PROCEEDINGS OF THE
AMERICAN MATHEMATICAL SOCIETY
Volume 126, Number 9, September 1998, Pages 2691–2697
S 0002-9939(98)04365-2

# A LIMITING EXAMPLE FOR THE LOCAL "FUZZY" SUM RULE IN NONSMOOTH ANALYSIS

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(Communicated by Dale Alspach)

ABSTRACT. We show that assuming all the summand functions to be lower semicontinuous is not sufficient to ensure a (strong) fuzzy sum rule for subdifferentials in any infinite dimensional Banach space. From this we deduce that additional assumptions are also needed on functions for chain rules, multiplier rules for constrained minimization problems and Clarke-Ledyaev type mean value inequalities in the infinite dimensional setting.

Let $X$ be a real Banach space with continuous dual $X^*$ whose closed unit ball is denoted by $B_X$. Let $f : X \to \bar{R} := R \cup \{+\infty\}$ be a lower semicontinuous function. We say that $x^*$ is a $\beta$-subderivative of $f$ at $x$ provided that there exists a locally Lipschitz $\beta$-smooth function $g$ such that $f - g$ attains a local minimum at $x$ and $x^* = \nabla g(x)$. We denote the $\beta$-subdifferential, the set of all $\beta$-subderivatives, of $f$ at $x$, by $D_\beta f(x)$.

Here $\beta$-smooth can mean any bornological smoothness or $s$-Hölder smoothness concept (see e.g. [P] for details). Some frequently used ones are: Gâteaux-smooth ($\beta = G$): the function $g$ is Gâteaux differentiable and $\nabla g$ is weak-star continuous in a neighborhood of $x$; Fréchet smooth ($\beta = F$): the function $g$ is Fréchet differentiable and $\nabla g$ is norm continuous in a neighborhood of $x$; $s$-Hölder smooth ($\beta = H(s)$, $s \in (0, 1]$): the function $g$ is $s$-Hölder differentiable and $\nabla g$ is norm continuous in a neighborhood of $x$. In a Hilbert space, $D_{H(1)}$ is the proximal subdifferential.

Subdifferentials of a function provide information on its local behavior. Fuzzy sum rules for subdifferentials are very important in discussing problems that relate to the local behavior of functions. The following fuzzy sum rule in finite dimensional Banach spaces was proved by Ioffe [I2].

**Theorem 0.** *Let $X$ be a finite dimensional Banach space and let $f_n : X \to \bar{R}, n = 1, 2, ..., N$ be lower semicontinuous functions. Suppose $f_1 + f_2 + \ldots + f_N$ attains a local minimum at 0. Then, for any $\varepsilon > 0$, there exist $x_n \in \varepsilon B_X, n = 1, 2, ..., N$,*

Received by the editors January 30, 1997.
1991 *Mathematics Subject Classification.* Primary 26B05, 49J50, 49J52.
The first author's research was partially supported by a Walla Walla College Faculty Development Grant.
The second author's work was partially supported by a grant from the Faculty Research and Creative Activities Support Fund, Western Michigan University.
Research for this note was completed while the authors were visiting Simon Fraser University. The authors thank J.M. Borwein and the CECM for their hospitality.

*such that*

$$0 \in \sum_{n=1}^{N} D_\beta f_n(x_n) + \varepsilon B_{X^*}.$$

A natural and important question is whether it is possible to extend Theorem 0 to infinite dimensional (smooth) Banach spaces without any additional assumption on the summand functions. Indeed, the quest for sufficient conditions that ensure a fuzzy sum rule in infinite dimensional (smooth) Banach spaces can be traced to Ioffe [I3], where the sufficient condition is that all but one of the summand functions are locally Lipschitz. Deville and Haddad [DH] reduced the Lipschitz condition to uniform continuity. Borwein and Ioffe [BI] and Ioffe and Rockafellar [IR] proposed a sequential uniform semicontinuity condition that unifies the Deville and Haddad condition in the case when $X$ is finite dimensional. A slightly weaker topological form of this sequential uniform semicontinuity condition called local uniform lower semicontinuity is given in Borwein and Zhu [BZ] to ensure fuzzy sum rules in smooth Banach spaces. Another way to extend Theorem 0 to smooth infinite dimensional spaces is to relax the arbitrary small ball $\varepsilon B_{X^*}$ to an arbitrary weak-star neighborhood; such extensions are usually referred to as weak fuzzy sum rules. The first weak fuzzy sum rule was established by Ioffe in [I1] for the Dini subdifferential. Systematic generalizations to bornologically smooth subdifferentials with some refinements can be found in [BZ].

