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A TOPOLOGY ON LATTICE ORDERED GROUPS

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Abstract. We show that a lattice ordered group can be topologized in a natural way. The topology depends on the choice of a set C of admissible elements (C-topology). If a lattice ordered group is 2-divisible and satisfies a version of Archimedes' axiom (C-group), then we show that the C-topology is Hausdorff. Moreover, we show that a C-group with the C-topology is a topological group.

In section 1 we recall the definition as well as the elementary properties of lattice ordered groups, especially the properties of the norm N on such groups.

In section 2 we introduce the notion of a set of admissible elements C (definition 2) in a lattice ordered 2-divisible group A. It is proved (lemma 2) that the sets $U_{x_0,r} = \{x \in A : r - N(x - x_0) \in C\}$ constitute a base of a topology on A (the C-topology).

In section 3 we describe the properties of the C-topology on lattice ordered 2-divisible C-Archimedean groups (such groups are called C-groups). It is proved that C-groups are Hausdorff topological groups (theorem 1 and theorem 2).

1. Lattice ordered groups

A lattice ordered group [1, Ch. VI, §§8, 9] is an ordered group A such that there exist $\sup(x, y)$ and $\inf(x, y)$, for every $x, y \in A$. Note that:

$$\inf(x, y) = -\sup(-x, -y) \tag{1}$$

Definition 1 ([1, VI, definition 4]). The norm N on a lattice ordered group A is the function $N : A \to A$ defined by $N(x) = \sup(x, -x)$.

Lemma 1. *Let N be the norm on a lattice ordered group A. Then:*

(i) $N(x) = \sup(x, 0) - \inf(x, 0)$ *for every* $x, y \in A$.
(ii) $N(x) = x$ *if and only if* $x \geq 0$. *In particular,* $N(N(x)) = N(x)$, *for every* $x \in A$.
(iii) $N(x) \geq 0$, *for every* $x \in A$.
(iv) $(N(x) = 0) \Leftrightarrow (x = 0)$, *for every* $x \in A$.
(v) $N(mx) = |m|N(x)$, *for every* $x \in A$ *and for every* $m \in \mathbf{Z}$.
(vi) $N(x + y) \leq N(x) + N(y)$, *for all* $x, y \in A$.

Proof. See [1, VI, Proposition 9 and corollary 4 of Proposition 11]. □

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Directly from definition 1 we conclude that:

$$N(x) \leq \epsilon \Leftrightarrow -\epsilon \leq x \leq \epsilon \tag{2}$$

for all $\epsilon \geq 0, x \in A$. In particular, $-N(x) \leq x \leq N(x)$, for every $x \in A$.

2. The C-topology

In this section we suppose that A is a lattice ordered 2-divisible group. Note that in a lattice ordered group there is no nontrivial torsion element. Namely, $nx = 0$, for $n \in \mathbf{N}$, implies $nN(x) = 0$ or equivalently $(n-1)N(x) = -N(x)$; hence $N(x) = 0$, and so $x = 0$ (lemma 1 (iii), (iv) and (v)). Therefore, $\frac{x}{2}$ is uniquely determined, for every $x \in A$. It is easily seen (by lemma 1, (ii) and (v)) that $x \geq 0$ implies $\frac{x}{2} \geq 0$.

Definition 2. A *set of admissible elements* in a lattice ordered 2-divisible group A is any nonempty subset C of the set A^+ of all positive elements having the following properties:

(i) $0 \notin C$,
(ii) $(x \in C \wedge y \geq x) \Rightarrow (y \in C)$,
(iii) $(x, y \in C) \Rightarrow (\inf(x, y) \in C)$,
(iv) $(x \in C) \Rightarrow (\frac{x}{2} \in C)$.

It is obvious that $C \subseteq A^+ \setminus \{0\}$. If there exist at least two coprime elements in the group, then the given inclusion is strict (by (iii), (i) of definition 2 and the fact that coprime elements are necessarily positive). Recall that $x, y \in A$ are coprime if $\inf(x, y) = 0$ ([1, V, definition 5]).

Remark. Suppose that a is a strictly positive element of A. Put $A_a = \{x \in A : x \geq a\}$. Then A_a satisfies (i), (ii), (iii) of definition 2. Denote $A_{a,n} = \frac{1}{2^{n-1}} A_a$, for $n \in \mathbf{N}$. Then we have $A_{a,n+1} \supseteq A_{a,n}$, for every $n \in \mathbf{N}$. Put $C = \bigcup_{n \in \mathbf{N}} A_{a,n}$. Then C is a set of admissible elements. This is the minimal set of admissible elements containing a.

