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SUFFICIENT CONDITIONS FOR ONE DOMAIN TO CONTAIN ANOTHER IN A SPACE OF CONSTANT CURVATURE

JIAZU ZHOU

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Abstract. As an application of the analogue of C-S. Chen's kinematic formula in the 3-dimensional space of constant curvature ϵ, that is, Euclidean space $\mathbb{R}^3$, 3-sphere S^3, hyperbolic space $\mathbb{H}^3$ ($\epsilon = 0, +1, -1$, respectively), we obtain sufficient conditions for one domain to contain another domain in either an Euclidean space $\mathbb{R}^3$, or a 3-sphere S^3 or a hyperbolic space $\mathbb{H}^3$.

§1. Introduction

For a long time one is faced with the problem of fitting a rugby into a box. Generalizing this problem we ask: given two domains in a space $\mathbb{E}^n_\epsilon$ of constant curvature when can one be moved by an *isometry* "inside" the other. The answer should only depend on geometric invariants of the domains, such as volumes, surface areas and curvature integrals of the boundaries of domains. In the case of $\mathbb{E}^2$ sufficient conditions were given by Hadwiger [8], [9]. By using integral geometry, Grinberg, Ren and Zhou [12] were able to give a fairly simple sufficient condition for domains in a plane $\mathbb{E}^2_\epsilon$ of constant curvature, and one that is derived simultaneously for all ϵ. The key idea is to use the isoperimetric deficit which was first invoked by Ren [3] in the Euclidean plane. Higher dimensional analogues were found by Zhou [14], [15], [16], [17] and Zhang [5] in the Euclidean case. As one might expect the inequalities involve many invariants.

The purpose of this paper is to give sufficient conditions for one domain to contain another domain in a 3-dimensional space $\mathbb{E}^3_\epsilon$ of constant curvature. We use the analog of C-S. Chen's formula [4], [17], i.e., kinematic formula for curvature in the space $\mathbb{E}^3_\epsilon$ of constant curvature. In general the domains D_i and D_j are assumed to be C^2-smooth. We also place an upper bound on the Euler-Poincaré characteristics of the intersections $D_i \cap gD_j$. The latter requirement can be avoided if one just considers smooth convex bodies D_i and D_j.

§2. Analogue of C-S. Chen's kinematic formula in $\mathbb{E}^3_\epsilon$

Let M_i, M_j be two submanifolds in a homogeneous space G/H and I an invariant of the intersection submanifold $M_i \cap gM_j$. Let dg be suitably normalized kinematic

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density of G. Evaluating the integral

$$\int_{\{g\in G: M_i\cap gM_j\neq\emptyset\}} I(M_i\cap gM_j)\,dg \tag{1}$$

in terms of invariants of M_i and M_j we obtain the so-called **kinematic formula** [1], [2], [3], [6], [14], [17].

Let Σ be a closed surface of class C^2 in a 3-dimensional Euclidean space $\mathbb{R}^3$. Denote by R, H, respectively, the Gaussian curvature, the mean curvature of Σ. The total Gaussian curvature, the total square mean curvature are, respectively, defined by

$$\tilde{R} = \int_\Sigma R\,d\sigma, \qquad \tilde{H}^2 = \int_\Sigma H^2\,d\sigma. \tag{2}$$

Let M_i and M_j be two surfaces of class C^2 in $\mathbb{R}^3$. Assume that M_i is fixed, gM_j is moving under the isometry g. Let dg be the kinematic density for $\mathbb{R}^3$, so normalized that the measure of all positions about a point is $8\pi^2$. Let κ_{Γ_g} denote the curvature of the intersection curve $\Gamma_g = M_i\cap gM_j$. Denote by F_k the surface area, and $\tilde{R}_k$, $\tilde{H}_k^2$ the total Gaussian curvature, the total square mean curvature of $M_k\,(k=i,j)$, respectively. Then we have the following C-S. Chen's kinematic formula [4], [17]

$$\int_G \left(\int_{M_i\cap gM_j} \kappa_{\Gamma_g}^2 ds\right) dg = 2\pi^3\left\{(3\tilde{H}_i^2-\tilde{R}_i)\,F_j + (3\tilde{H}_j^2-\tilde{R}_j)\,F_i\right\}. \tag{3}$$

Let $\mathbb{E}_\epsilon^3$ be the 3-dimensional simply connected space of constant curvature ϵ and IS_ϵ^3 the group of orientation preserving isometries of $\mathbb{E}_\epsilon^3$ with the normalized invariant density dg. Let $\Sigma\subset\mathbb{E}_\epsilon^3$ be a C^2-smooth surface and W the **Weingarten transformation**. That is, if $\mathbf{n}$ is the unit normal to Σ then W is the linear map defined on the tangent bundle $T(\Sigma)$ by

$$WX := -\nabla_X \mathbf{n},$$

where ∇ is the Riemannian connection of $\mathbb{E}_\epsilon^3$ and $X\in T(\Sigma)$ a tangent field.