Unfortunately, a weak fuzzy sum rule often is not accurate enough, and the uniform continuity type conditions are too restrictive in certain applications. Nevertheless, the question as to whether the uniform continuity type condition could be removed completely has remained opened. We show in Theorem 1 that the answer is negative. (After the submission of this paper we became aware that R. Deville and M. Ivanov have independently constructed a different example in [DI] showing that a fuzzy sum rule does not hold under only lower semicontinuity assumptions on the summand functions). Obviously, in constructing such an example one needs only to consider the sum of two functions and the Gâteaux subdifferential, which is the largest among the $\beta$-subdifferentials. In fact, we will use the following slightly weaker notion.

**Definition.** We will write $x^* \in \partial_G f(x)$ if $x^* = \nabla g(x)$, where $g$ has the following properties:
(a) $g$ is Gâteaux differentiable at $x$, and
(b) $\limsup_{\|h_i\|\to 0} |g(x+h_i) - g(x)|/\|h_i\| < \infty$, and
(c) $f - g$ attains a local minimum at $x$.

Notice that $\partial_G$ is larger than the Gâteaux subdifferential usually studied because (b) is weaker than locally Lipschitz (and is automatic when $g$ is Fréchet differentiable at $x$).

**Theorem 1.** *Let $X$ be an infinite dimensional Banach space, and let $c > 0$, $d > 0$. Then there are lower semicontinuous bounded-below functions $f_1$ and $f_2$ such that:*
*(a) $(f_1 + f_2)(0) = \inf f_1 + f_2$, and*
*(b) $\|x_1^* + x_2^*\| \geq c$ whenever $x_n^* \in \partial_G f_n(x_n)$ and $\|x_n\| \leq d$, $n = 1, 2$.*

*Proof.* We will prove this in the case that $X$ has a normalized monotone Schauder basis $\{e_k\}_{k=1}^{\infty}$ (the usual bases of $\ell_p$ and $c_0$ are such bases), and from this deduce the result in general. Now, $\|e_k\| = 1$ for each $k$ and each $x \in X$ can be uniquely represented as $x = \sum_{k=1}^{\infty} a_k e_k$; for $P_n(x) := \sum_{k=1}^{n} a_k e_k$, one has $\|P_m(x)\| \leq \|P_n(x)\|$ for $m \leq n$, in particular $\|P_n\| \leq 1$ for each $n$. Now $a_k \to 0$ as $k \to \infty$, and so $\|x\|_\infty := \max\{|a_k| : 1 \leq k < \infty\}$ exists. Moreover, for $k_0$ such that $|a_{k_0}| = \|x\|_\infty$, we have $|a_{k_0}| = \|P_{k_0+1}(x) - P_{k_0}(x)\| \leq 2\|x\|$. Thus $\|\cdot\|_\infty$ is 2-Lipschitz. We will also use $x(i)$ to denote the $i$-th coordinate of $x$, that is, $x(i) = a_i$.

To construct the functions, first let $F_n = \{x : \|x\| \leq 3d,\ x(i) \geq 0 \text{ and } x(i) = 0 \text{ if } i \bmod 3 \neq 0 \text{ or } i < 3n\}$. Now define

$$f_1(x) = \begin{cases} 0 & \text{if } x = 0; \\ -\frac{1}{\sqrt{n}} - c\|y\|_\infty & \text{if } x = \frac{1}{n} e_{3n-1} + y, y \in F_n; \\ +\infty & \text{otherwise;} \end{cases}$$

and

$$f_2(x) = \begin{cases} 0 & \text{if } x = 0; \\ -\frac{1}{\sqrt{n}} - c\|y\|_\infty & \text{if } x = \frac{1}{n} e_{3n-2} + y, y \in F_n; \\ +\infty & \text{otherwise.} \end{cases}$$

First observe that $\mathrm{dom}(f_1) \cap \mathrm{dom}(f_2) = 0$ by the uniqueness of basis representations; thus (a) is clear. From the definitions it also follows that $f_1$ and $f_2$ are both bounded below by $-1 - 6cd$, since $c\|\cdot\|_\infty$ is $2c$-Lipschitz.