Recall that the open ball in a normed space is defined by the relation $r - |x - x_0| > 0$. In our case we have

Definition 3. Let A be a lattice ordered 2-divisible group and let C be a set of admissible elements of A . The *open C-ball of radius $r \in C$, with the centre $x_0 \in A$*, is the set of all $x \in A$ such that $r - N(x - x_0) \in C$. We denote this set by $U_{x_0,r}$.

Lemma 2. *Let A be a lattice ordered 2-divisible group. Then open C-balls constitute a base of a topology on A (we called this topology the C-topology).*

Proof. Since $x_0 \in U_{x_0,r}$, for all $r \in C$, we get that open C-balls form an open cover of the space A. Let $z_0 \in U_{x_0,r} \cap U_{y_0,R}$ be arbitrary. By the definition of C-balls, we have $r - N(z_0 - x_0) = c_1$ and $R - N(z_0 - y_0) = c_2$, for some $c_1, c_2 \in C$. Then $U_{x_0,\epsilon} \subseteq U_{x_0,r} \cap U_{y_0,R}$, for $\epsilon = \inf(c_1, c_2)$. Namely, for $x \in U_{x_0,r}$, we have $\epsilon - N(x - x_0) = c_3$, for some $c_3 \in C$. Applying the triangle inequality (lemma 1, (vi)), we get $r - N(x - x_0) \geq r - N(x_0 - z_0) + \epsilon - N(z_0 - x) - \epsilon = c_1 + c_3 - \epsilon \in C$. Similar reasoning holds for the second ball. This completes the proof. □

Remark. Lemma 2 is valid even if A is not necessarily 2-divisible. Hence C does not need to satisfy (iv) of definition 2.

It is easy to see that if $V_{x_0,r} = \{x \in A : N(x - x_0) < r\}$ and $F_{x_0,r} = \{x \in A : N(x - x_0) \leq r\}$, then we have $U_{x_0,r} \subseteq V_{x_0,r} \subseteq F_{x_0,r}$.

Example 1. Let $A = \mathbf{R}^2$ with the relation of order defined by $(a,b) \le (c,d) \Leftrightarrow (a \le c \wedge b \le d)$. Then A is a divisible lattice ordered group such that

$$\sup\left((a,b),(c,d)\right) = \left(\sup(a,c), \sup(b,d)\right), \qquad A^+ = \{(a,b) : a \ge 0 \wedge b \ge 0\},$$

and $C = \{(a,b) : a > 0 \wedge b > 0\}$ is the set of admissible elements. It is easy to see that:

$U_{(x_0,y_0),r} = \{(x,y) \in \mathbf{R}^2 : -r < x - x_0 < r \wedge -r < y - y_0 < r\}$;

$F_{(x_0,y_0),r} = \{(x,y) \in \mathbf{R}^2 : -r \le x - x_0 \le r \wedge -r \le y - y_0 \le r\}$;

$V_{(x_0,y_0),r} = F_{(x_0,y_0),r} \setminus \{(x_0+r, y_0+r), (x_0+r, y_0-r), (x_0-r, y_0+r), (x_0-r, y_0-r)\}$.

Thus, the C-topology on A is equivalent to the standard topology on $\mathbf{R}^2$.

It can be shown that the set $\{(x,y) \in \mathbf{R}^2 : (x \ge 0) \wedge (y > 0)\}$ is a set of admissible elements, too. Of course, in this case the corresponding C-topology is not equivalent to the previous one.

Note that the norm N is a continuous function with respect to the C-topology. Namely, by the triangle inequality, we have $-N(x-y) \le Nx - Ny \le N(x-y)$, so by (2), $N(Nx - Ny) \le N(x-y)$.

Lemma 3. *Let A be a lattice ordered 2-divisible group, and let C be a set of admissible elements. Then:*

(i) *C is an open set in the C-topology.*

(ii) *$A = C - C$.*

Proof. (i) The inequality $N(x-c) \le \frac{c}{2}$ is, by (2), equivalent to the inequalities $\frac{c}{2} \le x \le \frac{3c}{2}$, for every $c \in C$. Hence, $U_{c,\frac{c}{2}} \subseteq C$.