The principal curvature of Σ is defined as the eigenvalue of W. The *mean curvature* H_Σ and the *extrinsic Gauss curvature* R_Σ^e are, respectively, defined by

$$H_\Sigma = \frac{1}{2}\,\mathrm{Tr}(W) = \frac{\lambda_1+\lambda_2}{2}, \qquad R_\Sigma^e = \det(W) = \lambda_1\lambda_2, \tag{4}$$

where λ_1, λ_2 are eigenvalues of W.

By the *Gauss equation* in $\mathbb{E}_\epsilon^3$ the Gauss curvature R_Σ and the extrinsic Gauss curvature R_Σ^e are related by (see [7], [13])

$$R_\Sigma = R_\Sigma^e + \epsilon. \tag{5}$$

The transfer principle [6] tells us that *the form of kinematic formula in a homogeneous space G/H does not depend on the full group of isometries G, but only on the invariant theory of the isotropy subgroup H*. For example, this transfer principle allows us to obtain a new kinematic formula in the space $\mathbb{E}_\epsilon^n$ from a kinematic formula which has been proved in a Euclidean space $\mathbb{R}^n$ without changing invariants that are independent of the curvature ϵ of the space. Therefore by the *transfer principle* Chen's kinematic formula holds in the 3-dimensional space $\mathbb{E}^3_\epsilon$ in the following form:

$$\int_{IS^3_\epsilon}\left(\int_{M_i\cap gM_j}\kappa_\Gamma^2\,ds\right)dg = 2\pi^3\left((3\tilde{H}_j^2-\tilde{R}_j^e)F_i+(3\tilde{H}_i^2-\tilde{R}_i^e)F_j\right), \tag{6}$$

where κ_Γ is the geodesic curvature of curve $\Gamma_g = M_i\cap gM_j$.

By use of the Gauss equation, the formula (6) can be rewritten in the following form[1]:

$$\begin{aligned}&\int_{IS^3_\epsilon}\left(\int_{M_i\cap gM_j}\kappa_\Gamma^2\,ds\right)dg\\ &\quad= 2\pi^3\left((3\tilde{H}_j^2-\tilde{R}_j)F_i + (3\tilde{H}_i^2-\tilde{R}_i)F_j+2\epsilon F_iF_j\right).\end{aligned} \tag{7}$$

If two surfaces M_i, M_j are closed then by use of the Gauss-Bonnet formula

$$\tilde{R}_\Sigma = 2\pi\chi(\Sigma), \tag{8}$$

where $\chi(\cdot)$ is the Euler-Poincaré characteristic, we obtain the following type kinematic formula.

Theorem 1. *Let $M_k\,(k=i,j)$ be two closed C^2-smooth surfaces in $\mathbb{E}^3_\epsilon$. Denote by F_k, H_k the area and the mean curvature of M_k, respectively. Then we have*

$$\begin{aligned}&\int_{IS^3_\epsilon}\left(\int_{M_i\cap gM_j}\kappa_\Gamma^2\,ds\right)dg\\ &\quad= 2\pi^3\left\{3\left(F_i\tilde{H}_j^2+F_j\tilde{H}_i^2\right)-2\pi(F_i\,\chi(M_j)+F_j\,\chi(M_i))+2\epsilon F_iF_j\right\}.\end{aligned} \tag{9}$$

§3. Hadwiger's theorem in space $\mathbb{E}^3_\epsilon$

Let D_i and D_j be two connected domains in space $\mathbb{E}^3_\epsilon$, bounded by surfaces ∂D_i and ∂D_j, which we assume to be of class C^2. In the case of the 3-sphere S^3, ∂D_i and ∂D_j are assumed to be connected and simply connected. Moreover, we assume that D_i and D_j are such that, for all $g\in G$, the group of orientation preserving isometries of $\mathbb{E}^3_\epsilon$, the Euler-Poincaré characteristic $\chi(D_i\cap gD_j)$ of the intersection $D_i\cap gD_j$ has an upper bound N_0, a finite integer. Let $V_k\,(k=i,j)$, F_k, $\tilde{H}_k$ and $\tilde{H}_k^{(2)}$ be the volume of D_k, the surface area of D_k, the total mean curvature and the total square mean curvature of ∂D_k, respectively. We have