We now prove that $f_1$ is lower semicontinuous (the proof for $f_2$ is similar). Suppose $x_n \in \mathrm{dom}(f_1)$ and $x_n \to x$. If $x = 0$, we may assume $x_n \neq 0$ and so $x_n = \frac{1}{k_n} e_{3k_n - 1} + y_n$, $y_n \in F_{k_n}$. Now $k_n \to \infty$ and $y_n \to 0$, and so $-\frac{1}{\sqrt{k_n}} - \|y_n\|_\infty \to 0$. If $x \neq 0$, we know that $k_n \not\to \infty$. Indeed, if $k_n \to \infty$, then for each $i$, we have $x_n(i) \to 0$ as $n \to \infty$ because $x_n(i) = 0$ for all $i \leq 3k_n - 1$. Because the norm and pointwise limit must agree if they both exist, we conclude that $x_n$ converges to 0 in norm. Now because $k_n \not\to \infty$, we know that $k_n = n_0$ for all large $n$ (because, when $n \neq m$, $\|\frac{1}{n} e_{3n-1} + y_n - (\frac{1}{m} e_{3m-1} + y_m)\| \geq \max\{\frac{1}{n}, \frac{1}{m}\}$ for $y_m \in F_m$, $y_n \in F_n$ by the monotonicity of the basis). Therefore, $x_n = \frac{1}{n_0} e_{3n_0 - 1} + y_n$, $y_n \in F_{n_0}$, for all large $n$. This implies $y_n \to \bar{y} \in F_{n_0}$, which with the continuity of $\|\cdot\|_\infty$ implies

$$f_1(x_n) = -\frac{1}{\sqrt{n_0}} - c\|y_n\|_\infty \to -\frac{1}{\sqrt{n_0}} - c\|y\|_\infty = f_1(x).$$

This proves the lower semicontinuity of $f_1$ (and similarly of $f_2$).

It remains to show (b). Let $g_i$ be the function associated to $x_i^*$, $i = 1, 2$, as in the definition. Now observe that $\partial_G f_1(0)$ is empty because

$$n[f_1(0 + \frac{1}{n} e_{3n-1}) - f_1(0)] \leq n[-\frac{1}{\sqrt{n}}] - 0 = -\sqrt{n};$$

similarly $\partial_G f_2(0)$ is empty. Thus we can write $x_1 = \frac{1}{m} e_{3m-1} + y_1$ and $x_2 = \frac{1}{n} e_{3n-1} + y_2$, where $y_1 = \sum_{k=m}^{\infty} a_k e_{3k} \in F_m$ and $y_2 = \sum_{k=n}^{\infty} b_k e_{3k} \in F_n$. We will prove that $\|x_1^* + x_2^*\| \geq c$ in the case $m \leq n$ (the proof for $m \geq n$ is similar). Let $b_{k_0} = \max_{k \geq n}\{b_k\}$; then $0 \leq b_{k_0} \leq 2\|x_2\| \leq 2d$, and thus $y_2 + te_{3k_0} \in F_n$ for $0 \leq t \leq d$, and $c\|y + te_{3k_0}\|_\infty = c(\|y\|_\infty + t)$. Therefore

$$\frac{g_2(x_2 + te_{3k_0}) - g_2(x_2)}{t} \leq \frac{f_2(x_2 + te_{3k_0}) - f_2(x_2)}{t} = -c.$$

Now, because $m \le n$, we have $y_1 + te_{3k_0} \in F_m$, and because $a_{3k_0} \ge 0$ we know that $\|y_1 + te_{3k_0}\|_\infty \ge \|y_1\|_\infty$. Consequently

$$\frac{g_1(x_1 + te_{3k_0}) - g_1(x_1)}{t} \le \frac{f_1(x_1 + te_{3k_0}) - f_1(x_1)}{t} \le 0.$$

Therefore, $\langle x_2^*, e_{3k_0}\rangle \le -c$, while $\langle x_1^*, e_{3k_0}\rangle \le 0$. This shows that $\|x_1^* + x_2^*\| \ge c$, as desired. This completes the proof in the case that $X$ has a monotone Schauder basis.