(ii) By lemma 1, (i) and (1), we have

$$x = \sup(x,0) - \sup(-x,0) = (\sup(x,0) + c) - (\sup(-x,0) + c)$$

for all $x \in A, c \in C$. □

3. C-groups

In this section we will assume that A is a lattice ordered 2-divisible C-Archimedean group. This means that in the group A the following version of Archimedes' axiom holds:

$$(\forall x \in C)(\forall y \in C)(\exists n \in \mathbf{N})(n \cdot y > x). \tag{3}$$

One can see that (3) is equivalent to

$$(\forall x \ge 0)(\forall y \in C)(\exists n \in \mathbf{N})(n \cdot y > x). \tag{4}$$

A consequence of the given assumption is that $U \cap C \ne \emptyset$, for every neighbourhood U of zero. Namely, if U is the open C-disc around zero of radius y and if $x \in C$ is arbitrary, then there exists $n \in \mathbf{N}$ such that $y - \frac{x}{2^n} \in C$. Hence $\frac{x}{2^n} \in U \cap C$.

Definition 4. We say that A is a *C-group* if A is a lattice ordered, 2-divisible, C-Archimedean group.

Example 2. Let A be $\mathbf{R}^2$ as in example 1. If we choose $C = \{(x,y) : x > 0, y > 0\}$, then A becomes a C-group. If we choose $C = \{(x,y) : x \ge 0, y > 0\}$, then A is not a C- group.

Example 3. The group $\mathbf{Q}_2$ of all dyadic numbers, with the standard ordering, is a C-group (C is the set of strictly positive dyadic numbers). The closure $\mathrm{Cl}\mathbf{Q}_2$ is the additive group of real numbers. Note that every C-group is a module over the dyadic numbers.

Lemma 4. *Let A be a C-group. Then for all $x \in A$ and $c \in C$ there exists $n \in \mathbf{N}$ such that $\frac{x}{2^n} + c \in C$.*

Proof. Let $c \in C, x \in A$. Then, by lemma 3 (ii), there exist $c_1, c_2 \in C$ such that $x = c_1 - c_2$. Hence, $\frac{x}{2^n} = \frac{c_1}{2^n} - \frac{c_2}{2^n}$, for every $n \in \mathbf{N}$. If we choose n such that $c - \frac{c_2}{2^n} > 0$ (this is possible because the group A is C-Archimedean), then we have $\frac{x}{2^n} + c = \frac{c_1}{2^n} + (c - \frac{c_2}{2^n}) \in C$. □

Recall that, by the definition, $x \in A$ is a limit of the sequence (x_n) if the following holds: $(\forall \epsilon \in C)(\exists n_0 \in \mathbf{N})(\forall n \geq n_0)\,(\epsilon - N(x - x_n) \in C)$.

Lemma 5. (i) *Let $F \subseteq A$. Then the closure of F is* $\mathrm{Cl}F = \{\lim x_n : x_n \in F\}$.
(ii) $A^+ = \mathrm{Cl}C$.

Proof. (i) Let the set of all limits of the sequences with terms from F be denoted by X.

Suppose that $x \in X$. Then every open neighbourhood of x cuts F. Therefore, $X \subseteq \mathrm{Cl}F$. Suppose that $x \in \mathrm{Cl}F$ and that $\epsilon \in C$. Let U_n be an open C-disc around x with radius $\frac{\epsilon}{2^n}$. By definition, $U_n \cap F \neq \emptyset$, for every $n \in \mathbf{N}$. Choose $x_n \in U_n \cap F$, for every $n \in \mathbf{N}$. Then $x = \lim x_n$. Therefore $\mathrm{Cl}F \subseteq X$. (ii) Suppose that $x \in A^+$. Choose an arbitrary $c \in C$. Then $\lim(x + \frac{c}{2^n}) = x$ and $x + \frac{c}{2^n} \in C$ for all $n \in \mathbf{N}$. Hence, $x \in \mathrm{Cl}C$.

