{l+j} \otimes BA. \end{aligned}$$

Hence $\mathcal{C}$ is a Lie algebra.

2. Main results and proofs

In this section, we will give our main results and their proofs. In each case we exhibit (using Lemma 1) a nontrivial 2-cocycle on the Lie algebra under consideration. The 2-cocycle is analogous to the standard nontrivial 2-cocycle on the Lie algebra obtained from Laurent polynomial algebra. Then for any given 2-cocycle on the Lie algebra, we reduce the 2-cocycle to a 2-cocycle which is equivalent to the original one and takes value 0 whenever the standard 2-cocycle takes value 0. We use Lemma 2 to show that the reduced 2-cocycle is a multiple of the standard one.

Theorem 1. *$dim H^2(\mathcal{A}, \mathbb{C}) = 1$.*

Proof. Let β be a 2-cocycle on $\mathcal{A}$. Define a linear function $f_\beta : \mathcal{A} \to \mathbb{C}$ by

$$f_\beta(g(t)D) = \beta\left(D, \int g(t)D\right) \quad \text{for} \;\; g(t) \in \mathbb{C}((t)),\; Res\; g(t) = 0$$

and

$$f_\beta\left(t^{-1}D\right) = \frac{1}{2}\beta\left(t^{-1}D, tD\right).$$

Then $\beta_1 = \beta - \alpha_{f_\beta}$ is a 2-cocycle on $\mathcal{A}$ which is equivalent to β.

For $f(t) = \sum\limits_{i \neq 0} a_i t^i \in \mathbb{C}((t))$,

$$\begin{aligned} &\beta_1(D, f(t)D) \\ &= \beta(D, f(t)D) - f_\beta([D, f(t)D]) \\ &= \beta(D, f(t)D) - f_\beta(f'(t)D) = 0, \end{aligned} \tag{3}$$

and

$$\begin{aligned} &\beta_1\left(t^{-1}D, tD\right) \\ &= \beta\left(t^{-1}D, tD\right) - f_\beta\left([t^{-1}D, tD]\right) \\ &= \beta\left(t^{-1}D, tD\right) - f_\beta\left(2t^{-1}D\right) = 0. \end{aligned} \tag{4}$$

Lemma 3. *$\beta_1(D, \mathcal{A}) = 0$ and $\beta_1(tD, \mathcal{A}) = 0$.*

Proof of Lemma 3. From (3) and $\beta_1(D, D) = -\beta_1(D, D)$, we have $\beta_1(D, \mathcal{A}) = 0$. For $f(t) \in \mathbb{C}((t))$ and $Res\; f(t) = 0$,

$$\begin{aligned} &\beta_1(tD, f(t)D) \\ &= \beta_1\left(tD, \left[D, \int f(t)D\right]\right) \\ &= \beta_1\left([tD, D], \int f(t)D\right) + \beta_1\left(D, \left[tD, \int f(t)D\right]\right) = 0. \end{aligned}$$

$\beta_1(tD, \mathcal{A}) = 0$ follows from this and (4). □

Lemma 4. $\beta_1(t^2D, \mathcal{A}) = 0$.

Proof of Lemma 4. For $f(t) \in \mathbb{C}((t))$ and $Res\ f(t) = 0$, we have

$$\begin{aligned}
&\beta_1(t^2D, f(t)D)\\
&= \beta_1\left(t^2D, \left[D, \int f(t)D\right]\right)\\
&= \beta_1\left([t^2D, D], \int f(t)D\right) + \beta_1\left(D, \left[t^2D, \int f(t)D\right]\right) = 0.
\end{aligned}$$

Also

$$\begin{aligned}
&\beta_1\left(t^2D, t^{-1}D\right)\\
&= \beta_1\left(t^2D, -\frac{1}{2}\left[tD, t^{-1}D\right]\right)\\
&= -\frac{1}{2}\beta_1\left(\left[t^2D, tD\right], t^{-1}D\right) - \frac{1}{2}\beta_1\left(tD, \left[t^2D, t^{-1}D\right]\right)\\
&= -\frac{1}{2}\beta_1\left(-t^2D, t^{-1}D\right) = \frac{1}{2}\beta_1\left(t^2D, t^{-1}D\right).
\end{aligned}$$

This implies that $\beta_1\left(t^2D, t^{-1}D\right) = 0$. ☐

Lemma 5. *If $f(t) \in \mathbb{C}((t))$ and $Res\ f(t) = 0$, then $\beta_1(t^3D, f(t)D) = 0$.*