Now suppose $X$ is an arbitrary infinite dimensional Banach space. Then $X$ has a closed subspace $Y$ which under an equivalent norm has a monotone Schauder basis, which we are free to normalize (see [D]). The construction above can be done on $Y$ and then extended to the whole space by defining the functions $f_i$ to be $+\infty$ off of $Y$ (also, since this construction works for all $c > 0$, $d > 0$ with an equivalent norm, it also works for all such constants with the original norm). □

As mentioned above, fuzzy sum rules hold under certain additional conditions on the functions. We refer the reader to [BZ] for such a result proved under a local uniform lower semicontinuity condition; however, [Z1] provides a simple example showing the conclusion of the fuzzy sum rule can still hold when this condition fails. Thus in spite of Theorem 1, this suggests that the search for weaker (more applicable) conditions under which fuzzy sum rules hold should continue.

For the remainder of this note, we outline some implications of Theorem 1 regarding conditions needed in chain rules, multiplier rules and mean value inequalities. Indeed, weak fuzzy sum rules are closely related to chain rules and necessary conditions for constrained minimization problems. Theorem 1 shows that one cannot achieve an arbitrary norm neighborhood of fuzziness in these results in infinite dimensional spaces under only lower semicontinuity and continuity assumptions. More specifically, consider the following minimization problem $\mathcal{P}$:

$$\text{minimize } f_0(x)$$

$$\text{subject to } f_n(x) \le 0, \quad n = 1, 2, \dots, M,$$

$$f_n(x) = 0, \quad n = M+1, \dots, N, \ x \in C.$$

In what follows we will state a fuzzy multiplier rule for this minimization problem derived in [BTZ] (in a slightly different form) that generalizes the Lagrange multiplier rule. In this fuzzy multiplier rule we need the Fréchet subdifferentials for the functions or their negatives in the equality constraints. To simplify notation, we use the quantities $\tau_n, n = M+1, \dots, N$, that equal either 1 or $-1$. Then $\tau_n f_n$ will either be the function itself or its negative. We denote by $d(S, a) := \inf\{\|s - a\| : s \in S\}$ the distance from $a \in X$ to $S \subset X$.

**Theorem 2** (Multiplier Rule). *Let $X$ be a reflexive Banach space, let $C$ be a closed subset of $X$, let $f_i$ be lower semicontinuous for $n = 0, 1, \dots, M$ and let $f_n$ be continuous for $n = M+1, \dots, N$. Assume that $\bar{x}$ is a local solution of $\mathcal{P}$ and*

$$\liminf_{x \to \bar{x}} d(D_F f_n(x), 0) > 0$$

*for $n = 1, ..., M$ and*

$$\liminf_{x \to \bar{x}} d(D_F f_n(x) \cup D_F(-f_n)(x), 0) > 0$$

*for $n = M+1, ..., N$. Then, for $\varepsilon > 0$ and any weak-star neighborhood $U$ of $0$ in $X^*$, there exist $(x_n, f_n(x_n)) \in (\bar{x}, f_n(\bar{x})) + \varepsilon B_{X\times R}, n = 0, 1, .., N$, $x_{N+1} \in \bar{x} + \varepsilon B_X$ and $\mu_n > 0, n = 1, \ldots, N$, such that*

$$0 \in D_F f_0(x_0) + \sum_{n=1}^{M} \mu_i D_F f_n(x_n) + \sum_{n=M+1}^{N} \mu_i D_F(\tau_n f_n)(x_n) + N_F(C, x_{N+1}) + U.$$

Let $f_1, ..., f_N : X \to \bar{R}$ be lower semicontinuous functions and let $f : R^N \to \bar{R}$ be a lower semicontinuous function nondecreasing for each of its variables. Then one can check that $(\bar{x}, (f_1(\bar{x}), ..., f_N(\bar{x})))$ is a local solution to the following minimization problem (on $X \times R^N$):

$$\text{minimize } f(y), \quad \text{subject to } f_n(x) - y_n \le 0, \quad n = 1, \ldots, N.$$

Applying Theorem 2 yields the following chain rule.