Suppose that $x \in \mathrm{Cl}C$. Then $x = \lim x_n$, for $x_n \in C$. Applying the continuity of the norm N and lemma 1 (ii), we get $Nx = N(\lim x_n) = \lim(Nx_n) = \lim x_n = x$. Hence, $x \geq 0$. □

Lemma 6. *Suppose that $x \geq 0$ and that $x < c$, for every $c \in C$. Then $x = 0$.*

Proof. Let $x = \lim x_n, x_n \in C$ for all $n \in \mathbf{N}$ (this is possible by lemma 5, (ii)). Therefore, $N(x - x_n) < \epsilon$, for every sufficiently large $n \in \mathbf{N}$, and for arbitrary $\epsilon \in C$. Hence, $-\epsilon < x - x_n < \epsilon$, and so, $x_n < x + \epsilon < 2\epsilon$, for every sufficiently large $n \in \mathbf{N}$. Thus, (x_n) converges to zero; hence, $x = 0$. □

Theorem 1. *A is a Hausdorff space.*

Proof. Suppose that $x, y \in A$ and $x \neq y$. If $N(x - y) = \epsilon \in C$, then $\frac{\epsilon}{4}$-neighbourhoods around x and around y are disjoint. Suppose that $N(x - y) \in A^+ \setminus C$. After a translation we can assume that $y = 0$ and $Nx \in A^+ \setminus C$ (note that $Nx \neq 0$). If we show that 0 and Nx can be separated by open C-balls, then we can conclude that 0 and x can be separated, too. Namely, if U is an open C-ball around 0 and W an open C-ball around Nx which are disjoint, then U is disjoint from an open C-ball V around x such that $NV \subset W$ (V exists because N is a continuous function). If not, there exists $z \in U \cap V$. Then we get $Nz \in N(U \cap V) \subseteq NU \cap NV \subseteq U \cap V$, a contradiction (note that by the definition of C-balls we have $NU \subset U$).

Therefore, we can suppose that $x \in A^+ \setminus C$ and $x \neq 0$. It can be easily seen that 0 can be separated from x. If not, we get that $x \in U$, for every open neighbourhood U around zero. Applying lemma 6, we conclude that $x = 0$ (a contradiction). Let's prove that x can be separated from 0. Choose $U = U_\epsilon$ an open C-ball of radius ϵ around zero, $\epsilon \in C$, such that $x \notin U_{2\epsilon}$. Then $x \in V = U_{x+\frac{\epsilon}{2},\epsilon}$. We claim that $U \cap V = \emptyset$. If not, there exists $z \in U \cap V$; hence, $Nz < \epsilon$ and $N(z - x - \frac{\epsilon}{2}) < \epsilon$. Therefore, $x + \frac{\epsilon}{2} = N(x + \frac{\epsilon}{2}) \leq N(x + \frac{\epsilon}{2} - z) + Nz < 2\epsilon$; hence, $x < \frac{3\epsilon}{2}$. This contradicts the assumption $x \notin U_{2\epsilon}$. The theorem is proved. □

Theorem 2. *Every C-group is a topological group.*

Proof. It is easy to see that the mapping $A \to A$, $x \mapsto -x$ is continuos. We have to show that the mapping $f : A \times A \to A$, $(x, y) \mapsto x + y$ is continuous, too. Let U be an open C-ball of radius ϵ around $x_0 + y_0$, and let V and W be open C-balls of radius $\frac{\epsilon}{2}$ around x_0 and y_0, respectively. Then $V \times W$ is an open neighbourhood around (x_0, y_0). Take $(x, y) = z \in V \times W$. Then $\epsilon - N(x_0 + y_0 - (x + y)) \geq \frac{\epsilon}{2} - N(x_0 - x) + \frac{\epsilon}{2} - N(y_0 - y) \in C$. Since (x, y) can be chosen arbitrarily, we have $f(U \times V) \subset U$, so the continuity is proved. According to theorem 1, A is a Hausdorff space. Therefore, A is a topological group. □

Concluding remarks

One can define an analogue of Cauchy sequence in a C-group (C-Cauchy sequence). It can be shown in a standard manner (but not so easily) that every C-group can be "completed". The "completion" $\hat{A}$ of a C-group A is a C-group with the properties:

(i) A is dense in $\hat{A}$,

(ii) $\hat{A}$ is C-complete (every C-Cauchy sequence with terms from $\hat{A}$ has limit in $\hat{A}$).

Moreover, it can be shown that $\hat{A}$ has a structure of ordered linear real space with semilinear topology. Such spaces are of special interest (see, for example [2]).

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