Proof of Lemma 5. We have

$$\begin{aligned}
&\beta_1(t^3D, f(t)D)\\
&= \beta_1\left(t^3D, \left[D, \int f(t)D\right]\right)\\
&= \beta_1\left(\left[t^3D, D\right], \int f(t)D\right) + \beta_1\left(D, \left[t^3D, \int f(t)D\right]\right)\\
&= \beta_1\left(-3t^2D, \int f(t)D\right) = 0.
\end{aligned}$$ ☐

Define $\alpha : \mathcal{A} \times \mathcal{A} \to \mathbb{C}$ by

$$\alpha\left(\sum_i a_i t^{i+1}D, \sum_j b_j t^{j+1}D\right) = \sum_i a_i b_{-i}(i^3 - i)$$

for $\sum_i a_i t^{i+1}D, \sum_j b_j t^{j+1}D \in \mathcal{A}$. Note that the sum on the right-hand side is finite and α is a 2-cocycle on $\mathcal{A}$. Let S be the subset of $\mathcal{A}$ given by

$$S = \left\{t^{-1}D\right\} \cup \left\{f(t)D \in \mathcal{A} \middle| Res\ f(t) = 0\right\}.$$

Now for $f(t) \in \mathbb{C}((t))$, $Res\ f(t) = 0$, we have

$$t^{-1}D = \left[\frac{1}{2}t^{-1}D, tD\right],$$

$$f(t)D = \left[D, \int f(t)D\right],$$

$$\alpha\left(t^{-1}D, tD\right) = 0,$$

and

$$\alpha\left(D, \int f(t)D\right) = 0.$$

Since α is nonzero (in fact, $\alpha(t^3D, t^{-1}D) = 6$), Lemma 1 shows that α is nontrivial. Also $\alpha = \alpha_1$. Applying Lemma 3, Lemma 4 and Lemma 5 to α, we have

$$\alpha(D, \mathcal{A}) = \alpha(tD, \mathcal{A}) = \alpha(t^2D, \mathcal{A}) = 0,$$

and

$$\alpha\left(t^3D, f(t)D\right) = 0 \quad \text{for} \quad f(t) \in \mathbb{C}((t)), \quad \mathit{Res}\ f(t) = 0.$$

Suppose that $\beta_1(t^3D, t^{-1}D) = 6r$ for some $r \in \mathbb{C}$. Define $\beta_2 = \beta_1 - r\alpha$; then we have

$$\beta_2(D, \mathcal{A}) = \beta_2(tD, \mathcal{A}) = \beta_2(t^2D, \mathcal{A}) = 0 \tag{5}$$

and

$$\beta_2(t^3D, \mathcal{A}) = 0. \tag{6}$$

We now show that $\beta_2 = 0$, completing the proof of Theorem 1.

Let ad: $\mathcal{A} \to \mathcal{A}$, $\mathrm{ad}(a)b = [a, b]$, be the adjoint operator; then

$$\begin{aligned} &\beta_2\left(\mathrm{ad}D(f(t)D), g(t)D\right) \\ &= \beta_2\left([D, f(t)D], g(t)D\right) \\ &= \beta_2\left([D, g(t)D], f(t)D\right) + \beta_2\left(D, [f(t)D, g(t)D]\right) \\ &= -\beta_2\left(f(t)D, \mathrm{ad}D(g(t)D)\right). \end{aligned} \tag{7}$$

Similarly,

$$\beta_2\left(\mathrm{ad}(tD)(f(t)D), g(t)D\right) = -\beta_2\left(f(t)D, \mathrm{ad}(tD)(g(t)D)\right), \tag{8}$$

$$\beta_2\left(\mathrm{ad}(t^3D)(f(t)D), g(t)D\right) = -\beta_2\left(f(t)D, \mathrm{ad}(t^3D)(g(t)D)\right). \tag{9}$$

Now we want to use formulas (7), (8) and (9) to construct two linear endomorphisms E and F on $\mathcal{A}$ so that we can use Lemma 2. Set

$$\begin{aligned} E &= (\mathrm{ad}D)^2\mathrm{ad}(t^3D) - (\mathrm{ad}tD)^3 - 3(\mathrm{ad}tD)^2 + 4\mathrm{ad}tD, \\ F &= -\mathrm{ad}(t^3D)(\mathrm{ad}D)^2 + (\mathrm{ad}tD)^3 - 3(\mathrm{ad}tD)^2 - 4\mathrm{ad}tD. \end{aligned}$$