Theorem 2. *For the domains in a space $\mathbb{E}^3_\epsilon$ of nonnegative curvature ϵ (that is, space with constant curvature $\epsilon = 0,+1$), let $F_{\max}=\frac{1}{2}\max\{F_i,F_j\}$. In the case of S^3, we assume that $F_k<4\pi\,(k=i,j)$. Then a sufficient condition for D_i to contain D_j or for D_j to contain D_i is*

$$\begin{aligned}&8\pi(V_i\chi(D_j)+V_j\chi(D_i))+2(F_i\tilde{H}_j+F_j\tilde{H}_i)-\frac{N_0\cdot 2\pi^2}{2\pi-\epsilon F_{\max}}\\ &\quad\times\{F_iF_j[3(\tilde{H}_i^2F_j+\tilde{H}_j^2F_i)-2\pi(F_i\chi(\partial D_j)+F_j\chi(\partial D_i))+2\epsilon F_iF_j]\}^{\frac{1}{2}}>0.\end{aligned} \tag{10}$$

Moreover, if $V_j\le V_i$, then D_j can be contained in D_i.

[1] This formula, i.e., the analogue of C-S. Chen's formula, is due to R. Howard.

Proof. S. S. Chern's fundamental formula, Santaló's formula, respectively, read [1]

$$\int_{IS_\epsilon^3} \chi(D_i \cap gD_j)\, dg = 8\pi^2(V_i\chi(D_j) + V_j\chi(D_i)) + 2\pi(F_i\tilde{H}_j + F_j\tilde{H}_i), \tag{11}$$

$$\int_{IS_\epsilon^3(\partial D_i \cap g\partial D_j)} L_{\Gamma_g}\, dg = 2\pi^3 F_i F_j, \tag{12}$$

where L_{Γ_g} is the volume of the intersection curve $\Gamma_g = \partial D_i \cap g\partial D_j$, i.e., the arc length of Γ_g. The curve Γ_g may be composed of several components (i.e., each of those is a simply closed curve).

For a disktype surface of area F bounded by a simple C^2-smooth curve Γ, we have the following Fenchel-Teufel formula [11]:

$$\int_\Gamma |\kappa_\Gamma|\, ds \geq 2\pi - \epsilon\, F, \tag{13}$$

where κ_Γ is the geodesic curvature of Γ.[2]

By Hölder's inequality we have

$$\begin{aligned} 2\pi - \epsilon F_{\max} &\leq \int_{\Gamma_g} |\kappa_{\Gamma_g}| ds \leq \left(\int_{\Gamma_g} 1^2 \cdot ds\right)^{\frac{1}{2}} \left(\int_{\Gamma_g} \kappa_{\Gamma_g}^2 ds\right)^{\frac{1}{2}} \\ &= (L_{\Gamma_g})^{\frac{1}{2}} \left(\int_{\Gamma_g} \kappa_{\Gamma_g}^2 ds\right)^{\frac{1}{2}}, \end{aligned} \tag{14}$$

i.e.,

$$2\pi - \epsilon F_{\max} \leq (L_{\Gamma_g})^{\frac{1}{2}} \left(\int_{\Gamma_g} \kappa_{\Gamma_g}^2 ds\right)^{\frac{1}{2}}. \tag{15}$$

Integrating both sides of the above inequality with respect to kinematic density dg and using Hölder's inequality again we get

$$\begin{aligned} (2\pi - \epsilon F_{\max}) \int_{IS_\epsilon^3(\partial D_i \cap g\partial D_j \neq \emptyset)} dg &\leq \int_{IS_\epsilon^3(\partial D_i \cap g\partial D_j)} (L_{\Gamma_g})^{\frac{1}{2}} \left(\int_{\Gamma_g} \kappa_{\Gamma_g}^2 ds\right)^{\frac{1}{2}} dg \\ &\leq \left(\int_{IS_\epsilon^3(\partial D_i \cap g\partial D_j)} L_{\Gamma_g} dg\right)^{\frac{1}{2}} \left(\int_{IS_\epsilon^3(\partial D_i \cap g\partial D_j)} \left(\int_{\Gamma_g} \kappa_{\Gamma_g}^2 ds\right) dg\right)^{\frac{1}{2}}. \end{aligned} \tag{16}$$