**Theorem 3.** *Let $X$ be a reflexive Banach space. Suppose that $f_1, ..., f_N : X \to \bar{R}$ are lower semicontinuous functions and $f : R^N \to \bar{R}$ is a lower semicontinuous function nondecreasing for each of its variables. Suppose that $f(f_1, ..., f_N)$ attains a local minimum at $\bar{x}$. Then, for any positive number $\varepsilon > 0$ and any weak-star neighborhood $U$ of $0$ in $X^*$, there exist $(x_n, f_n(x_n)) \in (\bar{x}, f_n(\bar{x})) + \varepsilon B_{X\times R}, n = 0, 1, .., N$, $(y, f(y)) \in (\bar{y}, f(\bar{y})) + \varepsilon B_{R^{N+1}}$ where $\bar{y} = (f_1(\bar{x}), ..., f_N(\bar{x}))$ and $\mu = (\mu_1, ..., \mu_N) \in D_F f(y) + \varepsilon B_{R^N}$, such that*

$$0 \in \sum_{n=0}^{N} D_F(\mu_n f_n)(x_n) + U.$$

Applying Theorem 3 to $f(f_1, ..., f_N) = \sum_{n=1}^{N} f_n$ yields the following sum rule.

**Theorem 4.** *Let $X$ be a reflexive Banach space and let $f_1, ..., f_N : X \to \bar{R}$ be lower semicontinuous functions. Suppose that $\sum_{n=1}^{N} f_n$ attains a local minimum at $x$. Then, for any $\varepsilon > 0$ and any weak-star neighborhood $U$ of $0$ in $X^*$, there exist $(x_n, f_n(x_n)) \in (x, f(x)) + \varepsilon B_{X\times R}$ and $x_n^* \in D_F f_n(x_n), n = 1, ..., N$, and $\mu_n \in (1-\varepsilon, 1+\varepsilon)$ such that*

$$0 \in \sum_{n=1}^{N} \mu_n D_F f_n(x_n) + U.$$

From Theorem 1 we can see that the weak-star neighborhood $U$ of $0$ in $X^*$ in Theorem 4 cannot be improved to an arbitrary norm neighborhood. Therefore the same can be said about the chain rule and the multiplier rule.

Recently Clarke and Ledyaev proved a very useful multidirectional mean value inequality [CL]. The following result from [Z2] is a form of the mean value inequality that is valid in smooth Banach spaces.

**Theorem 5.** *Let $X$ be a Banach space with a $\beta$-smooth equivalent norm. let $Y$ be a nonempty, closed, bounded and convex subset of $X$, let $x \in X$, and let $f : X \to \bar{R}$ be a lower semicontinuous function. Suppose that, for some $h > 0$, $f$ is bounded below on $[x, Y] + hB_X$ and*

$$\lim_{\eta \to 0} \inf_{y \in Y + \eta B_X} f(y) - f(x) > r.$$

*Then, for any $\varepsilon > 0$, there exist $z \in [x, Y] + \varepsilon B_X$ and $z^* \in D_\beta f(z)$ such that*

$$r < \langle z^*, y - x \rangle \text{ for all } y \in Y.$$

As our final application of Theorem 1, we show that the condition

$$\lim_{\eta\to 0}\inf_{y\in Y+\eta B_X} f(y)-f(x)>r$$

cannot be replaced by

$$\inf_{y\in Y} f(y)-f(x)>r$$

in an infinite dimensional Banach space. (The two conditions are the same when $X$ is finite dimensional.) In fact, following the method in [C2], we show that if one assumes the contrary, then one could deduce a fuzzy sum rule assuming that $f_1$ and $f_2$ are only lower semicontinuous. For this, suppose that $f_1+f_2$ attains a minimum at 0 over $hB$ for some $h>0$. Consider $X\times X$ with the Euclidean product norm. Define $f: X\times X\to \bar{R}$ by $f(x_1,x_2)=f_1(x_1)+f_2(x_2)$. Let $\varepsilon<h$ be an arbitrary positive constant. Applying Theorem 5 with $Y=\{(x,x): x\in(\varepsilon/2)B_X\}$ and $x=(0,0)$, we obtain that for $r=-\varepsilon^2/2$ there exist $z=(x_1,x_2)\in Y+(\varepsilon/2)B$ and $z^*=(x_1^*,x_2^*)\in D_\beta f(z)\subset D_\beta f_1(x_1)\times D_\beta f_2(x_2)$ such that

$$-\varepsilon^2<\langle z^*,y\rangle \quad \text{for all} \quad y\in Y.$$

That is to say, $-\varepsilon^2/2<\langle x_1^*+x_2^*,x\rangle$ for all $x\in(\varepsilon/2)B_X$, or $\|x_1^*+x_2^*\|<\varepsilon$.

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