Then, using (7), (8) and (9), we have

$$\beta_2(E(f(t)D), g(t)D) = \beta_2(f(t)D, F(g(t)D)). \tag{10}$$

For $f(t)=\sum_i a_i t^{i+1}, g(t)=\sum_j b_j t^{j+1} \in \mathbb{C}((t))$, we have

$$\begin{aligned}
&(\mathrm{ad}D)^2\mathrm{ad}(t^3D)(f(t)D)\\
&=\left[D,\left[D,\left[t^3D,f(t)D\right]\right]\right]\\
&=\left[D,\left[D,\sum_i (i-2)a_it^{i+3}D\right]\right]\\
&=\left[D,\sum_i (i-2)(i+3)a_it^{i+2}D\right]\\
&=\sum_i (i-2)(i+3)(i+2)a_it^{i+1}D\\
&=\sum_i (i^3+3i^2-4i-12)a_it^{i+1}D.
\end{aligned}$$

Also

$$(\mathrm{ad}tD)^k(f(t)D)=\sum_i i^k a_i t^{i+1}D$$

for all $k\in\mathbb{N}$. This implies that $E(f(t)D)=-12f(t)D$ for all $f(t)\in\mathbb{C}((t))$. Thus E is invertible. Similarly, for $g(t)=\sum_j b_j t^{j+1}\in\mathbb{C}((t))$, we have

$$-\mathrm{ad}(t^3D)(\mathrm{ad}\,D)^2(g(t)D)=\sum_j(-j^3+3j^2+4j)b_jt^{j+1}D.$$

Thus $F(g(t)D)=0$ for all $g(t)\in\mathbb{C}((t))$. From Lemma 2, we have $\beta_2=0$ or $\beta_1=r\alpha$. □

Theorem 2. *dim*$H^2(\mathcal{B},\mathbb{C})=1$.

Proof. For any 2-cocycle ψ on $\mathcal{B}$, define

$$f_\psi\left(g(t)D^l\right)=-\frac{1}{l+1}\psi\left(t,g(t)D^{l+1}\right)$$

for $g(t)\in\mathbb{C}((t))$. Then $\psi_1=\psi-\alpha_{f_\psi}$ is a 2-cocycle and equivalent to ψ. We have

$$\psi_1\left(t,g(t)D^{l+1}\right)=0$$

for all $g(t)\in\mathbb{C}((t))$ and $l\in\mathbb{N}$.

Lemma 6. *If* $g(t)\in\mathbb{C}((t))$ *and Res* $g(t)=0$, *then* $\psi_1\left(t,g(t)\right)=0$.

Proof of Lemma 6. For $f(t)\in\mathbb{C}((t))$ and *Res* $f(t)=0$, we have

$$\begin{aligned}
&\psi_1\left(t,f(t)\right)\\
&=\psi_1\left([tD,t],f(t)\right)\\
&=\psi_1\left(tD,[t,f(t)]\right)+\psi_1\left([tD,f(t)],t\right)\\
&=-\psi_1\left(t,tf'(t)\right).
\end{aligned}$$

Therefore, $\psi_1(t,f(t)+tf'(t))=0$. Note that every element $g(t)$ in $\mathbb{C}((t))$ with *Res* $g(t)=0$ can be written in the form $f(t)+tf'(t)$ for some $f(t)\in\mathbb{C}((t))$ with *Res* $f(t)=0$. □

Define $\varphi : \mathcal{B} \times \mathcal{B} \to \mathbb{C}$,

$$\varphi\left(\sum_m a_m t^{l+m} D^l, \sum_n b_n t^{k+n} D^k\right) = \sum_m a_m b_{-m} (-1)^l l! k! \binom{m+l}{l+k+1}.$$

Then φ is a 2-cocycle on $\mathcal{B}$. Let S be the subset of $\mathcal{B}$ given by

$$S = \left\{ f(t)D^l \middle| l \in \mathbb{N}, f(t) \in \mathbb{C}((t)) \right\}.$$