Therefore by (9), (12) and (16) we have

$$\begin{aligned} \int_{IS_\epsilon^3(\partial D_i \cap g\partial D_j \neq \emptyset)} dg \leq \frac{2\pi^3}{2\pi - \epsilon F_{\max}} \{F_iF_j[3(F_i\tilde{H}_j^2 + F_j\tilde{H}_i^2) \\ - 2\pi(F_i\chi(\partial D_j) + F_j\chi(\partial D_i)) + 2\epsilon F_iF_j]\}^{\frac{1}{2}}. \end{aligned} \tag{17}$$

From

$$\int_{IS_\epsilon^3(D_i \cap gD_j)} \chi(D_i \cap gD_j)\, dg \leq N_0 \int_{IS_\epsilon^3(D_i \cap gD_j \neq \emptyset)} dg, \tag{18}$$

[2] In the case of Euclidean space $\mathbb{R}^3$, the inequality is Fenchel's inequality. Inequality is due to Teufel when Γ is a curve in a 3-sphere S^3.

we obtain the kinematic measure

$$
\begin{aligned}
&m\{g\in IS^3_\epsilon:\, gD_j\subseteq D_i \quad\text{or}\quad gD_j\supseteq D_i\}\\
&\quad = m\{g\in IS^3_\epsilon:\, D_i\cap gD_j\neq\emptyset\} - m\{g\in IS^3_\epsilon:\, \partial D_i\cap g\partial D_j\neq\emptyset\}\\
&\quad = \int_{IS^3_\epsilon(D_i\cap gD_j\neq\emptyset)} dg - \int_{IS^3_\epsilon(\partial D_i\cap g\partial D_j\neq\emptyset)} dg \qquad (19)\\
&\quad \geq \frac{1}{N_0}[8\pi^2(V_i\chi(D_j)+V_j\chi(D_i))+2\pi(F_i\tilde H_j+F_j\tilde H_i)] - \frac{2\pi^3}{2\pi-\epsilon F_{\max}}\\
&\quad \times\{F_iF_j[3(\tilde H_i^2F_j+\tilde H_j^2F_i)-2\pi(F_i\chi(\partial D_j)+F_j\chi(\partial D_i))+2\epsilon F_iF_j]\}^{\frac12}.
\end{aligned}
$$

If this lower bound is nonnegative then one domain can be moved to contain another. We complete the proof of Theorem 2. □

Theorem 3. *Let D_i, D_j be two domains in a hyperbolic space $\mathbb{H}^3$. Then a sufficient condition for D_i to contain D_j or for D_j to contain D_i is*

(20)

$$
\begin{aligned}
&8\pi(V_i\chi(D_j)+V_j\chi(D_i))+2(F_i\tilde H_j+F_j\tilde H_i)\\
&\quad -\pi N_0\left\{F_iF_j[3(F_i\tilde H_j^2+F_j\tilde H_i^2)-2\pi(F_i\chi(\partial D_j)+F_j\chi(\partial D_i))-2F_iF_j\right\}^{\frac12}>0.
\end{aligned}
$$

Moreover, if $V_j\leq V_i$, then D_j can be contained in D_i.

Proof. For a disktype surface of area F bounded by a C^2-smooth curve Γ in the 3-dimensional hyperbolic space $\mathbb{H}^3$, we have the following Brickell-Hsiung inequality [18]:

$$\int_\Gamma |\kappa_\Gamma| ds \geq 2\pi + F. \qquad (21)$$

By Hölder's inequality and the weaker type of inequality (21) we have

$$
\begin{aligned}
2\pi &\leq \int_{\Gamma_g}|\kappa_{\Gamma_g}|ds \leq \left(\int_{\Gamma_g}1^2\cdot ds\right)^{\frac12}\left(\int_{\Gamma_g}\kappa^2_{\Gamma_g}ds\right)^{\frac12} \qquad (22)\\
&\leq \left(L_{\Gamma_g}\right)^{\frac12}\left(\int_{\Gamma_g}\kappa^2_{\Gamma_g}ds\right)^{\frac12},
\end{aligned}
$$

i.e.

$$2\pi \leq \left(L_{\Gamma_g}\right)^{\frac12}\left(\int_{\Gamma_g}\kappa^2_{\Gamma_g}ds\right)^{\frac12}. \qquad (23)$$

□

Integrating both sides of inequalitiy (23) over $IS^3_{-1}(\partial D_i\cap g\partial D_j\neq\emptyset)$ and using the analog of C-S. Chen's formula and Hölder inequality again we have

$$
\begin{aligned}
\int_{IS^3_{-1}(\partial D_i\cap g\partial D_j\neq\emptyset)} dg &\leq \pi^2\{F_iF_j[3(F_i\tilde H_j^2+F_j\tilde H_i^2) \qquad (24)\\
&\quad -2\pi(F_i\chi(\partial D_j)+F_j\chi(\partial D_i))-2F_iF_j]\}^{\frac12}.
\end{aligned}
$$

By (18), (11) and (24) we have

$$
\begin{aligned}
(25)\qquad & m\{g \in IS^3_{-1} : gD_j \subseteq D_i \quad \text{or} \quad gD_j \supseteq D_i\} \\
&= m\{g \in IS^3_{-1} : D_i \cap gD_j \neq \emptyset\} - m\{g \in IS^3_{-1} : \partial D_i \cap g\partial D_j \neq \emptyset\} \\
&= \int_{IS^3_{-1}(D_i \cap gD_j \neq \emptyset)} dg - \int_{IS^3_{-1}(\partial D_i \cap g\partial D_j \neq \emptyset)} dg \\
&\geq \frac{1}{N_0}[8\pi^2(V_i\chi(D_j) + V_j\chi(D_i)) + 2\pi(F_i\tilde{H}_j + F_j\tilde{H}_i)] \\
&- \pi^2\{F_iF_j[3(\tilde{H}_i^2F_j + \tilde{H}_j^2F_i) - 2\pi(F_i\chi(\partial D_j) + F_j\chi(\partial D_i)) - 2F_iF_j]\}^{\frac{1}{2}}.
\end{aligned}
$$

Inequality (25) immediately leads to the proof of Theorem 3.

As in euclidean space we say that a subset S of a 3-sphere S^3 or a 3-hyperbolic space $\mathbb{H}^3$ is convex if P, $Q \in S$ implies that the geodesic segment joining P to Q also lies in S. A nonempty, compact, convex set of $\mathbb{E}^3_\epsilon$ is called a convex body. If D_i and D_j are convex bodies, we have $\chi(\partial D_i) = \chi(\partial D_j) = 2$ and $\chi(D_i) = \chi(D_j) = \chi(D_i \cap gD_j) = 1$ for almost all $g \in IS^3_\epsilon$. Then from Theorem 2 and Theorem 3 we immediately obtain the following

Theorem 4. *For convex bodies D_i and D_j in a space $\mathbb{E}^3_\epsilon$ of nonnegative constant curvature ϵ (i.e., $\epsilon = 0, +1$), let $F_{\max} = \frac{1}{2}\min\{F_i, F_j\}$. In the case of S^3, we assume $F_k < 4\pi\,(k = i, j)$. Then a sufficient condition for D_i to contain, or to be contained in, D_j is*

$$
\begin{aligned}
(26)\qquad & 8\pi(V_i + V_j) + 2(F_i\tilde{H}_j + F_j\tilde{H}_i) - \frac{2\pi^2}{2\pi - \epsilon F_{\max}} \\
&\times \{F_iF_j[3(\tilde{H}_i^2F_j + \tilde{H}_j^2F_i) - 4\pi(F_i + F_j) + 2\epsilon F_iF_j]\}^{\frac{1}{2}} > 0.
\end{aligned}
$$

Moreover, if $V_i \leq V_j$, then D_i can be contained in D_j.

Theorem 5. *Let D_i and D_j be convex bodies in a hyperbolic space $\mathbb{H}^3$. Then a sufficient condition for D_i to contain, or to be contained in, D_j is*

$$
\begin{aligned}
(27)\qquad & 8\pi(V_i + V_j) + 2(F_i\tilde{H}_j + F_j\tilde{H}_i) \\
&- \pi\left\{F_iF_j[3(F_i\tilde{H}_j + F_j\tilde{H}_i) - 4\pi(F_i + F_j) - 2F_iF_j]\right\}^{\frac{1}{2}} > 0.
\end{aligned}
$$

Moreover, if $V_j \leq V_i$, then D_j can be contained in D_i.

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Department of Mathematics, Sultan Qaboos University, P.O.Box 36, Al-Khod 123, Sultanate of Oman

Current address: Department of Mathematics, Lehigh University, Bethlehem, Pennsylvania 18015-3174

E-mail address: jiz3@lehigh.edu