For any $l \in \mathbb{N}$ and $f(t) \in \mathbb{C}((t))$,

$$f(t)D^l = -\frac{1}{l+1}\left[t, f(t)D^{l+1}\right]$$

and

$$\varphi\left(t, f(t)D^{l+1}\right) = 0.$$

From Lemma 1 and the fact that $\varphi(t, t^{-1}) = 1$, we have that φ is nontrivial. If $\psi_1(t, t^{-1}) = s$, we define $\psi_2 = \psi_1 - s\varphi$. Then using Lemma 6, we have $\psi_2(t, \mathcal{B}) = 0$. Note that

$$\begin{aligned} & \psi_2\left(\mathrm{ad}t(f(t)D^l), g(t)D^k\right) \\ &= \psi_2\left(\left[t, f(t)D^l\right], g(t)D^k\right) \\ &= \psi_2\left(\left[t, g(t)D^k\right], f(t)D^l\right) + \psi_2\left(t, \left[f(t)D^l, g(t)D^k\right]\right) \\ &= -\psi_2\left(f(t)D^l, \mathrm{ad}t(g(t)D^k)\right). \end{aligned} \tag{11}$$

For $f(t) = \sum_m a_m t^{l+m} \in \mathbb{C}((t))$,

$$[t, f(t)D^l] = -l f(t) D^{l-1}.$$

Therefore the operator ad$t$ is surjective and locally nilpotent. Let $E = \mathrm{ad}\, t$ and $F = -\mathrm{ad}\, t$. From equation (11) and Lemma 2, we have $\psi_2 = 0$. This gives $\psi_1 = s\varphi$. □

Remark. This method gives a simplified proof of Theorem 2.1 of [L].

Consider the Lie algebra $\mathcal{C} = \mathcal{B} \otimes gl_n(\mathbb{C})$. Define a bilinear map $\phi : \mathcal{C} \times \mathcal{C} \to \mathbb{C}$ by

$$\begin{aligned} & \phi\left(\sum_m a_m t^{l+m} D^l \otimes A, \sum_n b_n t^{k+n} D^k \otimes B\right) \\ = & \sum_m a_m b_{-m} (-1)^l l! k! \binom{m+l}{l+k+1} tr(AB) \end{aligned}$$

for $\sum_m a_m t^{l+m}, \sum_n b_n t^{k+n} \in \mathbb{C}((t))$, $l, k \in \mathbb{N}$ and $A, B \in gl_n(\mathbb{C})$. Then ϕ is a 2-cocycle on $\mathcal{C}$. Using a method similar to the proof of Theorem 2.2 of [L], we have

Corollary. $dim H^2(\mathcal{C}, \mathbb{C}) = 1$.

ACKNOWLEDGMENT

The authors are grateful to James Lepowsky for his helpful comments. In addition, the first author is grateful to Professor Lepowsky for his continuous encouragement as thesis advisor.

References

[Bl] R. E. Block, On the Mills-Seligman axioms for Lie algebras of classical type, *Trans. Amer. Math. Soc.* **121** (1966), 378-392. MR **32:**5795

[Bo] R. E. Borcherds, Vertex algebras, Kac-Moody algebras, and the Monster, *Proc. Natl. Acad. Sci. USA* **83** (1986), 3068-3071. MR **87m:**17033

[C] H. -J. Chang, Über Wittsche Lie-Ringe, *Abh. Math. Sam. Hansischen* **14** (1941), 151-184. MR **3:**101A

[F] B. L. Feigin, The Lie algebra $gl(\lambda)$ and the cohomology of the Lie algebra of differential operators, *Usp. Math. Nauk.* **43**, No. 2 (1988), 157-158. MR **90b:**17024

[FKRW] E. Frenkel, V. Kac, A. Radul, W. Wang, $\mathcal{W}_{1+\infty}$ and $\mathcal{W}(gl_N)$ with central charge N, *Commun. Math. Phys.* **170** (1995), 337-357. MR **96i:**17024

[FLM] I. B. Frenkel, J. Lepowsky and A. Meurman, *Vertex Operator Algebras and the Monster*, Pure and Appl. Math., Vol. 134, Academic Press, Boston, 1988. MR **90h:**17026

[GF] I. M. Gelfand, D. B. Fuchs, Cohomology of the Lie algebra of vector fields on a circle, *Funct. Anal. Appl.* **2** (1968), 92-93. English translation: *Funct. Anal. Appl.* **2** (1968), 342-343. MR **39:**6348a

[KP] V. Kac, D. H. Peterson, Spin and wedge representations of infinite-dimensional Lie algebras and groups. *Proc. Natl. Acad. Sci. USA* **78** (1981), 3308-3312. MR **82j:**17019

[KR] V. Kac, A. Radul, Quasifinite highest weight modules over the Lie algebra of differential operators on the circle, *Commun. Math. Phys.* **157** (1993), 429-457. MR **95f:**81036

[L] W. Li, 2-cocycles on the algebra of differential operators, *J. Algebra* **122** (1989), 64-80. MR **90d:**17018

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